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Effect of materials and design on the bending stiffness of tennis rackets

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Abstract

Lawn tennis rackets have changed considerably since the origins of the game. Early rackets were wooden, making them heavier and more flexible than modern designs made from fibre-polymer composites. The fundamental frequency of a freely suspended tennis racket is often used as an analogue to stiffness, despite it being dependent on mass. We estimate the bending stiffness of 525 rackets, dating from 1874 to 2017, using a uniform beam model that accounts for mass. The model suggests composite rackets are typically about twice as stiff as their wooden predecessors. Applying typical values of Young's modulus, density and second moment of area, the model was used to demonstrate the benefits of fibre-polymer composites for making lightweight, stiff rackets. Undergraduate students could make use of our large dataset of tennis rackets to explore these patterns themselves. They could also go on to measure the dimensions, mass and fundamental frequency of tennis rackets and model them as a uniform beam. Students could also apply the theory to other implements, like badminton and squash rackets or baseball and cricket bats.

Keywords: vibration, Young's Modulus, second moment of area, wood, fibre-polymer composite.

1. Introduction

Lawn tennis rackets have developed since the origins of the game in the 1870s [1, 2, 3]. Until the 1970s, most tennis rackets were wooden with small heads. Wooden rackets have solid cross-sections and masses of about 330 to 440 g. Since the 1980s, most high-end rackets have been made from fibre-polymer composites, which offer more design freedom than wood. Composites have enabled modern rackets to have larger heads than their wooden predecessors, and large hollow cross-sections to make them light and stiff, with masses of about 280 to 350 g (Figure 1).

The larger strung surface in modern tennis rackets makes it easier to play the game, and allows proficient players to apply more topspin to the ball. A wider head increases the polar moment of inertia, or "twist-weight", of the racket, which is defined as the resistance to angular acceleration about the long axis [4, 5] (see Figure 1a for the position of the long axis). A racket with a high twist-weight will rotate less about the long axis if the ball is hit off-centre [6, 7]. Modern rackets also tend to have a lower moment of inertia about an axis passing sideways through the handle in the plane of the frame [2, 3], commonly named as "swing-weight" (if the axis is ~10 cm from the butt) (Figure 1a). The reduced swing-weight allows these modern rackets to have higher accelerations for a given torque [4, 5, 8, 9, 10]. A lower swing-weight and faster swing does not necessarily give a faster shot, as the higher racket speed can be offset by a lower effective mass striking the ball [11, 12]. Given that advanced players can detect differences in swing-weight as small as 2.5% [13], racket selection is likely to be influenced by many factors, such as preference, playing style and experience, with lightweight modern designs being easier to manoeuver.

A racket with a stiffer frame will bend less, vibrate faster and absorb less energy if the ball is hit away from the

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vibration node, which is close to the centre of the string bed [14]. With less kinetic energy being lost to frame vibration in a stiffer racket, the ball rebounds faster [6, 15]. A stiffer racket should allow the player to serve faster, as the ball is hit towards the tip away from the vibration node [2]. The fundamental (lowest) frequency (f_1) of a freely suspended racket is often used as an analogue to frame stiffness [16]. Indeed, a freely suspended racket vibrates in a similar way to a hand-held racket [17, 18]. Wooden rackets have an f_1 of ~80 to 120 Hz when freely suspended, with composite rackets vibrating faster at ~140 to 180 Hz [2, 3]. As f_1 is dependent on mass, it is not ideal for comparing the stiffness of traditional and modern rackets.

Researchers have modelled rackets as a uniform beam, so they can study the basic mechanics and predict how design parameters, like mass, swing-weight and stiffness, may influence tennis. We use a uniform beam model [14] as a simple tool to demonstrate that fibre-polymer composites are superior to wood for making lightweight, stiff rackets. First, we apply the beam model to 525 rackets dating from 1874 to 2017 [3], to quantify how much stiffer composite designs are than their wooden predecessors. We then apply values for typical materials properties for wood and fibre-polymer composites and frame cross-sectional dimensions to the beam model, demonstrating the mass and stiffness benefits from a large hollow cross-section. Our aim is to make students aware of the importance of cross-section shape on the stiffness-to-mass ratio of beams.

2. Frequency as a proxy for bending stiffness

The flexural rigidity or bending stiffness (*EI*) of a uniform beam is the product of Young's modulus (*E*), the established measure of material stiffness, and the second moment of area (*I*). Wood and fibre-polymer composites used in tennis rackets are anisotropic and inhomogeneous, so *E* depends on both the measurement direction and location. The *EI* calculation assumes a constant *E* (in Nm^{-2}) for the tennis racket material. Second moment of area is a measure of the distance of the material from the neutral axis (in m^4) (Figure 2). As such, a racket will have a high *EI* if it is made from stiff material and it has a large *I*. Fibre-polymer composites have higher strength-to-mass and stiffness-to-mass ratios than wood [19], and allow for large, hollow cross-sections with a large *I* and low mass.

