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On asset pricing in a binomial model with fixed and proportional transaction costs, portfolio constraints and dividends

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Abstract
We extend the classical binomial model proposed by Cox, Ross, and Rubinstein for derivative security pricing to encompass both fixed and proportional transaction costs, portfolio constraints including margin requirements, and dividend-paying assets. Our focus is on studying option hedging within this enriched framework. Initially, we establish the existence of a hedging strategy in this context. Subsequently, we determine the optimal hedging strategy and its associated initial cost by decomposing the problem into a sequence of hedging problems. To illustrate our approach, we present a numerical example within a 3-period binomial model.

Keywords Binomial model · Self-financing condition · Transaction costs · Hedging · Portfolio constraints · Dividends

JEL Classification G10 · G11 · G12 · G13 · C61 · C65 · C67

1 Introduction

The binomial model, first introduced by Cox et al. (1979), commonly known as the CRR binomial model, is a widely used approach for modelling financial asset prices across various market conditions. It simplifies the complex dynamics of financial markets by assuming that the price of an asset can only take two possible values at any given time, and these values are determined by the underlying market conditions.

In recent years, there has been significant progress in developing binomial models, both in terms of theoretical advancements and practical applications. For example, Shvimer and Herbon (2020) conducted empirical studies on binomial call-option pricing using S&P 500 data, while Muroi and Suda (2022) introduced a discrete

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cosine transform method to enhance option pricing in models such as jump-diffusion processes. Kim et al. (2019) extended the classical CRR model by incorporating time-dependent parameters and introduced a trinomial model to improve hedging strategies. Breton et al. (2023) proposed a q-binomial extension of the CRR model with time-dependent switching probabilities, and He et al. (2019) developed a nonparametric predictive inference framework, providing more flexibility in option pricing. Several studies have also focused on addressing transaction costs. For instance, Belze et al. (2019) examined fair value adjustments in the presence of transaction costs, while Ratibenyakool and Neammanee (2019) explored the convergence rates of binomial models. In addition, binomial models have been applied beyond traditional markets, such as by Liu and Ronn (2020) for renewable energy investments and Yeh and Lien (2019) for real estate development.

Transaction costs, which include fees, commissions, bid-ask spreads, and other trading expenses, can significantly affect asset pricing. These costs complicate the determination of an asset's true value, as they erode profits and affect market liquidity. To address this, various approaches have been developed to incorporate transaction costs into pricing models. For example, proportional transaction costs, where a fixed percentage is charged on each trade, have been modelled in both continuous-time and discrete-time frameworks. For instance, Leland (1985) applied a continuous-time framework based on the Black-Scholes model (Black and Scholes 1973) to derive an approximation for option prices that accounts for proportional transaction costs. To account for transaction costs in the binomial model, various approaches have been developed. Boyle and Vorst (1992) extended the CRR binomial model allowing for proportional transaction costs in replicating perfectly a given option. The study showed that the presence of transaction costs can significantly affect the optimal replication strategy and the corresponding option prices. Palmer (2001) adjusted the conditions under which there is a unique replicating strategy in the Boyle-Vorst model for an arbitrary contingent claim. Bensaid et al. (1992) provided conditions under which the cost of the replicating portfolio does not exceed the cost of any super replicating portfolio where the results were extended by Stettner (2000) to the case of asymmetric transaction costs. They noted that perfect replication might not always be optimal for hedging options. Melnikov and Petrachenko (2005) developed a binomial option-pricing model to cover the case of proportional transaction costs for one risky asset with different interest rates on which they presented an explicit formulas for self-financing strategies. Roux et al. (2008) developed an algorithm for computing the ask and bid prices of options with arbitrary payoffs in an arbitrary discrete model under proportional transaction costs of any magnitude.

The introduction of both fixed and proportional transaction costs adds further complexity to option pricing. Fixed costs are typically a flat fee charged per transaction, while proportional costs are a percentage of the transaction value. Several studies have investigated this transaction cost structure, particularly within the binomial model. Studies such as Bank and Dolinsky (2019) have shown that super-replication prices in continuous-time markets with fixed transaction costs can be prohibitively high. Edirisinghe et al. (1993) proposed a framework for minimizing replication costs under transaction costs and trading constraints. Numerous papers have addressed option pricing with both fixed and proportional transaction costs within a continuous-time framework. For example (Subramanian 2001) explores European Option Pricing in a market with short-selling constraints and transaction costs, covering proportional and fixed costs. Using stochastic impulse control theory, it solves two stochastic impulse control problems for option pricing. Zakamouline (2006) offers a systematic approach to utility-based option pricing and hedging in markets with fixed and proportional transaction costs. They extend the framework developed by Davis et al. (1993) and propose a numerical procedure for computing option prices and optimal hedging strategies.

Building on these foundational studies, we adopt the approach used by Bensaid et al. (1992) and extend it in several key directions. Specifically, our model incorporates both random fixed transaction costs and proportional transaction costs, with different rates for long and short positions. In addition to these transaction costs, we allow for short selling under certain constraints, including margin requirements, and we account for assets that pay dividends with potentially different random dividend rates for long and short positions. These extensions address key gaps in the existing literature, which often simplifies or omits such practical features. However, this advancement introduces technical challenges. To navigate these, we impose very general assumptions and employ the super-replicating argument, as linear self-financing conditions are not applicable and perfect replication is not feasible in the presence of such transaction costs. This adds complexity to pricing methodologies, but it also reflects more accurately the realities of markets with trading frictions. We present a three-step numerical example to illustrate our results and algorithms. Although our example simplifies certain conditions, such as assuming fixed interest rates and stable transaction costs over time, the limitations of perfect replication are still evident.

Our model is a specific adaptation of the von Neumann-Gale framework, applied in a setting with two assets-a bank account and a risky asset. Both fixed and proportional transaction costs are incorporated into this model. The theory of von Neumann-Gale dynamics, originally introduced by Von Neumann (1937) and Gale (1956), was initially developed in the context of economic growth modeling. Over time, these systems have been extended by various scholars (see, for example, Rockafellar (1967); Dynkin (1971); Radner (1970)), with more recent contributions by Evstigneev and colleagues (Evstigneev and Schenk-Hoppé 2006, 2008; Bahsoun et al. 2008). Although originally designed for modeling economic growth, these dynamics have since been adapted for financial modeling, leading to new research directions (Dempster et al. 2006; Evstigneev and Zhitlukhin 2013; Babaei 2024). This interdisciplinary approach draws parallels with economic growth models but diverges from traditional stochastic analysis by relying on intuition and techniques rooted in growth theory. For instance, Babaei et al. (2020a, b, 2021) applied von Neumann-Gale dynamics to capital growth theory under proportional transaction costs, contributing significantly to the advancement of this field.

While the binomial model has been a staple in option pricing literature for decades, its classical nature does not detract from its ongoing importance and practical applicability. Despite its simplicity, the binomial model serves as a versatile framework for understanding and pricing derivative securities, especially when considering real-world complexities such as transaction costs, portfolio constraints, and dividend payments. Our extension of the classical CRR binomial model to incorporate these factors fills a significant gap in the literature, providing a robust framework for addressing practical challenges in option pricing. Additionally, some theoretical and empirical studies have demonstrated the effectiveness of the binomial model in capturing market dynamics and accurately pricing options across various asset classes and market conditions (see e.g., the work of Kociński (2004), Shvimer and Herbon (2020), Kim and Park (2006) and others). Furthermore, the binomial model serves as a benchmark against which more complex models can be compared and validated.

In financial models that incorporate transaction costs, managing the trade-off between rebalancing frequency and transaction costs is essential. Continuous rebalancing, as assumed in continuous-time models, would lead to infinite costs due to transaction fees (Palmer 2001). Discrete-time models, such as the binomial model used in this paper, offer a more practical alternative by limiting rebalancing frequency. This paper uses assumptions that constrain transaction costs. This approach aligns with other works in the field that show how discrete-time models can balance transaction costs and hedging accuracy, as demonstrated by Koehl et al. (1999) and El Bernoussi and Rockinger (2023). By carefully structuring transaction costs, this model avoids the issue of infinite rebalancing costs and ensures practical applicability in real markets.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 states the general assumptions and provides some basic results about the model. Section 4 describes the hedging strategies and their existence in the market model. Section 5 defines optimal hedging strategies and introduces an algorithm to find an optimal hedging strategy. Section 6 provides a three-step numerical example to illustrate the results and algorithms. Section 7 concludes this paper.

2 The model

In this section we consider a specialized model for a financial market with both fixed and proportional transaction costs, and portfolio constraints which is a special model based on von Neumann-Gale dynamical systems.

