Deficit or difference? The role of students' epistemologies of mathematics in their interactions with proof

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ABSTRACT
The ability to handle proof is the focus of a number of well-documented complaints regarding students' difficulties in encountering degree-level mathematics. However, in addition to observing that proof is currently marginalised in the UK pre-university mathematics curriculum with a consequent skills deficit for the new undergraduate mathematics student, we need to look more closely at the nature of the gap between expert practice and the student experience in order to gain a full explanation. The paper presents a discussion of first year undergraduate students' personal epistemologies of mathematics and mathematics learning with illustrative examples from twelve student interviews. Their perceptions of the mathematics community of practice and their own position in it with respect to its values, assumptions and norms support the view that undergraduate interactions with proof are more completely understood as a function of institutional practices which foreground particular epistemological frameworks while obscuring others. It is argued that enabling students to access the academic proof procedure in the transition from pre-university to undergraduate mathematics is a question of fostering an epistemic fluency which allows them to recognise and engage in the process of creating and validating mathematical knowledge.

KEY WORDS: Epistemic fluency; epistemologies; mathematics community of practice; proof; transition from pre-university to undergraduate mathematics

1. INTRODUCTION

The concept of epistemic fluency (Morrison and Collins, 1996) has been developed within academic literacy and instructional design contexts to describe the extent to which individuals negotiate new social and cultural contexts in order to operate successfully within them in terms of 'reading' the values and attitudes of a social practice and what counts as knowledge and appropriate behaviour within it. Epistemic fluency enables learners to shift between practices by means of the ability to 'identify and use different ways of knowing, to understand their different forms of expression and evaluation, and to take the perspective of others who are operating within a different epistemic framework' (Morrison and Collins, 1996, p.109). It is therefore key to success in transition states such as the move from pre-university to university level mathematics learning.

Students will cross the boundaries between practices on a number of occasions during their mathematics learning careers as they move between different contexts (for example, the 'real world' versus the mathematised world of the classroom), across disciplines (for example in the case of mathematics applications in science and engineering), within the different branches of mathematics as they use terms and tools in new problem situations and - if they continue beyond the years of compulsory mathematics education - between educational sectors. Student performance in a new mathematical context such as the first year at university draws on, and makes new demands on, levels of epistemic fluency developed through pre-university experience. The match or mismatch between students' developing epistemological beliefs about mathematics and those implied in degree-level and research-level mathematics will be of central importance, as will individual students' more general beliefs about their own learning. Furthermore, their experiences of the pedagogic practices of successive educational communities as they move through school or college to university will have a role to play in fostering or inhibiting epistemic fluency. In this paper I explore the implications of a focus on epistemic fluency for our understanding of student difficulties with proof at degree level. I will briefly examine the general literature on epistemic beliefs and communities of practice before turning to that which identifies particular problems in students' interactions with proof. Using interviews with twelve first year undergraduate mathematics students to illustrate points from the literature, I will then explore the more specific issue of epistemological beliefs about mathematics and the influence of pedagogical styles on their development. Although the students interviewed differ in their motivations for studying mathematics and their aspirations for the future, their personal epistemologies of mathematics and their related perceptions of proof share common characteristics which highlight particular pedagogical issues in university-level mathematics study.

2. LEARNING DISCIPLINARY KNOWLEDGE: THE ROLE OF STUDENTS' EPISTEMOLOGIES WITHIN THE INSTITUTIONAL COMMUNITY OF PRACTICE

Students' conceptions of knowledge and learning have been extensively researched since Perry's (1970) recognition that these 'personal epistemologies' (Hofer and Pintrich, 2002) are related to their educational achievements. Within the specific context of mathematics learning, research has identified the importance of individuals' beliefs about how they learn and their beliefs about the nature and validation of knowledge claims within the discipline, together with the centrality of learner identities (for example, Kloosterman, 1996; Sierpinska and Lerman, 1996; Ernest, 1999; Boaler and Greeno, 2000; De Corte et al., 2002).

The relationship between beliefs about mathematics, beliefs about learning, and learner identities is not a straightforward one, however. For example, it is possible for an individual to hold beliefs about their own learning which conflict with their beliefs about a discipline, as Schoenfeld (1988) has shown. It is broadly recognised that our understanding of these complexities and contradictions is enhanced by including in the picture one further area of research: the powerful role of the classroom context of learning mathematics (Cobb and Yackel, 1998; Solomon, 1989; Boaler, 1999; Schoenfeld, 1992). To this end, I also draw in this paper on Wenger's (1998) analysis of communities of practice to provide a theoretical framework which enables us to make sense of the interface between an individual's beliefs about a discipline and their self-positioning within the community, be it school classroom or university lecture theatre.