The f_1 of a freely suspended uniform beam is given by,

$$f_1 = 3.561\sqrt{\frac{EI}{ML^3}} \qquad (1)$$

where *M* is mass and *L* is length [14] (Figure 1a). Equation 1 shows that f_1 increases with *EI*, and decreases with *M* and *L*. Whilst f_1 can only be used as a proxy for racket *EI*, since it is dependent on mass and length, Equation 1 is beneficial, as it is often simpler to measure f_1, mass and length than *EI*, which requires a specialist rig or device (e.g. Babolat Racquet Diagnostic Centre with proprietary units). The f_1 of a racket is typically measured ~~by modal analysis,~~ as follows: i) hang it by a long string; ii) lightly strike the string bed with a ball while recording the vibrations with a sensor on the handle; and iii) apply a discrete Fourier transform to convert the signal from the time to the frequency domain (see [3]).

When applying Equation 1 to model tennis rackets as uniform beams and estimating f_1, Cross [14] set the length to 70 cm, which is about right for most designs [3]. Cross modelled beams with masses of 250, 300 and 350 g, which covers most modern designs but is lighter than many wooden rackets [3]. Varying *EI* from 100 to 400 Nm^2, Cross [14] altered f_1 of his beam from just over 100 to almost 250 Hz, covering values for most modern rackets [3].

Rearranging Equation 1 to

$$EI = f_1^2 \frac{ML^3}{12.681} \qquad (2)$$

allows racket *EI* to be estimated from measured values of f_1, mass and length. Despite being cubed in Equation 2, length tends to have little influence on tennis racket *EI*, since most are about 68 cm long and tend to only vary by a few centimeters (mean and standard deviation (SD) of 68.5 ± 1 cm for rackets from [3] spanning >140 year timespan). Indeed, Equation 3 can be used to approximate tennis racket *EI* to within ~10% of Equation 2, if the length is within a couple centimeters of 68 cm.

$$EI \approx 0.25 f_1^2 M \qquad (3)$$

Using either Equation 2 or 3 to estimate *EI* is better than using f_1 as a proxy, given that f_1 is also dependent on mass which can vary by >100 g between rackets of a similar age [3]. Note that f_1 has more influence on *EI* than mass, as f_1 is squared in Equation 2 and can vary from ~80 to >200 Hz between rackets [3].

Figure 3a shows *EI* of 525 tennis rackets from Taraborrelli et al. [3] as predicted from Equation 2 inputting known values of mass, length and frequency. Taraborrelli et al. [3] included the racket measurements in their open access supplementary material, and we have included the *EI* values in the supplementary material for this paper. The *EI* values of the composite rackets mainly fell between 100 and 400 Nm^2 [14], with the older, mostly wooden, rackets (pre-1960s) having lower values from ~50 to 150 Nm^2. The reason Equation 2 predicted many of these older rackets to have an *EI* below the lowest value of 100 Nm^2 applied by Cross [14] was because they had an f_1 <100 Hz (Figure 3b), despite many exceeding 350 g (Figure 3c).

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Based on Equation 3, Figure 3b shows how *EI* of a uniform beam increases with f_1 for masses of 250 g, 325 g and 400 g. By comparing *EI* values for the dataset (Equation 2) to those of the uniform beams, we can visualise the effect of variability in the dataset, and determine how well f_1 approximates the stiffness of diverse rackets of varying mass. As the dataset follows the general trends of Equation 3, f_1 is a reasonable proxy for stiffness, particularly for wooden rackets (Figure 3b). Figure 3c shows how *EI* of a uniform beam increases with mass for f_1 values of 100, 150 and 200 Hz. Beams with a higher f_1 see larger increases in *EI* with increases in mass. The wooden rackets have low f_1 values (~80 to 120 Hz) and followed the general trend of Equation 3 for a 100 Hz beam (Figure 3c), presumably because their material properties are all similar with heavier designs having larger, and hence stiffer, cross-sections (Figure 2). The composite rackets tend to have higher and more varied f_1 values and do not follow the trends of Equation 3 for any of the beams in Figure 3c, exhibiting a wide range in ratios of *EI* to mass (~250 to 1500 Nm^2/kg, Figure 3d). These results indicate that f_1 is a reasonable proxy for the stiffness of wooden rackets, but less so for composite rackets as they have higher f_1 values and more varied stiffness-to-mass ratios (Figure 3d). The properties of fibre-polymer composites can depend on many factors, including i) the fibres used and the polymer matrix that bonds them, ii) the fibre volume ratio and iii) the orientation and placement of fibres within the matrix. The extra design freedom offered by composites over wood has allowed engineers to make more diverse rackets, with variation in material properties, cross-section shapes and overall frame shapes.