Let us first define some notations we will use in this paper. For a real number r, let $r_+ = \max\{r, 0\}$ and $r_- = \max\{-r, 0\}$. Let $|\cdot|$ denote the norm of a vector in a finite-dimensional space, defined as the sum of the absolute values of its coordinates. For a finite-dimensional vector x, we will denote by $\mathbb{B}(x, r)$ the ball $\{y : |y - x| \leq r\}$. If $x = (\alpha, \beta)$, then we define $x_+ = (\alpha_+, \beta_+)$ and $x_- = (\alpha_-, \beta_-)$.

We consider a market where 2 assets can be traded at dates $t = 0, 1, \ldots, T$; asset 1 represents cash deposited with a bank account, and the second asset represents holdings in shares of a stock. The positions of a portfolio $x = (\alpha, \beta) \in \mathbb{R}^2$ will be measured in terms of their value; α is the amount invested in a bank account and β is the amount invested on the stock.

The space of states of the world consists of two elements, u and d ("up" and "down"). Let $a_t \in \{u, d\}$ be the state of the world influencing the market at time t. Sequences $\omega = \omega^T = (a_1, \ldots, a_T)$ is called the history of the market (may also be viewed as possible scenarios of the market development over the time period). For each $t = 1, \ldots, T-1$, the sequence $\omega^t = (a_1, \ldots, a_t)$ is called the partial history or partial scenario (up to time t). There are 2^T histories and 2^t partial histories for each

$t \leq T-1$. Let $\mathbb{P}$ be the "real-world" probability measure;

$$\mathbb{P}(a_1) = \begin{cases} p & a_1 = u \\ 1-p & a_1 = d \end{cases}$$

where we assume that $p > 0$. Let us define a basis probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ in the following way. Let $\Omega = \{u, d\}^T$ be the space of outcomes of ω, $\mathbb{F}$ be the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t$ is the σ-Algebra generated by ω^t, $t = 1, \ldots, T$, and $\mathcal{F}_T = \mathcal{F}$. We will omit ω^t in the notation where it does not lead to ambiguity.

The following random variables are given:

- $\mathcal{F}_{t-1}$-measurable random variables $0 < r_t^+ \leq r_t^-$, $t = 1, \ldots, T$, representing *risk-free* interest rates for lending and borrowing money, respectively.
- $\mathcal{F}_0$-measurable random variable S_0 representing the price of the stock at $t = 0$. The price $S_t(\omega^t)$ of the stock at time t has the following structure

$$S_t = S_0 Z(a_1) Z(a_2) \ldots Z(a_t),$$

where the values of the function $Z(a)$ on $\{u, d\}$ are two numbers $0 < Z(d) < 1 < Z(u)$, and for each t and ω^t, $Z(u) > 1 + r_t^-(\omega^t)$. Thus $S_t = S_{t-1} Z(a_t)$ are $\mathcal{F}_t$-measurable and

$$S_t = \begin{cases} S_{t-1} Z(u) & a_t = u \\ S_{t-1} Z(d) & a_t = d \end{cases}$$

Therefore, the price can either "jump up" or "jump down".
- $\mathcal{F}_t$-measurable random variables $0 \leq \lambda_t^+ < 1, \lambda_t^- \geq 0, t = 0, \ldots, T$, representing transaction cost rates for selling and buying the stock, respectively.
- $\mathcal{F}_t$-measurable random variables $0 \leq D_t^+ \leq D_t^-$, $t = 1, \ldots, T$, representing dividends for long and short positions on the stock, respectively.
- $\mathcal{F}_t$-measurable random variable $C_t \geq 0$, $t = 1, \ldots, T-1$, representing fixed transaction costs.

The possibility to have different dividend rates for long and short positions may be used due to the presence of taxes on dividends, e.g. when the stock pays dividends in a currency different from asset 1 and there is a bid–ask spread in the exchange rates. Let $R_t = S_t / S_{t-1}$ denote the return on the stock on the time period of $[t-1, t)$. Thus,

$$R_t = \begin{cases} Z(u) & a_t = u \\ Z(d) & a_t = d \end{cases}$$

Portfolio constraints in the model are specified by the cones[1]

$$X_t(\omega^t) = \left\{x = (\alpha, \beta) \in \mathbb{R}^2 : \chi_t(x) \geq 0\right\}, \tag{1}$$

where

$$\chi_t(x) = \alpha_+ + (1 - \lambda_t^+)\beta_+ - \mu_t \left(\alpha_- + (1 + \lambda_t^-)\beta_-\right), \tag{2}$$

$\mu_t > 1$ is constant which can be interpreted as a margin requirement coefficient. According to (1), a trader must be able to liquidate the long positions of her portfolio to cover the short positions with excess determined by μ_t.

Trading in the model at hand goes on as follows. At each date t, $t = 1, \ldots, T-1$, a trader pays C_t as fixed transaction costs. We assume that $C_0 = C_T = 0$. Then she receives the interest and the dividend on her portfolio $x_{t-1}(\omega^{t-1}) = (\alpha^{t-1}, \beta^{t-1})$ that she purchased at the previous date. The amount of interest and dividend is specified by $\delta_t(x_{t-1}) = r_t(\alpha^{t-1}) + d_t(\beta^{t-1})$, where

$$r_t(\alpha^{t-1}) = r_t^+ \alpha_+^{t-1} - r_t^- \alpha_-^{t-1}$$
$$d_t(\beta^{t-1}) = D_t^+ \beta_+^{t-1} - D_t^- \beta_-^{t-1}.$$

Here $D_t^{\pm}$ specify the amount of dividend received or returned[2] for amount of money invested in asset 2. The amount of dividend received or returned for 1 physical unit of the stock will be $D_t^{\pm} S_{t-1}$.

After that, the trader rearranges her portfolio $x_{t-1}(\omega^{t-1}) = (\alpha^{t-1}, \beta^{t-1})$ with added dividend and interest to a portfolio $x_t(\omega^t) = (\alpha^t, \beta^t)$ subject to the self-financing constraint. The possibility of rearrangement is specified by the following inequality

$$\begin{aligned}(\alpha^{t-1} - \alpha^t)_+ + (1 - \lambda_t^+)(R_t\beta^{t-1} - \beta^t)_+ + \delta_t(x_{t-1}) \geq \\ (\alpha^{t-1} - \alpha^t)_- + (1 + \lambda_t^-)(R_t\beta^{t-1} - \beta^t)_- + C_t.\end{aligned} \tag{3}$$

The left hand-side of (3) is the amount of money the trader receives for selling assets and for the dividends and interests, the right hand-side is the amount of money she pays for buying assets, including fixed and proportional transaction costs. This inequality means that the trader does not use external funds to rearrange her portfolio, so it can be regarded as a self-financing condition.

For any portfolio $x = (\alpha, \beta) \in X_t$, we use $\phi_t(x)$ defined below as a liquidation value function

$$\phi_t(x) = \alpha + \varphi_t(\beta), \tag{4}$$

[1] A set X in a linear space is called a cone if it contains with any its elements x, y any non-negative linear combination $\lambda x + \mu y$ ($\lambda, \mu \geq 0$) of these elements. The cone X is called pointed if the inclusions $x \in X$ and $-x \in X$ imply $x = 0$.

[2] We assume that dividend on short positions must be returned.

where

$$\varphi_t(\beta) = (1 - \lambda_t^+)\beta_+ - (1 + \lambda_t^-)\beta_-, \tag{5}$$

for any $\beta \in \mathbb{R}$.

Let $x_{t-1} = (\alpha^{t-1}, \beta^{t-1}) \in X_{t-1}$, $x_t = (\alpha^t, \beta^t) \in X_t$ and

$$\psi_t(x_{t-1}, x_t) = (\alpha^{t-1} - \alpha^t) + \varphi_t(R_t\beta^{t-1} - \beta^t) + \delta_t(x_{t-1}). \tag{6}$$

Then inequality (3) can be described as $\psi_t(\omega^t, x_{t-1}, x_t) \geq C_t$. The above description of the model corresponds to the sets

$$Z_t(\omega^t) := \left\{(x_{t-1}, x_t) \in X_{t-1}(\omega^{t-1}) \times X_t(\omega^t) : \psi_t(x_{t-1}, x_t) \geq C_t\right\}, \quad t = 1, \dots, T-1. \tag{7}$$

Observe that Z_t is a convex set but not a cone: it is convex, since the function $\psi_t(x, y)$ is concave as follows from the representation

$$\begin{aligned}\psi_t(x_{t-1}, x_t) = &[(\alpha^{t-1} - \alpha^t) + r_t^+\alpha^{t-1} + (1 - \lambda_t^+)(R_t\beta^{t-1} - \beta^t) + D_t^+\beta^t] \\ &- [(\lambda_t^- + \lambda_t^+)(R_t\beta^{t-1} - \beta^t)_- + (r_t^- - r_t^+)\alpha_-^{t-1} + (D_t^- - D_t^+)\beta_-^{t-1}],\end{aligned}$$

where the first sum is a linear function of x_{t-1}, x_t and the second sum is a convex function of x_{t-1}, x_t. However, it does not contain with any vector (x_{t-1}, x_t) all vectors $\lambda(x_{t-1}, x_t)$, where $\lambda \geq 0$.