Within this theoretical framework, we can understand students' epistemologies as central to an analysis of the nature of undergraduate learning and the shifts that are required in moving from pre-university to university mathematics. Their apprehension of the discipline-based ground rules which define not a set of skills but rather the structure of values, attitudes and ways of thinking and doing necessary for success within a particular community of practice will play a fundamental part in how they engage with it and act as learners within it. To use Morrison and Collin's (1996) terminology, a student's success in a particular discipline is a function of the extent to which they are aware of and able to engage in the epistemic games of that discipline. Thus a key difference between experts and novices is their approach to, and understanding of, novel situations and problems. This is not just a matter of accumulated knowledge, however; membership of a community of practice involves sharing common values, assumptions, purposes and rules of engagement and communication. Within mathematics, Schoenfeld's (1992) work on problem solving demonstrates the qualitatively different approaches of experts and novices in terms of their strategies of exploration, analysis, planning, implementation and verification. Burton's (1999, 2002) work on research mathematicians similarly shows the range, depth and complexity of their practices. Undergraduates, on the other hand, frequently bring to learning a constrained range of epistemic beliefs and corresponding mathematical practices which are fostered by earlier classroom experiences. Research evidence on this cluster of beliefs presents a common view of mathematics:

> Mathematics is associated with certainty, and with being able to give quickly the correct answer; doing mathematics corresponds to following rules prescribed by the teacher; knowing math means being able to recall and use the correct rule when asked by the teacher; and an answer to a mathematical question or problem becomes true when it is approved by the authority of the teacher. (De Corte et al., 2002, p.305)

The following items are added to this view by Schoenfeld (1992, p.359):

> Mathematics problems have one and only one right answer.
> There is only one correct way to solve any mathematics problem – usually the rule the teacher has most recently demonstrated to the class.
> Mathematics is a solitary activity, done by individuals in isolation.
> The mathematics learned in school has little or nothing to do with the real world.
> Formal proof is irrelevant to processes of discovery or invention.

Mis-match between expert and student perceptions of the nature and role of proof, authority and certainty in mathematics are detailed further by a number of researchers (Dreyfus, 1999; Almeida, 2000; Hanna, 2000; Recio and Godino, 2001). Relatedly, experts and novices differ in terms of their beliefs about themselves as learners or problem-solvers (ie, their motivational and self-regulatory beliefs), and their beliefs about the mathematical problem-solving context itself and their role or identity within it (see Carlson, 1999; Schoenfeld, 1992). These beliefs are embedded within particular communities of practice and constitute, and are constituted by, the identities of the participants.

Much of the existing literature on the growth of mathematical understanding focuses on development within one specific educational context, community or sector. We know little about the role of student epistemologies in transition from one learning community to another, and in particular from pre-university courses into university. Entry into university makes new learning demands: even if a student is pursuing a subject which they have already studied, its treatment in the university context will be different, particularly with respect to the ways in which claims are made and defended. Success in entering into a novel community of practice will depend on students' levels of epistemic fluency in terms of their awareness of the existence of epistemic games (ways of co-constructing knowledge) which involve different kinds of epistemic forms (target knowledge structures which are characteristic of the community), and their metacognitive awareness of their own success in accessing these new ground rules. In Wenger's communities of practice model, individuals differ in their relationship to the ground rules: identities of participation and non-participation reflect the extent to which individuals are able to control and negotiate meanings within the practice, and even to create new meanings. For the novice, the crucial development is to see oneself as a 'legitimate peripheral participant' (Lave and Wenger, 1992) who, while not yet a full participant, has the *potential* to become one. But as researchers in school mathematics have shown, learners can be, and often are, excluded from the negotiation of meaning or even the beginnings of it, developing instead an identity of non-participation and marginalisation. Their lack of ownership generates and is generated by compliance with authority and an emphasis on following pre-set procedures which are reflected in the epistemologies of mathematics noted above. This theoretical juxtaposition of identities and epistemologies generated within the classroom community of practice can provide an insight into the issue of proof.

2.1 The case of proof

The ability to handle proof is a classic example of transfer issues between school and university sectors, and is the focus of a number of well-documented complaints regarding students' difficulties in encountering degree-level mathematics (Anderson, 1996; Almeida, 2000; Cox, 2001; Kyle, 2002). However, in addition to observing that proof is currently marginalised in UK pre-university mathematics curricula with a consequent skills deficit for the undergraduate mathematician, Almeida (2000) also makes a further important point regarding expert practice which refines the view: while, as MacLane (1994, p.191) points out, the research mathematician handles proof procedures which include as a matter of course *intuition, trial, error, speculation, conjecture, proof*, undergraduate *teaching* stresses instead a much simpler and very different model of *definition, theorem, proof*, as discussed in Moore (1994). Hanna (1995) too has observed that whereas mathematical practice uses proof to justify and verify, mathematics teaching uses it to explain. As Sierpinska (1994) points out, this is an important distinction: there are substantive epistemological differences between proof and explanation. From the point of view of the theoretical framework laid out above, this is by far the more important disjunct, because it involves much more than a skills deficit. The exclusion of students from the knowledge construction process results in undergraduates who 'exhibit a lack of concern for meaning, a lack of appreciation of proof as a functional tool and an inadequate epistemology' (Alibert and Thomas, 1991, p.215) and who are outside of the formal proof culture of academic mathematicians (Harel and Sowder, 1998). Crawford et al.(1994) observe the related effect that first year mathematics students see mathematics learning simply as a rote learning task. Thus undergraduate students see proof in an instrumental and performance-related way rather than as an intrinsic component of being a mathematician subscribing to a mathematician's values, assumptions and practices.