The sets $X_t(\omega^t)$ and $Z_t(\omega^t)$ described above generates a stochastic dynamical system over the time interval $t = 0, 1, \cdots, T-1$. Let $\mathcal{L}_t^m$ $(t = 0, 1, ...)$ be a linear space of $\mathcal{F}_t$-measurable vector functions $x(\omega^t)$, with values in $\mathbb{R}^m$. We say that a vector function $x(\omega^t)$ is a *random state* of the system and write $x \in \mathcal{X}_t$ if $x \in \mathcal{L}_t^2$ and $x(\omega^t) \in X_t(\omega^t)$. A sequence of random states $x_t \in \mathcal{X}_t,\ t = 0, 1, \dots, T-1$ is called a *feasible trading strategy* if

$$(x_{t-1}(\omega^{t-1}), x_t(\omega^t)) \in Z_t(\omega^t),\ t = 1, \dots, T-1.$$

Note that a feasible trading strategy in the model is nothing but a *path* of the dynamical system under consideration. The above model is specified by random sets X_t and Z_t describing portfolio constraints and trading rules. In this model described above, sets X_t are closed cones, while Z_t are closed convex sets. This model is a version of the original von Neumann-Gale models in which both sets X_t and Z_t are closed cones, meaning that these models take into account proportional transaction costs in the most general way. Babaei (2024) establishes the asset pricing and hedging principle in a financial market model with both fixed and proportional transaction costs and trading constraints in a very general setting– not necessarily binomial models. The main results are hedging criteria stated in terms of consistent valuation systems, generalizing the notion of an equivalent martingale measure.

3 Some basic results

Let us introduce two basic assumptions that will be assumed to hold throughout the paper. Define $\Lambda_t^+ = 1 - \lambda_t^+$ and $\Lambda_t^- = 1 + \lambda_t^-$. Then we require the following to hold.

(**B1**) There exist constants $\underline{\Lambda}, \overline{\Lambda}, \overline{D}$, and $\overline{C}$ such that $0 < \underline{\Lambda} \le \Lambda_t^+(\omega^t), \Lambda_t^-(\omega^t) \le \overline{\Lambda}$, $D_t^-(\omega^t) \le \overline{D}$, and $C_t(\omega^t) \le \overline{C}$ for all t and ω^t.

(**B2**) For each t, we have $\mu_t > \nu$ where

$$\nu := \max\{(\overline{\Lambda} Z(u) + \overline{D})/\underline{\Lambda} Z(d); \overline{\Lambda}/\underline{\Lambda}\}.$$

These assumptions are not restrictive, and in fact, (**B1**) is always satisfied as the probability space is finite. In what follows, all equalities/inequalities between random vectors are understood coordinate-wise.

In the next proposition, we prove that the sets Z_t defining the self-finance constraints have non-empty interior.

Proposition 1 *For each $t = 1, 2, \ldots, T-1$, there exists a bounded $\mathcal{F}_t$-measurable vector function $\mathring{z}_t = (\mathring{x}_t, \mathring{y}_t)$ such that for all ω^t, we have*

$$\mathbb{B}(\mathring{z}_t, \varepsilon_t) \subseteq Z_t(\omega^t), \tag{8}$$

where $\varepsilon_t > 0$ is some constant.

We will need the following auxiliary result to prove Proposition 1.

Lemma 1 *(a) There exists a constant $\tau_t > 0$ such that if $x \in X_t(\omega^t)$ then $|x_+| - \nu|x_-| \ge \tau_t |x|$.*

(b) There exist positive constants $\kappa_{t,1}$ and $\kappa_{t,2}$ such that if $x \in X_{t-1}(\omega^{t-1})$, $y \in X_t(\omega^t)$ and $|y| \le \kappa_{t,1}|x| - \kappa_{t,2}$, then $(x, y) \in Z_t(\omega^t)$.

Proof (a) Observe that $X_t(\omega^t) \subseteq \tilde{X}_t$, where $\tilde{X}_t = \{x \in \mathbb{R}^2 : \mu_t |x_-| \le |x_+|\}$. Then for each $x \in X_t(\omega^t)$, we have $|x_+| - \nu|x_-| \ge (\mu_t - \nu)|x_-|$, and $|x_+| - \nu|x_-| \ge \frac{(\mu_t-\nu)}{\mu_t}|x_+|$. Then

$$2(|x_+| - \nu|x_-|) \ge \frac{(\mu_t - \nu)}{\mu_t}|x_+| + (\mu_t - \nu)|x_-| \ge \frac{(\mu_t - \nu)}{\mu_t}|x|.$$

This implies (a) with $\tau_t = (\mu_t - \nu)/2\mu_t$.

(b) Let $y = (\alpha^2, \beta^2) \in X_t(\omega^t)$. It is straightforward to check that for any numbers r, s we have $(r-s)_+ \ge r_+ - s_+$ and $(r-s)_- \le r_- + s_+$. Using this and conditions (**B1**) and (**B2**), we obtain for any $x = (\alpha^1, \beta^1) \in X_{t-1}(\omega^{t-1})$

$$\begin{aligned}\psi_t(x, y) &= \alpha_+^1(1 + r_t^+) - \alpha_-^1(1 + r_t^-) + \Lambda_t^+(R_t\beta^1 - \beta^2)_+ \\ &\quad - \Lambda_t^-(R_t\beta^1 - \beta^2)_- + (D_t^+\beta_+^1 - D_t^-\beta_-^1) - \alpha^2 \\ &\ge \alpha_+^1 Z(d) - \alpha_-^1 Z(u) + \underline{\Lambda}(Z(d)\beta_+^1 - \beta_+^2) - \overline{\Lambda}(Z(u)\beta_-^1 + \beta_+^2) - \overline{D}\beta_-^1 - \alpha_+^2\end{aligned}$$

$$\geq \underline{\Lambda}Z(d)|x_+| - (\overline{\Lambda}Z(u) + \overline{D})|x_-| - 2\overline{\Lambda}|y|$$
$$\geq \underline{\Lambda}Z(d)(|x_+| - \nu|x_-|) - 2\overline{\Lambda}|y|$$
$$\geq \tau_{t-1}\underline{\Lambda}Z(d)|x| - 2\overline{\Lambda}|y|,$$

where the third inequality follows from (**B2**). Then statement (b) can be fulfilled with the constant $\kappa_{t,1} = \tau_{t-1}\underline{\Lambda}Z(d)/2\overline{\Lambda}$ and $\kappa_{t,2} = \overline{C}/2\overline{\Lambda}$, since in that case $\psi_t(x, y) \geq C_t$, implying $(x, y) \in Z_t$. □

Proof of Proposition 1 Let $\mathring{x}_t = \frac{\kappa_{t,2}}{\kappa_{t,1}}(1, 1) \in \mathbb{R}^2$. Put $\mathring{z}_t = (\mathring{x}_t, \mathring{y}_t)$ with $\mathring{y}_t = (\kappa_{t,1}/4)\mathring{x}_t$. Note that $|\mathring{x}_t| = 2\kappa_{t,2}/\kappa_{t,1}$, and $|\mathring{y}_t| < \kappa_{t,1}|\mathring{x}_t| - \kappa_{t,2}$, thus statement (b) of Lemma 1 implies $\mathring{z}_t \in Z_t$. Observe that there exists $\epsilon_t > 0$ such that $\mathbb{B}(\mathring{z}_t, \epsilon_t) \subset \mathbb{R}^4_+$ and therefore $\mathbb{B}(\mathring{z}_t, \epsilon_t) \subset X_{t-1} \times X_t$. Since $|\mathring{y}_t| < \kappa_{t,1}|\mathring{x}_t| - \kappa_{t,2}$, then one can find $0 < \varepsilon_t \leq \epsilon_t$ such that $|y| \leq \kappa_{t,1}|x| - \kappa_{t,2}$ for any $(x, y) \in \mathbb{B}(\mathring{z}_t, \varepsilon_t)$. Indeed, we have

$$|y| \leq |y - \mathring{y}_t| + |\mathring{y}_t| \leq \varepsilon_t + \kappa_{t,2}/2 = \varepsilon_t + 2\kappa_{t,2} - 3\kappa_{t,2}/2$$
$$\leq \varepsilon_t + \kappa_{t,1}|x| + \kappa_{t,1}\varepsilon_t - 3\kappa_{t,2}/2 \leq \kappa_{t,1}|x| - \kappa_{t,2}.$$

The third inequality holds because $|x| \geq |\mathring{x}_t| - \varepsilon_t$, and the last inequality holds as long as $\varepsilon_t \leq \kappa_{t,2}/2(1 + \kappa_{t,1})$. Hence, $\mathring{z}_t$ and ε_t satisfy conditions of proposition. □

Proposition 2 *For any $x = (\alpha, \beta) \in X_t$, we have $\phi_t(x) \geq 0$, and $\phi_t(-x) \leq 0$. In particular, if $x \neq 0$, $\phi_t(x) > 0$, and $\phi_t(-x) < 0$.*

Proof This follows from conditions (**B1**), (**B2**) and Lemma 1. Indeed,

$$\phi_t(x) = \alpha_+ - \alpha_- + \Lambda_t^+\beta_+ - \Lambda_t^-\beta_-$$
$$\geq \underline{\Lambda}|x_+| - \overline{\Lambda}|x_-| \geq \underline{\Lambda}(|x_+| - \nu|x_-|) \geq \underline{\Lambda}\tau_t|x|.$$

In this chain, the first inequality follows from (**B1**), the second follows from (**B2**), and the last one follows from Lemma 1. We also have

$$-\phi_t(-x) = \alpha + \Lambda_t^-\beta_+ - \Lambda_t^+\beta_- \geq \alpha + \Lambda_t^+\beta_+ - \Lambda_t^-\beta_- = \phi_t(x).$$

This proves Proposition 2. □

Note that $-\phi_t(-x)$ is the minimum amount that one needs to construct portfolio x at time t, and this amount is more than the liquidation value of x.

4 Hedging in the market

Let us define the following sets describing possibilities of constructing initial portfolios and liquidating terminal ones.

$$V_0 = \{(v, x) \in \mathcal{L}_0^1 \times X_0 : v \geq -\psi_0(0, x)\}, \tag{9}$$

and

$$V_T(\omega) = \{(x, v) \in X_{T-1}(\omega^{T-1}) \times \mathcal{L}^1_T : \psi_T(x, 0) \geq v\}, \tag{10}$$

where $\psi_t(\cdot)$ is the function defined by (6). In order to construct a portfolio x at time 0, one needs at least an amount of $-\psi_0(0, x) = -\phi_0(-x)$, and when liquidating a portfolio $x \in X_{T-1}$ at time T, one should receive $\psi_T(x, 0)$ exceeding v. This motivates the definitions of the sets V_0 and V_T in (9) and (10).

As a consequence of Proposition 2, for each $x \in X_0$ we have $-\psi_0(0, x) \geq 0$ and if $x \neq 0$, $-\psi_0(0, x) > 0$. We also have for each $x = (\alpha, \beta) \in X_{T-1}$

$$\begin{aligned}
\psi_T(x, 0) &= \alpha + r_T(\alpha) + d_T(\beta) + \varphi_T(R_T\beta) \\
&\geq \underline{\Lambda} Z(d)|x_+| - (\overline{\Lambda} Z(u) + \overline{D})|x_-| \\
&\geq \underline{\Lambda} Z(d)(|x_+| - \nu|x_-|) \geq \tau_{T-1}\underline{\Lambda} Z(d)|x|,
\end{aligned}$$

where the second inequality follows from (**B2**) and the last one from Lemma 1. Thus, for each $x \in X_{T-1}$, we have $\psi_T(x, 0) \geq 0$, and if $x \neq 0$ then $\psi_T(x, 0) > 0$. We, therefore, notice that for constructing a non zero portfolio at time zero one needs to have positive endowment, and when liquidating a non zero portfolio at the expiry date, one gets a positive amount.

A sequence $(v_0, x_0, x_1, \ldots, x_{T-1}, v_T)$ is called a *hedging strategy* if

(a) $(x_0, x_1, \ldots, x_{T-1})$ is a feasible trading strategy,
(b) $v_0 \in \mathcal{L}^1_0$, and $(v_0, x_0) \in V_0$,
(c) $v_T \in \mathcal{L}^1_T$, and $(x_{T-1}, v_T) \in V_T$.

Let us say that an initial endowment $v_0 \in \mathcal{L}^1_0$ allows the hedging of a contingent claim $v_T \in \mathcal{L}^1_T$ if there exists a hedging strategy of the form $(v_0, x_0, x_1, \cdots, x_{T-1}, v_T)$. The main question is constructing hedging strategies for a given contingent claim $v_T \in \mathcal{L}^1_T$. Let $\mathcal{H}(v_T)$ be the set of

$$\{v_0 : v_0 \text{ allows the hedging of a contingent claim } v_T\}.$$

Suppose the set $\mathcal{H}(v_T)$ is non-empty and contains a smallest element. Then this element is called the *hedging price* of the contingent claim v_T and we denote it by $\rho(v_T)$. Let us first construct a hedging strategy for a given contingent claim $v_T \in \mathcal{L}^1_T$.

Theorem 1 *For any contingent claim $v_T \geq 0$, the set $\mathcal{H}(v_T)$ is non-empty.*

Proof Let $0 \leq v_T \in \mathcal{L}^1_T$. We construct a hedging strategy starting from the end and moving backwards, in the form of $(v_0, x_0, x_1, ..., x_{T-1}, v_T)$. Define

$$x_T^{\max}(\omega^{T-1}) = \max\{v_T(\omega^{T-1}, u), v_T(\omega^{T-1}, d)\},$$

and

$$x_{T-1} = \left(\frac{x_T^{\max}}{1 + r_T^+}, 0\right).$$

Note that $x_{T-1} \geq 0$, r_T^+ is $\mathcal{F}_{T-1}$-measurable, then $x_{T-1} \in X_{T-1}$. Thus, for any $\omega = (\omega^{T-1}, a_T) \in \Omega$

$$\psi_T(x_{T-1}(\omega^{T-1}), 0) = \frac{x_T^{\max}}{1 + r_T^+} \times (1 + r_T^+) \geq v_T(\omega),$$

showing that $(x_{T-1}, v_T) \in V_T$. Let us now construct $x_{T-2} \in X_{T-2}$. Define

$$x_{T-1}^{\max}(\omega^{T-2}) = \max\{x_T^{\max}(\omega^{T-2}, u), x_T^{\max}(\omega^{T-2}, d)\},$$

and

$$x_{T-2} = \left(\frac{x_{T-1}^{\max}}{\kappa_{T-1,1}}, \frac{\kappa_{T-1,2}}{\kappa_{T-1,1}} \right),$$

where $\kappa_{T-1,1}$ and $\kappa_{T-1,2}$ are constants introduced in Lemma 1. Note that $0 \leq x_{T-2} \in X_{T-2}$. Thus, for any $\omega^{T-1} = (\omega^{T-2}, a_{T-1})$,

$$|x_{T-2}| = \frac{|x_{T-1}^{\max}| + \kappa_{T-1,2}}{\kappa_{T-1,1}} \geq \frac{|x_{T-1}| + \kappa_{T-1,2}}{\kappa_{T-1,1}},$$

showing that $|x_{T-1}| \leq \kappa_{T-1,1}|x_{T-2}| - \kappa_{T-1,2}$, then by applying Lemma 1 we conclude that $(x_{T-2}, x_{T-1}) \in Z_{T-1}$. This procedure leads to the construction of a feasible strategy $(x_0, x_1, \ldots, x_{T-1})$. By defining $v_0 = -\psi(0, x_0)$, we have constructed a hedging strategy of the form $(v_0, x_0, x_1, \cdots, x_{T-1}, v_T)$. □

5 Optimal hedging

In Theorem 1 we constructed a hedging strategy $(v_0, x_0, \cdots, x_{T-1}, v_{T+1})$ for a given contingent claim $v_T \in \mathcal{L}_T^1$. Indeed v_0 is not the hedging price and $\rho(v_T) < v_0$. We call a hedging strategy $(v_0, x_0, \cdots, x_{T-1}, v_T)$ is *optimal* if $\rho(v_T) = v_0$. In this section, we propose an algorithm to construct an optimal hedging strategy based on the approach developed by Bensaid et al. (1992).

For each t, consider $x_t = (\alpha^t, \beta^t) \in X_t$. We rewrite the self-financing condition defined by (6) in the following form

$$(1 + r_t)\alpha^{t-1} \geq C_t + \alpha^t - \varphi_t(R_t\beta^{t-1} - \beta^t) - d_t(\beta^{t-1}). \tag{11}$$

Let $v_T \in \mathcal{L}_T^1$. We use the following algorithm which computes a sequential strategy by evaluating the sequential problem $\mathcal{Q}_t$, $t = 0, 1, \ldots, T$.