Almeida's (2000) study is illuminating in this respect because it directly accesses undergraduate perceptions of proof and proof practices. Responses to statements about proof from first to third year students show increasing awareness of key features of proof, but students tended to perceive it as an exercise to be undertaken in all circumstances, even when not required, thus suggesting pedagogic enculturation into proving as demonstration and explanation rather than as a means of gaining insight into a problem. Recio and Godino's (2001) examination of the contrast between institutional and personal meanings of proof provides related evidence that students spontaneously and by preference use empirically-based proof schemes, particularly when presented with new or more complex problems. They argue that students' simultaneous membership of a number of social institutions, each with different ways of defending knowledge claims, accounts for their failure to distinguish between appropriate uses of different types of argumentation. Thus they are familiar with empirical inductive proof in scientific situations, and with informal deductive proof in the classroom, but rarely encounter or participate in formal deductive proof. This analysis concords with the observation that pedagogic exposure to proof – as explanation of a finished product - is rather different from its use by mathematicians as part of a creative process replete with blind alleys and false starts and including both formal and informal proof schemes.

That students' epistemological confusions are the result of a disjunct between the practitioner's tacit knowledge and actual practice of proof and the way in which mathematics is taught and portrayed, and a corresponding mis-match of identity and approaches to problem solving, seems highly likely. As Rav (1997, p.36) points out, 'mathematics is a collective art' with proof playing a central role which goes beyond a purely logical-deductive function. Ernest (1998, 1999) similarly emphasises the social nature of proof and the discursive differences between pedagogic and research mathematics. Thus an analysis of proof presents an interesting case study of epistemic fluency: accessing the academic proof procedure is a question of students' ability to apprehend practices which are at best only implicit and at worst obscured by teaching practices which carry another message, coinciding with many students' pre-existing epistemological frameworks. Their ability to see through this pedagogic screen depends on an epistemic fluency which enables a wider perception of the purpose of university mathematics, a self-monitoring of their success in entering the community of practice in question, and a participative identity in relation to that set of practices.

The theory and research reviewed here indicates that students' and experts' conceptions of knowledge and knowing are a major feature of their engagement with a discipline. However, the exact nature of the relationship between personal epistemologies and student learning in mathematics is unclear, particularly in the transition to university. How do the mathematical epistemologies held by students at university compare to those observed in younger students? How do undergraduates reflect on and deal with the change in educational context? What motivational and self-regulatory characteristics come into play at university? Is there any evidence of shifts in identity and perceived authority? Attempting to answer these questions may shed some light on the issue of difficulties in transfer to university mathematics, including students' dealings with proof. In the remainder of this paper I will use extracts from interviews with twelve first year undergraduates to illustrate the potential of an examination of their personal epistemologies, returning to the issue of epistemic fluency and how it can be supported in the final section.

3. THE STUDY

While much research has used quantitative methods in the study of students' epistemologies, the complexity of the issues raised in the discussion of the literature above suggests that a qualitative approach may shed new light on the relationship between beliefs about mathematics, beliefs about learning, and learner identities in students entering a new community of practice.

3.1 Participants

The data presented here were collected in interviews with twelve first-year undergraduate mathematics students at an English university with a strong research culture. The students were self-selecting, having responded to a request delivered via their tutors to help with a project concerning mathematics learning in which they would get an opportunity to talk about their own study experiences. The participants represented a range of mathematics student profiles: ten respondents were aged 19-20, and included four women and six men; the eleventh was a twenty-three-year-old male mature student, and the twelfth was a thirty-four year old female mature student. Schools in England offer two mathematics qualifications between the ages of 16 and 18: students choosing to study mathematics take the Advanced Level General Certificate of Education in Mathematics, but some take in addition Advanced Level Further Mathematics, which builds on the material of the Advanced Level Mathematics syllabus. Of the regular age students all had taken Advanced Level Mathematics and one had taken Further Mathematics; both mature students had entered the university with a further education college access award in mathematics. Students at this university take up to three subjects in their first year of study, proceeding in their second and third years to study their chosen major subject or subjects (joint majors combining courses from two subject areas). All the participants were taking the basic first-year mathematics course offered at this university, but six were taking an additional mathematics option, compulsory for intending mathematics majors. Three students (one male, two female) were registered for a single major degree in mathematics, one (female) for a single major in environmental mathematics, two (both male) for a joint degree in mathematics combined with computer science, one (male) for a joint degree in mathematics and management, one (female) for a combined sciences degree with mathematics options and four (one female, three male) for major degrees in other subjects with mathematics as a minor subject.