We start with problem $\mathcal{Q}_T$ to construct $x_{T-1} = (\alpha^{T-1}, \beta^{T-1})$ as

$$\begin{aligned} \mathcal{Q}_T(\omega^{T-1}; v_T) = &\min_{x_{T-1}} -\phi_{T-1}(-x_{T-1}), \\ \text{s.t.} \quad &\psi_T(x_{T-1}, 0) \geq v_T^{\max}(\omega^{T-1}), \end{aligned} \tag{12}$$

$$\chi_T(x_{T-1}) \geq 0,$$

where $v_T^{\max}(\omega^{T-1}) = \max\{v_T(\omega^{T-1}, u), v_T(\omega^{T-1}, d)\}$. The sequential problems $\mathcal{Q}_t$, $t = 1, \ldots, T-2$ are defined as

$$\begin{aligned} \mathcal{Q}_t(\beta^{t-1}, \omega^t; v_T) = & \min_{x_t=(\alpha^t,\beta^t)} C_t + \alpha^t - \varphi_t(R_t\beta^{t-1} - \beta^t) - d_t(\beta^{t-1}), \\ & \text{s.t.} \quad (1 + r_{t+1})\alpha^t \geq \mathcal{Q}_{t+1}^{\max}(\beta^t, \omega^t; v_T), \\ & \qquad \chi_t(x_t) \geq 0, \end{aligned} \tag{13}$$

where $\mathcal{Q}_{t+1}^{\max}(\beta^t, \omega^t; v_T) = \max\{\mathcal{Q}_{t+1}(\beta^t, (\omega^t, u); v_T), \mathcal{Q}_{t+1}(\beta^t, (\omega^t, d); v_T)\}$, with

$$\mathcal{Q}_{T-1}(\beta^{T-2}, \omega^{T-1}; v_T) = C_{T-1} + \alpha^{T-1} - \varphi_{T-1}(R_{T-1}\beta^{T-2} - \beta^{T-1}) - d_{T-1}(\beta^{T-2}).$$

The problem at $t = 0$ is defined as

$$\begin{aligned} \mathcal{Q}_0(v_T) = & \min_{x_0=(\alpha^0,\beta^0)} -\phi(-x_0), \\ & \text{s.t.} \quad (1 + r_1)\alpha^0 \geq \max\{\mathcal{Q}_1(\beta^0, u), \mathcal{Q}_1(\beta^0, d)\}, \\ & \qquad \chi_0(x_0) \geq 0. \end{aligned} \tag{14}$$

We call a hedging strategy $(\mathcal{Q}_0(v_T), x_0, \ldots, x_{T-1}, v_T)$ *sequentially optimal* if

- $x_{T-1}(\omega^{T-1})$ solves program $\mathcal{Q}_T(\omega^{T-1}; v_T)$ for all ω^{T-1},
- for each $t = 1, \ldots, T-2$, $x_t(\omega^t)$ solves program $\mathcal{Q}_t(\beta^{t-1}, \omega^t; v_T)$ for all ω^t,
- x_0 solves program $\mathcal{Q}_0(v_T)$.

We will omit v_T in $\mathcal{Q}_t(\beta^{t-1}, \omega^t; v_T)$ where it does not lead to ambiguity. The relation between the $\rho(v_T)$ and $\mathcal{Q}_0(v_T)$ is given by the following result.

Theorem 2 *Let $v_T \in \mathcal{L}_T^1$. Then $\rho(v_T) = \mathcal{Q}_0(v_T)$.*

Proof Let $(\mathcal{Q}_0(v_T), x_0, \cdots, x_{T-1}, v_T)$ be a sequentially optimal hedging strategy, where $x_t = (\alpha^t, \beta^t)$, $t = 0, \ldots, T-1$. We observe that $x_{T-1} \in X_{T-1}$ and $(x_{T-1}, v_T) \in V_T$. For each $t = 1, \ldots, T-2$, we have

$$\begin{aligned} (1 + r_{t+1})\alpha^t & \geq \mathcal{Q}_{t+1}(\beta^t, (\omega^t, a_{t+1})) \\ & = C_{t+1} + \alpha^{t+1} - \varphi_{t+1}(R_{t+1}\beta^t - \beta^{t+1}) - d_{t+1}(\beta^t). \end{aligned}$$

By using self-financing condition (11), we observe that $(x_t, x_{t+1}) \in Z_{t+1}$. Hence, $(\mathcal{Q}_0(v_T), x_0, \ldots, x_{T-1}, v_T)$ is a feasible hedging strategy, which ensures that $\rho(v_T) \leq \mathcal{Q}_0(v_T)$.

Conversely, let $(v_0, x_0, \ldots, x_{T-1}, v_T)$ be a feasible hedging strategy, where $x_t = (\alpha^t, \beta^t)$, $t = 0, \ldots, T-1$. We observe that $\chi_t(x_t) \geq 0$ and $-\phi(-x_{T-1}) \geq \mathcal{Q}_T(\omega^T)$. By induction we show that for each $t = 1, \ldots, T-1$

$$C_t + \alpha^t - \varphi_t(R_t\beta^{t-1} - \beta^t) - d_t(\beta^{t-1}) \geq \mathcal{Q}_t(\beta^{t-1}, \omega^t). \tag{15}$$

This is true with equality for $t = T-1$, by the definition of $\mathcal{Q}_{T-1}$. Now suppose it is true at $t+1$. As $(v_0, x_0, \cdots, x_{T-1}, v_T)$ is a feasible hedging strategy, by using self-financing condition (11), we have

$$(1+r_{t+1})\alpha^t \geq C_{t+1} + \alpha^{t+1} - \varphi_{t+1}(R_{t+1}\beta^t - \beta^{t+1}) - d_{t+1}(\beta^t),$$

and so $(1+r_{t+1})\alpha^t \geq \mathcal{Q}_{t+1}(\beta^t, \omega^{t+1})$. This implies that $x_t = (\alpha^t, \beta^t)$ is $\mathcal{Q}_t$-admissible, and hence (15) holds. By induction, the equation also prevails at $t = 1$. A similar reasoning shows then that $(1+r_1)\alpha^0 \geq \mathcal{Q}_1(\beta^0, \omega^1)$, and so $\mathcal{Q}_0(v_T) \leq -\phi_0(-x_0) \leq v_0$, for all admissible hedging strategies. Therefore, necessarily, $\mathcal{Q}_0(v_T) \leq \rho(v_T)$. This completes the proof. □

According to Theorem 2, any sequentially optimal strategy is optimal. This describes a recursive algorithm for computing such a sequential strategy, starting from $t = T-1$ up to date $t = 0$. The following lemma will ensure the convexity of $\mathcal{Q}_t$.

Lemma 2 *For all $t = 1, \ldots, T-1$, $\mathcal{Q}_t(\beta^{t-1}, \omega^t)$ is convex with respect to β^{t-1}.*

Proof We prove the result by induction. First, we note that

$$\mathcal{Q}_{T-1}(\beta^{T-2}, \omega^{T-1}) = C_{T-1} + \alpha^{T-1} - \varphi_{T-1}(R_{T-1}\beta^{T-2} - \beta^{T-1}) - d_{T-1}(\beta^{T-2})$$

is convex as functions φ_t and d_t are concave.

Suppose $\mathcal{Q}_{t+1}(\beta^t, \omega^{t+1})$ is convex with respect to β^t. Then $\mathcal{Q}^{\max}_{t+1}(\beta^t, \omega^t)$ is convex with respect to β^t, being the maximum of two convex functions. Therefore, the set of admissible points for the problem $\mathcal{Q}_t(\beta^{t-1}, \omega^t)$ (which does not depend on β^{t-1}) is also convex. Let β'^{t-1}, $\beta^{t-1} \in \mathcal{L}^1_{t-1}$, (α^t, β^t), (α'^t, β'^t) be $\mathcal{Q}_t$-admissible, and $\lambda \in [0,1]$. Define $\Gamma^{t-1}_\lambda = \lambda\beta^{t-1} + (1-\lambda)\beta'^{t-1}$, $\Gamma^t_\lambda = \lambda\beta^t + (1-\lambda)\beta'^t$, and $\Theta^t_\lambda = \lambda\alpha^t + (1-\lambda)\alpha'^t$. Then we have

$$\begin{aligned}\mathcal{Q}_t(\Gamma^{t-1}_\lambda, \omega^t) &\leq C_t + \Theta^t_\lambda - d_t(\Gamma^{t-1}_\lambda) - \varphi_t\left(R_t\Gamma^{t-1}_\lambda - \Gamma^t_\lambda\right)\\ &\leq \lambda\left(C_t + \alpha^t - \varphi_t(R_t\beta^{t-1} - \beta^t) - d_t(\beta^{t-1})\right)\\ &\quad + (1-\lambda)\left(C_t + \alpha'^t - \varphi_t(R_t\beta'^{t-1} - \beta'^t) - d_t(\beta'^{t-1})\right).\end{aligned}$$