While all students enter the university in order to study a particular major or joint major degree, a small number opt to change their intended major at the end of their first year, pursuing instead another degree programme. Thus students are able to move from joint major degree schemes to single major schemes and vice versa, and also to change their major subjects altogether, providing they have taken the subject in question in their first year. Some of the students in the sample were intending to make these sorts of changes. The three mathematics single major students were intending to continue as mathematics majors into the second year of university and the environmental mathematics student was continuing in environmental science and taking statistics as a minor only. Of the three students who were combining mathematics with another subject as joint majors, one was continuing as joint, one was intending to take mathematics as a minor subject only, and one was intending to change from a joint degree in mathematics and computer science to a single major in mathematics. The remaining students – taking mathematics as a minor or as part of a general science degree - showed a similar variety of intentions. One notable instance was Richard's complete change of major from management to mathematics. These details are summarised in Table 1, which shows each participant's registered major on entry to the university, and their intended major for the second and third years of their degree.

Table 1: Student profiles

Student name[1]	**Male/ Female**	**Registered major on entry: Mathematics majors/joint majors are in bold**	**Intended second and third year major subjects: Mathematics majors/joint majors are in bold**
Carol	**F**	**Environmental mathematics**	Mathematics minor only
Debbie (mature)	**F**	**Single major mathematics** RS minor	**Single major mathematics**
Sarah	**F**	**Single major mathematics** Art minor	**Single major mathematics**
Larry	**M**	**Single major mathematics**	**Single major mathematics**
Pete (mature)	**M**	**Mathematics/computer science joint**	**Mathematics/computer science joint**
Steve	**M**	**Mathematics/computer science joint**	**Single major mathematics**
Joe	**M**	**Management/mathematics joint**	Statistics minor only
Sue	**F**	Combined sciences (includes mathematics options)	Combined sciences, including mathematics
Diane	**F**	Geography	Geography
Charlie	**M**	Computer science	Communication studies
Chris	**M**	Natural sciences	Statistics minor only
Richard	**M**	Management	**Single major mathematics**

3.2 Interviews

The students were contacted by e-mail and asked to come along to the interview with a selection of work, including a topic they had enjoyed and/or found easy, and a topic which they had disliked or found difficult to do. The interviews were semi-structured, lasting for approximately one hour each and focussing on the following issues: the students' 'mathematics histories' and comparisons between mathematics at school or college and at university, the effect of different teaching styles on their learning experiences, their experiences of getting 'stuck' and strategies for resolving problems, the topics they found easy or hard (students were asked to talk through the examples they

[1] All names are pseudonyms.

had brought with them), comparisons with other subjects in terms of the kind of work expected and how they approached the subject matter and tasks, the students' reasons for choosing mathematics at university, their views on what kind of approach would lead to success in mathematics, and their perceptions of research mathematics and of themselves as mathematicians. The students were interviewed individually when they were approximately two-thirds of the way through their first year at university. The interviews were audio-taped.

3.3 The analysis process

The interviews were transcribed in full and analysed thematically with assistance from Atlas-ti qualitative analysis software. This entailed assigning relevant pieces of text to categories initially generated from the theoretical framework outlined above; these focused on attitudes towards and definitions of mathematics, comparisons between mathematics at school and at university, beliefs about learning (with respect to students' own perceived learning styles but also with respect to mathematics learning generally), performance versus mastery orientations in motivation patterns, and negative or positive learner identities. Repeated exploration of these categories and the connections between them generated recurrent themes which were evident across the interviews and which are presented in section 4 with illustrative quotes to represent particular themes (see Seale, 2000, for an analysis of techniques similar to those employed here).

4. WHAT IS MATHEMATICS? STUDENTS' EPISTEMOLOGIES

A number of recurrent themes and connections emerge from the interviews which are indicative of particular patterns in the students' beliefs about the nature of mathematics and their own relationships to it as learners in the university mathematics community – both are encapsulated in their beliefs about proof. Their more general beliefs about learning, whether their own or that of others, are closely linked to these beliefs about mathematics, and are indicative of their levels of epistemic fluency in negotiating the boundaries of university mathematics. In this section I will show how these themes are interwoven in the students' accounts, so illustrating the effect of prevailing institutional relationships which act as a potential barrier to students' engagement in higher mathematics.