The first inequality holds as $(\Theta^t_\lambda, \Gamma^t_\lambda)$ is $\mathcal{Q}_t$-admissible, and the second inequality follows from the concavity of φ_t and d_t. Since this is true for all $\mathcal{Q}_t$-admissible $(\Theta^t_\lambda, \Gamma^t_\lambda)$, this leads to

$$\mathcal{Q}_t(\Gamma^{t-1}_\lambda, \omega^t) \leq \lambda\mathcal{Q}_t(\beta^{t-1}, \omega^t) + (1-\lambda)\mathcal{Q}_t(\beta'^{t-1}, \omega^t).$$

□

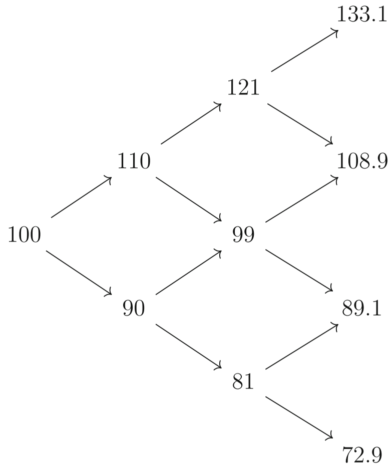

Fig. 1 Stock prices at different times

6 A three-step example

In this section, a numerical example will illustrate the theoretical results of our paper. Let us consider the following three-step model.

- There are three periods; $T = 3$.
- Interest rates for lending and borrowing money are constants; $r_t^+ = 5\%$, $r_t^- = 6\%$ for each $t = 1, 2, 3$ and ω^t.
- The price of the stock at $t = 0$ is $S_0 = 100$, and $Z(u) = 1.1$, $Z(d) = 0.9$. The prices of the stock at $t = 1, 2, 3$ are represented in the Fig. 1.
- Transaction cost rates for long and short positions on the stock are constant; $\lambda_t^+ = \lambda_t^- = 1\%$.
- Dividend rates for long and short positions on the stock are constant; $D_t^+ = D_t^- = 2\%$.
- Fixed transaction costs at $t = 1, 2$ are constant; $C_1 = C_2 = 1$.
- The marginal coefficients are fixed; $\mu_t = 1.5$.
- The contingent claim v_3 is a European call option with strike price $K = 105$ and maturity three years. The value of the option at t = 3 is given in Table 1.

$v_3(u^3)$	$v_3(u^2d)$	$v_3(ud^2)$	$v_3(d^3)$
28.1	3.9	0	0

By using this information, we calculate the constants introduced in conditions (**B1**), (**B2**) and Lemma 1, in Table 2.

Let us now construct an optimal hedging strategy $(v_0, x_0, x_1, x_2, v_3)$ by using the algorithm defined by programs (12), (13), and (14).

$\overline{\Lambda}$	$\underline{\Lambda}$	$\overline{D}$	$\overline{C}$	ν	τ_t	$\kappa_{t,1}$	$\kappa_{t,2}$
1.01	0.99	2%	1	1.269	0.077	0.034	0.495

We begin with constructing $x_2(\omega^2) = (\alpha^2, \beta^2)$ by solving the following problem, where ω^2 is either u^2, ud or d^2.

$$\begin{aligned} \mathcal{Q}_3(\omega^2; v_3) = &\min_{x_2} -\phi_2(-x_2), \\ \text{s.t.} \quad & \psi_3(x_2(\omega^2), 0) \geq v_3(\omega^2, u), \\ & \psi_3(x_2(\omega^2), 0) \geq v_3(\omega^2, d), \\ & \chi_2(x_2) \geq 0. \end{aligned}$$

At $\omega^2 = u^2$, the problem will be

$$\begin{aligned} \mathcal{Q}_3(u^2; v_3) = &\min_{(\alpha^2, \beta^2)} \alpha^2 + \overline{\Lambda}\beta_+^2 - \underline{\Lambda}\beta_-^2, \\ \text{s.t.} \quad & (1+r^+)\alpha_+^2 - (1+r^-)\alpha_-^2 + \overline{D}\beta^2 + \underline{\Lambda}Z(u)\beta_+^2 - \overline{\Lambda}Z(u)\beta_-^2 \geq v(u^3), \\ & (1+r^+)\alpha_+^2 - (1+r^-)\alpha_-^2 + \overline{D}\beta^2 + \underline{\Lambda}Z(d)\beta_+^2 - \overline{\Lambda}Z(d)\beta_-^2 \geq v(u^2 d), \\ & \alpha_+^2 + \underline{\Lambda}\beta_+^2 - \mu_t(\alpha_-^2 + \overline{\Lambda}\beta_-^2) \geq 0. \end{aligned}$$

We will split this non-linear program to a linear one by considering the following cases.

Case 1: $\alpha^2,\ \beta^2 \geq 0$; we have the following linear program

$$\begin{aligned} \mathcal{Q}_3(u^2; v_3) = &\min_{(\alpha^2, \beta^2)} \alpha^2 + 1.01\beta^2, \\ \text{s.t.} \quad & 1.05\alpha^2 + 0.02\beta^2 + 1.089\beta^2 \geq 28.1, \\ & 1.05\alpha^2 + 0.02\beta^2 + 0.891\beta^2 \geq 3.9, \\ & \alpha^2 + 0.99\beta^2 \geq 0. \end{aligned}$$

Solving this, we have $(\alpha^2, \beta^2) = (0, 25.338)$ and $-\phi_2(-x_2) = 25.592$.

Case 2: $\alpha^2 \geq 0,\ \beta^2 \leq 0$; we have the following linear program

$$\begin{aligned} \mathcal{Q}_3(u^2; v_3) = &\min_{(\alpha^2, \beta^2)} \alpha^2 + 0.99\beta^2, \\ \text{s.t.} \quad & 1.05\alpha^2 + 0.02\beta^2 + 1.111\beta^2 \geq 28.1, \\ & 1.05\alpha^2 + 0.02\beta^2 + 0.909\beta^2 \geq 3.9, \\ & \alpha^2 + 1.515\beta^2 \geq 0. \end{aligned}$$

Solving this, we get $(\alpha^2, \beta^2) = (26.762, 0)$ and $-\phi_2(-x_2) = 26.762$.

Case 3: $\alpha^2 \le 0,\ \beta^2 \ge 0$; we have the following linear program

$$\begin{aligned} \mathcal{Q}_3(u^2; v_3) = \ & \min_{(\alpha^2,\beta^2)} \alpha^2 + 1.01\beta^2, \\ \text{s.t.} \quad & 1.06\alpha^2 + 0.02\beta^2 + 1.089\beta^2 \ge 28.1, \\ & 1.06\alpha^2 + 0.02\beta^2 + 0.891\beta^2 \ge 3.9, \\ & 1.5\alpha^2 + 0.99\beta^2 \ge 0. \end{aligned}$$

Solving this, we get $(\alpha^2, \beta^2) = (-45.3, 68.637)$ and $-\phi_2(-x_2) = 24.023$. Note that both α^2, β^2 can not be negative. Therefore, for $\omega^2 = u^2$, the optimal portfolio is $(\alpha^2, \beta^2) = (-45.3, 68.637)$.

At $\omega^2 = ud$, the problem will be

$$\begin{aligned} \mathcal{Q}_3(ud; v_3) = \ & \min_{(\alpha^2,\beta^2)} \alpha^2 + \overline{\Lambda}\beta^2_+ - \underline{\Lambda}\beta^2_-, \\ \text{s.t.} \quad & (1+r^+)\alpha^2_+ - (1+r^-)\alpha^2_- + \overline{D}\beta^2 + \underline{\Lambda}Z(u)\beta^2_+ - \overline{\Lambda}Z(u)\beta^2_- \ge v(u^2d), \\ & (1+r^+)\alpha^2_+ - (1+r^-)\alpha^2_- + \overline{D}\beta^2 + \underline{\Lambda}Z(d)\beta^2_+ - \overline{\Lambda}Z(d)\beta^2_- \ge v(ud^2), \\ & \alpha^2_+ + \underline{\Lambda}\beta^2_+ - \mu_t(\alpha^2_- + \overline{\Lambda}\beta^2_-) \ge 0. \end{aligned}$$

Using the same method, we find the optimal portfolio as $(\alpha^2, \beta^2) = (-6.287, 9.526)$. Obviously, at $\omega^2 = d^2$, the optimal portfolio is $(\alpha^2, \beta^2) = (0, 0)$.