4.1 Beliefs about mathematics

Beliefs which are very similar to those observed by Schoenfeld (1992) and De Corte et al. (2002) are clearly discernible in the interviews. A general theme of certainty in mathematics emerged, coupled with an emphasis on the necessity of learning rules, reproducing solutions and working at speed to get correct answers. Correspondingly, the students thought that creativity in pure mathematics was not possible, although statistics afforded some opportunities in this respect. For example, Richard, who was changing from a management major to pursue single major mathematics, was emphatic about the certainty of mathematics in contrast with his management course: 'when I hand in an essay, who's the tutor to say that someone else's is better than mine? …. There's a right and wrong in maths … there's nothing that's open to the teacher's opinion'.

Steve, who was intending to change from a combined major to pursue single major mathematics, made similar comparisons within the mathematics curriculum itself, contrasting an assumed certainty in pure mathematics with a greater need for interpretation in statistics in which he ran the risk of 'not being quite right'. Pete (continuing mathematics and computer science joint major) also expressed a strong dislike of statistics because of its lack of precision, while Sarah (continuing single major mathematics) described statistics as 'a bit fiddly' in comparison to pure mathematics which is 'do this, do that'.

While these students disliked the need for interpretation in statistics, it was precisely this quality that the students who favoured statistics over pure mathematics liked. Both Chris (natural sciences major, retaining statistics as a minor only) and Joe (mathematics/management joint major, retaining statistics as a minor only) saw statistics as allowing for more autonomy and as being more meaningful. Thus Joe claimed that 'there is a form of arguing in stats' in contrast to pure mathematics which for him was void of argument and dominated by rote-produced solutions. Carol (environmental mathematics major, retaining statistics as a minor only) favoured statistics for the same reason, arguing that it offered more scope for creativity, 'whereas we're not going to discover anything new to do with pure maths'.

Students who were pro-statistics also appreciated it as having potential applications to real-world problems, arguing that, conversely, pure mathematics had no applications at all. Both Carol and Charlie (computer science major, changing to communication studies major and dropping mathematics) felt that this perceived aspect of pure mathematics detracted from any potential interest it might have. Sue (combined sciences major, retaining statistics options) also expressed a preference for statistics based on its more visible real-world applications in comparison to the difficulty and exclusivity of pure mathematics in which 'there doesn't seem to be any sort of reason'.

These views were shared generally among the group. Although five of the intending major students said that they actively preferred pure mathematics to statistics, they concurred with the others in making a clear distinction between statistics as useful and pure mathematics as generally little more than a puzzle with intrinsic value. This general emphasis on pure mathematics as a rule-bound and largely useless activity in which they could not participate in any creative sense spilled over into images of mathematics as an isolated pursuit and mathematicians as rather cranky individuals, very different from themselves. For example, Carol described research mathematicians as 'just sitting in their office with a calculator for 4 hours a week or something. … I can't really see them discovering anything new'.

As these extracts illustrate, the students tended towards epistemologies of mathematics which assumed certainty, irrelevance, rule-boundedness and lack of creativity potential in pure mathematics. For some this was tolerable, for others it was not. It may well be significant that all those students who favoured statistics on the grounds that as a branch of mathematics it was not prone to these particular perceived characteristics then proceeded to reject mathematics as a whole, changing to other major subjects in their second year. It seems unlikely that this is merely an issue of subject preference – all the pro-statistics students had chosen to study mathematics at degree level, three of them originally intending to study it as part of their major options. An alternative explanation is that their university experience did not foster a sense of how they could become part of the mathematics community of practice, with the result that they opted out altogether rather than remain on the margins. Indeed, the students seemed largely unaware of the existence of a mathematics community of practice which might have negotiable rules of communication and validation beyond the simple authority of the individual teacher-experts with whom they came into contact.

4.2 Beliefs about learning mathematics

In their account of beliefs about the self in relation to mathematics, De Corte et al. (2002, p.307) differentiate between goal orientation (I want to understand), task value beliefs (learning the material is important), control beliefs (learning is possible with proper study), and self-efficacy beliefs (I can understand difficult material). While we might anticipate that making a free choice to study mathematics at university would be associated with strong and positive versions of these beliefs, this was not evident in the students' accounts. Nor were they necessarily intrinsically motivated to study mathematics: Charlie, Richard, Joe, Chris and Larry all said that they had chosen mathematics at university for its value in the labour market. Sue, Debbie and Sarah also made reference to the relevance of mathematics to their career plans. Richard presented a particularly interesting case because he had lacked the confidence to major in mathematics at university, choosing management science instead as a good career option, despite the fact that mathematics was his best subject at school. However, his first year performance, in which he was consistently scoring higher than most other students, had persuaded him that he was 'quite good' at mathematics and that he should change his intended major. This decision was very much based on comparison with others and an explicit prioritisation of good marks over understanding:

> Sometimes I don't really understand the process, but if I can apply it I'm happy with that. If I was struggling I'd drop it. … I have to be the best. [*Doesn't that make you vulnerable?*] That could be the reason I didn't take it in the first place.

Comparison with the performance of others is a recurrent and explicit theme in Richard's account but it is also very evident in the other interviews. Pete invoked the familiar theme of speed of understanding as a defining characteristic of those who are good at mathematics, while Sarah illustrated its companion attribute of succeeding on the basis of very little work, claming that students who are good at mathematics 'usually don't do much work at all … they leave it to the last minute and they just do it and then they get full marks'. The emphasis on speed of understanding and working is linked to a predominant fixed ability belief which was visible in most accounts. For instance, Pete believed that 'at university it's requiring more innate abilities' while Sarah thought that the good students 'just have that skill … their mind is different or something'. As might be expected, control beliefs and self-efficacy beliefs were not particularly evident. Diane (geography major, dropping mathematics) did not believe that any amount of effort would enable her to achieve the same as the 'good students': 'I could be taught to do maths but I don't think I could be taught to be good at maths'. Larry, who was intending to continue with his single major in mathematics, was a lone voice in describing a high level of self-efficacy when it comes to solving a difficult problem; he was also the only student to describe mathematics as an integrated subject rather than a set of isolated facts: 'if you know a method - a method that makes sense - you can combine that to get something else'.

The emerging themes in the students' accounts of learning mathematics centre on fixed ability beliefs and a consequent tendency to focus on performance rather than mastery. They are closely related to their epistemologies of mathematics, focussing as these do on rule-bounded sterility, irrelevance and certainty in mathematics. This partnership of beliefs about mathematics and beliefs about mathematics learning relates also to a lack of epistemic fluency in the students. Although they did reflect on their learning, their beliefs about mathematics and about learning mathematics appeared to hamper the development of strategies which would enable them to cross the boundaries of university level mathematics and enter into the community of practice inhabited by their tutors. Most notably, with the exception of Sarah, they did not recognise the possibility that they might create mathematics by engaging with its values and assumptions; their marginalised position was linked to their experience of mathematics pedagogy, as the next section shows.

4.3 Negotiating the boundaries of higher mathematics – identity and authority

Cobb and Yackel (1998) describe how the social and socio-mathematical norms of classrooms prescribe teachers' and students' roles and the nature of knowledge, explanation and justification. In the transition to university it might seem reasonable to look for a shift in identity and perceived authority towards a more autonomous and participatory role for the student. There was little evidence of this, however. On the contrary, a strong theme in the students' accounts was one of disenfranchisement in the learning process, and adverse comparisons with their prior experience at school. Diane, for instance, recounted how her school mathematics experience had been far more participatory and connected than at university:

> [At school] because each of us understood different parts of it we [would say] "no, no, you're wrong" and "well explain yourself then". So one of us would be teaching and the rest of us would go "oh, I see". …. It puts it into context a lot better and it just gives the basis of the whole topic so that you can refer back to it and you think "oh, yes, I'm doing this because …". [At university] they should be trying to make you aware that you can bring all the other bits into the topic that you're doing at the moment.

Within the university itself, the students described themselves as outside of the mathematics community. Their relationships with the lecturers required them to engage only with mathematics as already created rather than with the disciplinary process of creation and validation of knowledge, and their experience of missing explanations and exclusivity placed them on the periphery of the community. Joe summed up his own experience and also that of Sue, Carol and Chris:

> The lecturers are always setting us more challenging work which we don't understand .. you never really feel like a mathematician because you don't understand how it works.

Although they were not necessarily able to articulate the difference, the students effectively described their university teachers as having two types of institutional relationships: their membership of the mathematics research community, and their pedagogic relationship with their students. The first of these was hidden from the students, and based on practices which they were only dimly aware of. The second was their public role as far as the students were concerned, in which they and their students acted in ways which coincided with the students' narrowest epistemic frameworks.

Sarah was the only student who described an attempt at creativity in mathematics, comparing it to art, which she also studied:

> At the end you have this nice thing and you have worked all through it … in a way I am not very creative in my maths, but in a way I am as well because sometimes I'm working and I think "oh maybe this could work", and I get all excited.

Sarah's position is an unusual one in the group. Her willingness to experiment demonstrates a different quality of epistemological belief about mathematics learning and mathematics itself. Unlike the other students, she sees herself as a potential contributor to mathematics knowledge, and in a position to engage with the community, albeit in a junior role. She recognises that it has rules and believes that she has the potential to access and manipulate these rules – despite her beliefs about innate ability reported above. Sarah thus demonstrates an identity of 'legitimate peripheral participant' and a degree of epistemic fluency which is lacking in the other students with the possible exception of Larry, who, as we have already seen, looks for usable connections within mathematics. Given the predominant pattern of institutional relationships and their corresponding epistemologies, it is unsurprising that the majority of the students were unclear about the central role of proof. As we shall see in the next section, Sarah's outlook led her to take a different and unusual stance on this matter too.