From the definition, we have

$$\mathcal{Q}_2(\beta^1, \omega^2; v_3) = C_2 + \alpha^2(\omega^2) - \varphi_2(R_2\beta^1 - \beta^2(\omega^2)) - d_2(\beta^1).$$

Recall that $\varphi_2(R_2\beta^1 - \beta^2) = \underline{\Lambda}(R_2\beta^1 - \beta^2)_+ - \overline{\Lambda}(R_2\beta^1 - \beta^2)_-$. Let us now construct $x_1 = (\alpha^1, \beta^1)$. We start with problem $\mathcal{Q}_1(\beta^0, u)$.

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = \ & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(d)\beta^0 - \beta^1) - d_1(\beta^0), \\ \text{s.t.} \quad & (1+r_2)\alpha^1 \ge \mathcal{Q}_2(\beta^1, u^2), \\ & (1+r_2)\alpha^1 \ge \mathcal{Q}_2(\beta^1, ud), \\ & \chi_1(x_1) \ge 0, \end{aligned}$$

For solving this, we consider the following cases.

Case 1: $\alpha^1, \beta^1 \ge 0$, $Z(u)\beta^1 - \beta^2(u^2) \ge 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = \ & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0, \\ \text{s.t.} \quad & (1+r^+)\alpha^1 \ge C_2 + \alpha^2(u^2) - \underline{\Lambda}(Z(u)\beta^1 - \beta^2(u^2)) - \overline{D}\beta^1, \\ & (1+r^+)\alpha^1 \ge C_2 + \alpha^2(ud) - \underline{\Lambda}(Z(d)\beta^1 - \beta^2(ud)) - \overline{D}\beta^1. \end{aligned}$$

We can easily check that in this case the optimal portfolio is $(\alpha^1, \beta^1) = (0, \beta^2(u^2)/Z(u)) = (0, 62.40)$ which is independent of β^0, and

$$\mathcal{Q}_1(\beta^0, u) = \begin{cases} C_1 + \alpha^1 - \underline{\Lambda}(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0 & Z(u)\beta^0 - \beta^1 \geq 0 \\ C_1 + \alpha^1 + \overline{\Lambda}(\beta^1 - Z(u)\beta^0) - \overline{D}\beta^0 & Z(u)\beta^0 - \beta^1 < 0 \end{cases}$$

Case 2: $\alpha^1, \beta^1 \geq 0$, $Z(u)\beta^1 - \beta^2(u^2) \leq 0$, and $Z(d)\beta^1 - \beta^2(ud) \geq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0, \\ \text{s.t.} \quad & (1 + r^+)\alpha^1 \geq C_2 + \alpha^2(u^2) + \overline{\Lambda}(\beta^2(u^2) - Z(u)\beta^1) - \overline{D}\beta^1, \\ & (1 + r^+)\alpha^1 \geq C_2 + \alpha^2(ud) - \underline{\Lambda}(Z(d)\beta^1 - \beta^2(ud)) - \overline{D}\beta^1. \end{aligned}$$

In this case the optimal portfolio is $(\alpha^1, \beta^1) = (0, 22.12)$.

Case 3: $\alpha^1, \beta^1 \geq 0$, and $Z(d)\beta^1 - \beta^2(ud) \leq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0, \\ \text{s.t.} \quad & (1 + r^+)\alpha^1 \geq C_2 + \alpha^2(u^2) + \overline{\Lambda}(\beta^2(u^2) - Z(u)\beta^1) - \overline{D}\beta^1, \\ & (1 + r^+)\alpha^1 \geq C_2 + \alpha^2(ud) + \overline{\Lambda}(\beta^2(ud) - Z(d)\beta^1) - \overline{D}\beta^1. \end{aligned}$$

In this case the optimal portfolio is $(\alpha^1, \beta^1) = (12.43, 10.58)$.

Case 4: $\alpha^1 \leq 0$, $\beta^1 \geq 0$, $Z(u)\beta^1 - \beta^2(u^2) \geq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0, \\ \text{s.t.} \quad & (1 + r^-)\alpha^1 \geq C_2 + \alpha^2(u^2) - \underline{\Lambda}(Z(u)\beta^1 - \beta^2(u^2)) - \overline{D}\beta^1, \\ & (1 + r^-)\alpha^1 \geq C_2 + \alpha^2(ud) - \underline{\Lambda}(Z(d)\beta^1 - \beta^2(ud)) - \overline{D}\beta^1 \\ & \mu\alpha^1 + \underline{\Lambda}\beta^1 \geq 0. \end{aligned}$$

In this case the optimal portfolio is $(\alpha^1, \beta^1) = (-41.18, 62.40)$.

Case 5: $\alpha^1 \leq 0$, $\beta^1 \geq 0$, $Z(u)\beta^1 - \beta^2(u^2) \leq 0$, and $Z(d)\beta^1 - \beta^2(ud) \geq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0, \\ \text{s.t.} \quad & (1 + r^-)\alpha^1 \geq C_2 + \alpha^2(u^2) + \overline{\Lambda}(\beta^2(u^2) - Z(u)\beta^1) - \overline{D}\beta^1, \\ & (1 + r^-)\alpha^1 \geq C_2 + \alpha^2(ud) - \underline{\Lambda}(Z(d)\beta^1 - \beta^2(ud)) - \overline{D}\beta^1 \\ & \mu\alpha^1 + \underline{\Lambda}\beta^1 \geq 0. \end{aligned}$$

In this case the optimal portfolio is $(\alpha^1, \beta^1) = (-38.28, 58)$.

Case 6: $\alpha^1 \leq 0$, $\beta^1 \geq 0$, and $Z(d)\beta^1 - \beta^2(ud) \leq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0, \\ \text{s.t.} \quad & (1+r^-)\alpha^1 \geq C_2 + \alpha^2(u^2) + \overline{\Lambda}(\beta^2(u^2) - Z(u)\beta^1) - \overline{D}\beta^1, \\ & (1+r^-)\alpha^1 \geq C_2 + \alpha^2(ud) + \overline{\Lambda}(\beta^2(ud) - Z(d)\beta^1) - \overline{D}\beta^1 \\ & \mu\alpha^1 + \underline{\Lambda}\beta^1 \geq 0 \end{aligned}$$

There is no solution for this case as all constraints are not satisfied.

Case 7: $\alpha^1 \geq 0$, $\beta^1 \leq 0$, and $Z(d)\beta^1 - \beta^2(ud) \leq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_1(\beta^0, u) = & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(u)\beta^0 - \beta^1) - \overline{D}\beta^0, \\ \text{s.t.} \quad & (1+r^+)\alpha^1 \geq C_2 + \alpha^2(u^2) + \overline{\Lambda}(\beta^2(u^2) - Z(u)\beta^1) - \overline{D}\beta^1, \\ & (1+r^+)\alpha^1 \geq C_2 + \alpha^2(ud) + \overline{\Lambda}(\beta^2(ud) - Z(d)\beta^1) - \overline{D}\beta^1 \\ & \alpha^1 + \mu\overline{\Lambda}\beta^1 \geq 0 \end{aligned}$$

In this case, the optimal portfolio is $(\alpha^1, \beta^1) = (23.83, 0)$.

By considering all cases, we observe that the global optimal portfolio is $(\alpha^1(u), \beta^1(u)) = (-38.28, 58)$ as it has the minimum value for $\mathcal{Q}_1(\beta^0, u)$ for any β^0.

Let us now consider problem $\mathcal{Q}_1(\beta^0, d)$ for finding the optimal portfolio $(\alpha^1(d), \beta^1(d))$. We have

$$\begin{aligned} \mathcal{Q}_1(\beta^0, d) = & \min_{x_1=(\alpha^1,\beta^1)} C_1 + \alpha^1 - \varphi_1(Z(d)\beta^0 - \beta^1) - d_1(\beta^0), \\ \text{s.t.} \quad & (1+r_2)\alpha^1 \geq \mathcal{Q}_2(\beta^1, ud), \\ & (1+r_2)\alpha^1 \geq \mathcal{Q}_2(\beta^1, d^2), \\ & \chi_1(x_1) \geq 0, \end{aligned}$$

With the same method, we can find that the global optimal portfolio is $(\alpha^1(d), \beta^1(d)) = (-6.68, 10.12)$.