4.4 Beliefs about mathematics and authority - the role of proof

None of the undergraduates in this study demonstrated a very clear understanding of proof and its role in mathematics. Although this may be unremarkable in itself – they simply haven't yet been taught, one could argue - their beliefs about mathematics and their perceived place in it form a powerful context for their comments, which are in line with the literature reviewed in section 2.1. Carol presents a common view of proof as something that lecturers - not students - do, as part of teaching:

> It's something that you get on the OHP when they're doing some new part of pure maths and you're being shown and then you could probably work it out again yourself but I don't think we've ever had to sit there and come up with a reason for the things that we're doing.

Joe talks about proof in terms of tutor demonstrations which can be ignored because they are not assessed:

> All the things we look at we're told how to prove it, but then we were told we didn't need to know how to prove it. So as soon as we were told that I never looked at it again, they would show us how to prove it and then they would say 'you'll never need this' - so that was that - so I just thought 'forget that'.

Chris too associates proofs with tutors and believes that they merely serve the function of giving the mathematics department's teaching an appearance of quality:

> I'm told "so and so and so and so is this" then I won't go and read and try and understand why. I just remember the result ... I think they just do it so they don't get criticism of just throwing it at you.

However, while Carol, Joe, Pete and Chris limit proof to tutor demonstration in Moore's *definition, theorem, proof* sense, Larry and Sue recognise that proof is something which they will have to do themselves, although they both felt inadequate for the task. Diane feared proof, describing considerable confusion as to its nature and practices:

> In the A-Levels the questions I'd always got stuck on would be "show that this equals this". I think, "How am I meant to? What am I meant to use? If I use any of these am I not just using the fact that this is equal to this to prove it?" … I see the word "prove" and think "Oh no".

Unlike the other students, Charlie, although not explicit on the nature or methods of proof, did associate it with doing 'proper' mathematics – by which he meant the use of insight and experience rather than 'cookbook' rule following:

> I quite enjoy doing that sort of stuff [proof]. … A lot of stuff, … you either know it or you don't, you just follow a set path and you do it … . In [proof] you're using maths to do it rather than just your memory.

Charlie's distinction between mathematics and memory indicates his awareness of mathematics as a social practice rather than a collection of algorithms. But although he enjoys proof, he describes mathematicians and their activities as separate from himself and his own interests and capabilities. Steve also placed himself outside of the mathematics community, but in a different way from Charlie: even though he was opting to study a single major in mathematics, he did not see proof as an integral part of doing mathematics. What is striking about his comment here is its illustration of his focus on career and his instrumental view of his university experience:

> I concentrate on just doing the methods, I accept whatever it is, I don't question, I don't really focus on proof…[*But isn't proof a part of maths?*] I'm not quite sure, I think it would be, it just depends whether it's useful after your degree, it depends what job you do ..It's interesting to see where it comes from, but I'm not too sure about after, applying proofs.

Debbie (continuing as a mathematics major) had a far more positive attitude to proof. Her interview illustrates the importance not only of students' beliefs about mathematics but also their beliefs about learning. While she did not feel able to engage with proof right now, Debbie anticipated being able to do so:

> It's weird because even though I didn't really understand it, it took me a while to get to understand certain things we have to do, I did sort of feel to myself "I think I'm going to like this". I like the concept of it, I like proof, I think I'm going to be all right with that.

In a way which is predictable from her comments on creativity, Sarah showed the most understanding of the role of proof in mathematics and an eagerness to participate in it. She is willing to try things out as we have already seen; while she recognises that she does not yet have the skills required she does not interpret this situation as excluding:

> Sometimes if someone is teaching me, or I am doing a problem, I might see a connection between some things and I will think "Oh, maybe this would work and then maybe I would be able to prove that, and this and the other" - which doesn't usually work but I do think about it.

What distinguishes Sarah, and to some extent Debbie, from the other students is the recognition of proof as a part of mathematics rather than as an optional extra. Both have beliefs about learning which enable them to sustain a self-image of potential participant, and in Sarah's case this extends to acting as a participant, albeit in the role of novice or legitimate peripheral participant. Her attitude to proof is part of her wider view of mathematics as a creative process which she can be part of. In this respect her outlook on mathematics is surprising given the institutional context which is so apparent in the interviews. She exhibits an epistemic fluency in terms of her understanding of the community of mathematicians: unlike the majority of the students she sees through teaching practices which compound their view of a sterile mathematics.

5. CONCLUSION: NEITHER DEFICIT NOR DIFFERENCE

The case studies reported here are striking in their portrayal of familiar themes when it comes to mathematical epistemologies and identities and as such they provide an insight into why undergraduate students struggle with proof. As the extracts reported in sections 4.1 to 4.3 show, students' beliefs about the nature of mathematics as a matter of certainty, rule-following, isolation, abstraction and lack of creativity differ little from those identified by researchers into school mathematics. Again in correspondence with school research, their beliefs about learning mathematics emphasise speed and fixed ability. Rather than reporting a shift in identity towards perceptions of themselves as negotiators or even *potential* negotiators of mathematical knowledge, these students described themselves as powerless, and university lecturers as aloof authority figures. These beliefs and identities are encapsulated in the specific case of proof: the dominant view described in section 4.4 was to see it as an irrelevance, or as an unattainable and excluding skill. There is therefore little evidence in the students' accounts of epistemic fluency in the sense of an awareness of the existence of discipline-specific ways of co-constructing knowledge or their own success in accessing these. Indeed, for the most part, the interviews are illustrative of learners who are highly marginalised (Wenger, 1998) and, to use Schoenfeld's (1988) terms 'passive consumers of others' mathematics'. From this point of view, these students are not simply demonstrating a deficit in their skills repertoire in their difficulties with proof. They are working within a different epistemological framework from that of the established mathematics community, a framework which does not include the creativity or ownership associated with proof. Importantly, though, they are not at odds with the public face of mathematics as they experience it within the institution of the university and as enacted by their teachers, and in this sense they exhibit neither deficit nor difference – both students and teachers are acting in accordance with the same institutional relationships and epistemic frameworks. This is particularly evident in the students' accounts of identity and authority reported in section 4.3 and their corresponding descriptions of proof in section 4.4 as mere gloss in the teaching context. Success in these circumstances ultimately relies on students' ability to see beyond this public image and engage in the creative practices which unsurprisingly strike them as so private. As the interviews discussed here show, this is rare and is not fostered by the institution.

These observations raise two major issues with implications for mathematics teaching. First, as a number of educators have observed, students' experiences of traditional mathematics teaching which emphasises 'received' mathematics are unlikely to engender attitudes and identities which enable them to take control of their own development as mathematicians (Alibert and Thomas, 1991; Boaler and Greeno, 2000). Even where students are encouraged to make their reasoning explicit in an attempt to 'make learning experiences more co-operative, more conceptual and more connected' (Dreyfus, 1999, p.85), lack of clarity on the part of both teachers, students and textbooks as to the relative status of different kinds of mathematical explanations militates against shared epistemologies as Dreyfus' research shows. As Harel and Sowder (1998, p.237) point out, 'we, their teachers, take for granted what constitutes evidence in their eyes. Rather than gradually refining students' conception of what constitutes evidence and justification in mathematics, we impose on them proof methods and implications rules that in many cases are utterly extraneous to what convinces them'. We might add to this that teaching which emphasises mathematics as already created rather than mathematics *in* creation will do little to contribute to this refinement. As the analysis of the data shows, an assumption of mathematics as already created, neatly packaged, and accessible only to the quick and able is pervasive and presents a major barrier to the development of a conception of even potential participation in the negotiation, construction or validation of knowledge.

Secondly, the normalisation of a university education and the expectation that most young people who enter post-16 education will go on to a university education and enter the job market as graduates has shifted undergraduates' perceptions of that education towards more instrumental views. The group interviewed here are no exception, as illustrated most clearly by their highly performance-oriented beliefs about learning in section 4.2 but also Steve's attitude to proof in section 4.4. Their instrumental approach may be exacerbated by the massification and related vocationalisation of the university system which alters the overall aspirations and expectations of undergraduates. At the same time, the shift to a marketised and managed system with growing numbers of students has changed the nature of work in academia. In these circumstances, a performance orientation which prevents deeper engagement with the subject is perhaps not surprising. As Dreyfus (1999, p.106) points out, students need to 'make the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures'. Since computation is related to performance, we might add, in the light of these interviews, that they need to make the transition from a performance-oriented and individualistic view of mathematics which is encouraged by market-driven education policies to a view of mathematics which emphasises construction, communication and community. Without such a view, students are unlikely to abandon their ideas of proof as definition and explanation for a more risky engagement with proof as involving intuition, trial, error and speculation. Sarah's insight is, as I have shown, not the norm, and Sue is not alone in her feeling that mathematics is a mystery outside of her own sphere of understanding when she says that 'In maths it seems they change the rules when they want…. Why don't they just tell you the truth?'. A pedagogic shift is required towards teaching in ways which allow for and foster epistemic fluency. This can be enabled through a sensitivity to students' epistemological beliefs as they enter successive educational phases and an explicit bridging into the new. As Burton (2002) argues, we should aim to reconcile, rather than eradicate, differences between different communities of practice. Understanding and building on students' own pre-existing epistemological resources appears to be the key to fostering their recognition of and engagement in the process of creating and validating mathematical knowledge, but at the same time, educators need to recognise the force of their projected epistemologies and the need to make their own practices transparent.

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