Finally, we construct $x_0 = (\alpha^0, \beta^0)$ by solving the following problem

$$\begin{aligned} \mathcal{Q}_0(v_3) = & \min_{x_0=(\alpha^0,\beta^0)} -\phi(-x_0), \\ \text{s.t.} \quad & (1+r_1)\alpha^0 \geq \mathcal{Q}_1(\beta^0, u), \\ & (1+r_1)\alpha^0 \geq \mathcal{Q}_1(\beta^0, d), \\ & \chi_0(x_0) \geq 0. \end{aligned}$$

Using the previous findings, we will solve the problem:

$$\mathcal{Q}_0(v_3) = \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \overline{\Lambda}\beta^0_+ - \underline{\Lambda}\beta^0_-,$$

$$\begin{aligned} \text{s.t.} \quad & (1+r_1)\alpha^0 \geq C_1 + \alpha^1(u) - \varphi_1(Z(u)\beta^0 - \beta^1(u)) - d_1(\beta^0), \\ & (1+r_1)\alpha^0 \geq C_1 + \alpha^1(d) - \varphi_1(Z(d)\beta^0 - \beta^1(d)) - d_1(\beta^0) \\ & \chi_0(x_0) \geq 0. \end{aligned}$$

We will, again, consider the following cases.

Case 1. $\alpha^0, \beta^0 \geq 0$, and $Z(u)\beta^0 - \beta^1(u) \geq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_0(v_3) = & \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \overline{\Lambda}\beta^0, \\ \text{s.t.} \quad & (1+r_1^+)\alpha^0 \geq C_1 + \alpha^1(u) - \underline{\Lambda}(Z(u)\beta^0 - \beta^1(u)) - \overline{D}\beta^0, \\ & (1+r_1^+)\alpha^0 \geq C_1 + \alpha^1(d) - \underline{\Lambda}(Z(d)\beta^0 - \beta^1(d)) - \overline{D}\beta^0. \end{aligned}$$

Solving this, we get $(\alpha^0, \beta^0) = (0, 52.73)$ and $v_0 = -\phi(-x_0) = 53.26$.

Case 2. $\alpha^0, \beta^0 \geq 0$, $Z(u)\beta^0 - \beta^1(u) \leq 0$, and $Z(d)\beta^0 - \beta^1(d) \geq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_0(v_3) = & \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \overline{\Lambda}\beta^0, \\ \text{s.t.} \quad & (1+r_1^+)\alpha^0 \geq C_1 + \alpha^1(u) + \overline{\Lambda}(\beta^1(u) - Z(u)\beta^0) - \overline{D}\beta^0, \\ & (1+r_1^+)\alpha^0 \geq C_1 + \alpha^1(d) - \underline{\Lambda}(Z(d)\beta^0 - \beta^1(d)) - \overline{D}\beta^0. \end{aligned}$$

Solving this, we get $(\alpha^0, \beta^0) = (0, 18.83)$ and $v_0 = -\phi(-x_0) = 19.02$.

Case 3. $\alpha^0, \beta^0 \geq 0$, and $Z(d)\beta^0 - \beta^1(d) \leq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_0(v_3) = & \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \overline{\Lambda}\beta^0, \\ \text{s.t.} \quad & (1+r_1^+)\alpha^0 \geq C_1 + \alpha^1(u) + \overline{\Lambda}(\beta^1(u) - Z(u)\beta^0) - \overline{D}\beta^0, \\ & (1+r_1^+)\alpha^0 \geq C_1 + \alpha^1(d) + \overline{\Lambda}(\beta^1(d) - Z(d)\beta^0) - \overline{D}\beta^0. \end{aligned}$$

Solving this, we get $(\alpha^0, \beta^0) = (8.17, 11.25)$ and $v_0 = -\phi(-x_0) = 19.53$.

Case 4. $\alpha^0 \leq 0$, $\beta^0 \geq 0$, and $Z(u)\beta^0 - \beta^1(u) \geq 0$. Then the problem will be

$$\begin{aligned} \mathcal{Q}_0(v_3) = & \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \overline{\Lambda}\beta^0, \\ \text{s.t.} \quad & (1+r_1^-)\alpha^0 \geq C_1 + \alpha^1(u) - \underline{\Lambda}(Z(u)\beta^0 - \beta^1(u)) - \overline{D}\beta^0, \\ & (1+r_1^-)\alpha^0 \geq C_1 + \alpha^1(d) - \underline{\Lambda}(Z(d)\beta^0 - \beta^1(d)) - \overline{D}\beta^0, \\ & \mu\alpha^0 + \underline{\Lambda}\beta^0 \geq 0. \end{aligned}$$

Solving this, we get $(\alpha^0, \beta^0) = (-34.80, 52.73)$ and $v_0 = -\phi(-x_0) = 18.46$.

Case 5. $\alpha^0 \leq 0$, $\beta^0 \geq 0$, $Z(u)\beta^0 - \beta^1(u) \leq 0$, and $Z(d)\beta^0 - \beta^1(d) \geq 0$. Then the problem will be

$$\mathcal{Q}_0(v_3) = \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \overline{\Lambda}\beta^0,$$

$$
\begin{aligned}
\text{s.t.} \quad & (1 + r_1^-)\alpha^0 \geq C_1 + \alpha^1(u) + \overline{\Lambda}(\beta^1(u) - Z(u)\beta^0) - \overline{D}\beta^0, \\
& (1 + r_1^-)\alpha^0 \geq C_1 + \alpha^1(d) - \underline{\Lambda}(Z(d)\beta^0 - \beta^1(d)) - \overline{D}\beta^0, \\
& \mu\alpha^0 + \underline{\Lambda}\beta^0 \geq 0.
\end{aligned}
$$

Solving this, we get $(\alpha^0, \beta^0) = (-32.59, 49.38)$ and $v_0 = -\phi(-x_0) = 17.28$.

Case 6. $\alpha^0 \leq 0$, $\beta^0 \geq 0$, and $Z(d)\beta^0 - \beta^1(d) \leq 0$. Then the problem will be

$$
\begin{aligned}
\mathcal{Q}_0(v_3) = & \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \overline{\Lambda}\beta^0, \\
\text{s.t.} \quad & (1 + r_1^-)\alpha^0 \geq C_1 + \alpha^1(u) + \overline{\Lambda}(\beta^1(u) - Z(u)\beta^0) - \overline{D}\beta^0, \\
& (1 + r_1^-)\alpha^0 \geq C_1 + \alpha^1(d) + \overline{\Lambda}(\beta^1(d) - Z(d)\beta^0) - \overline{D}\beta^0, \\
& \mu\alpha^0 + \underline{\Lambda}\beta^0 \geq 0.
\end{aligned}
$$

There is no solution for this case as all constraints are not satisfied.

Case 7. $\alpha^0 \geq 0$, $\beta^0 \leq 0$, and $Z(d)\beta^0 - \beta^1(d) \leq 0$. Then the problem will be

$$
\begin{aligned}
\mathcal{Q}_0(v_3) = & \min_{x_0=(\alpha^0,\beta^0)} \alpha^0 + \underline{\Lambda}\beta^0, \\
\text{s.t.} \quad & (1 + r_1^+)\alpha^0 \geq C_1 + \alpha^1(u) + \overline{\Lambda}(\beta^1(u) - Z(u)\beta^0) - \overline{D}\beta^0, \\
& (1 + r_1^+)\alpha^0 \geq C_1 + \alpha^1(d) + \overline{\Lambda}(\beta^1(d) - Z(d)\beta^0) - \overline{D}\beta^0, \\
& \alpha^0 + \mu\overline{\Lambda}\beta^0 \geq 0.
\end{aligned}
$$

Solving this, we get $(\alpha^0, \beta^0) = (20.29, 0)$ and $v_0 = -\phi(-x_0) = 20.29$.

After considering all cases, we observe that the global initial optimal portfolio is $(\alpha^0, \beta^0) = (-32.59, 49.38)$, and the minimum initial endowment required to hedge v_3 is $v_0 = 17.28$.

7 Conclusions

In this paper, we extend the classical CRR binomial model by incorporating fixed and proportional transaction costs, portfolio constraints, and dividend payments. As a foundational framework, we use von Neumann-Gale dynamics and adapt the approach developed by Bensaid et al. (1992), allowing us to model realistic market conditions more comprehensively. Our model introduces both random fixed and proportional transaction costs, with different rates for long and short positions, and accounts for dividend payments with potentially different random rates for long and short positions. Additionally, we allow short selling under specific margin constraints. By incorporating these complexities, we provide a robust framework for pricing and hedging strategies that aligns more closely with real-world financial scenarios.

A key contribution of this paper is the development of an algorithm to compute optimal hedging strategies in this enriched framework. By using a recursive, sequentially optimal approach, we derive hedging prices and demonstrate the practical application of the model through a detailed three-step numerical example. The results show that while transaction costs introduce significant complexity, our model can still provide effective solutions for pricing and hedging in markets with both fixed and proportional costs.

The findings in this paper suggest several directions for future research. A key area for further development would be to improve the computational efficiency of the algorithms, particularly when dealing with different random rates, which were simplified in the presented example, and for handling more steps in the binomial model. Addressing the discretization error issue is also crucial, as it influences the trade-off between transaction costs and rebalancing frequency. Optimizing this balance could significantly enhance the accuracy of the model.

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