3. The uniform beam model

3.1 Modelling the racket as a uniform rectangular beam

The uniform beam model is useful for comparing *EI* across a wide range of racket designs. For example, applying typical values for a wooden racket of 100 Hz and 350 g to Equation 3 gives *EI* of ~90 Nm^2, whereas values of 150 Hz and 300 g, as typical for a modern racket, almost double *EI* to ~170 Nm^2. It is also interesting to compare specific rackets, particularly unusual designs and those that are associated with famous players. For example, the Dunlop Maxply was a prominent wooden racket first released in the early 1930s, produced in various forms for 50 years and used by famous players like Rod Laver, Virginia Wade and John McEnroe[1]. The *EI* of a "Dunlop Maxply Fort" from 1971 (Figure 1d) was calculated as 124 Nm^2 (See Appendix A1). In contrast, an "Original Widebody 280 Hz" composite racket from 1991 (Figure 1e) had the highest *EI* of the dataset at >500 Nm^2. As a "widebody" racket [1, 20], the frame depth of the "Original Widebody 280 Hz" was ~1.5 times that of the "Dunlop Maxply Fort" [3] (Figure 4). We will now demonstrate the importance of cross-section shape and construction material on racket *EI*.

3.2 Second moment of area of a uniform beam

For simplicity, tennis rackets were assumed to have almost rectangular cross-sections. Based on this assumption, the standard equation for *I* of either a solid (Equation 4) or a hollow (Equation 5) rectangle was applied to model wooden and composite rackets, respectively:

$$I = \frac{bd^3}{12} \quad (4)$$

$$I = \frac{\left[bd^3 - (b-2t)(d-2t)^3\right]}{12} \quad (5)$$

where *b* is the width, *d* is the depth and *t* is the wall thickness [21] (Figure 2). According to these equations, for a given volume of material, *EI* can be maximised by increasing the ratio of frame depth to width, and by prioritising frame depth over frame wall thickness in composite rackets. Frame depth can be almost double the frame width at the widest point of the head, as reported by Taraborrelli et al. [3][2] (Figure 1a), in wooden tennis rackets and almost six times the frame width in composite rackets (Figure 4) [3]. To estimate *EI* as an input for uniform beams that approximate wooden and composite rackets, representative values of *E* and the mass per unit volume, or density (ρ), are needed for these materials. While ρ does not directly influence *EI*, by using representative values of ρ for a given material, the mass of the beam can be set to that of the actual racket, by adjusting the volume.

3.3 Young's modulus and density of racket material

Various woods have been used in tennis rackets, including ash, maple, birch and beech [1]. These woods typically have *E* in a direction along the grain of ~7 to 15 GPa, and ρ of ~370 to 660 kg/m^3 [1]. Widing and Moeinzadeh [22] applied a constant *E* of 12.5 GPa in a finite element model (FEM) of an ash tennis racket. Unfortunately, they did not state the cross-sectional dimensions of the racket, so it is unfeasible to estimate their value of *I* and calculate *EI*. Studies modelling wooden baseball bats, including those made from ash, maple and birch, have applied *E* from 14 to 16 GPa in a direction along the grain, reducing to ~1 GPa across the grain [23, 24]. Indeed, as a natural material the specific properties of a wooden racket can depend on many factors, including i) how various woods were layered together, ii) how the grains were orientated iii) any imperfections and defects, iv) whether reinforcements were applied and v) any degradation or change in moisture content with age. Fortin-Smith et al. [25] reported ρ of ash and maple, as typically used in baseball

[1] www.dunlopsports.com/our-story

[2] Note: what we call frame width here is defined as "frame thickness" in Figure 1c of Taraborrelli et al. [3].

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bats, to fall between ~550 and 850 kg/m^3. As such, ranges in E and ρ that could be applied to the beam model to represent wooden rackets were estimated to vary from 5 to 15 GPa and 400 to 900 kg/m^3, respectively [19, 26].

Fibre-polymer composite tennis rackets can employ various fibres, including glass (E-glass) and carbon, amongst others. Glass and carbon fibres can have E from ~70 to 600 GPa [1, 27, 28, 29], with the polymer matrix having a lower stiffness of ~1 to 6 GPa [1, 28, 29]. For unidirectional fibre-polymer composites (all fibres in one direction), E is typically ~40 to 400 GPa [1, 24, 28-29] in a direction along the fibres, reducing to ~3 to 15 GPa across the fibres [24, 28, 29, 30]. As composite tennis rackets contain fibres in different orientations [1], E in any given direction will be lower than in the stiffest direction for a unidirectional equivalent. The ρ of the fibre-polymer composites typically used in rackets are ~1,500 to 2,000 kg/m^3 [1]. Much like with wooden rackets and bats, FEM studies of fiber-polymer composite rackets can provide an insight into typical material properties, as well as typical values for wall thickness.

In modelling studies, a constant E is often set by tuning f_1 of the model to match the actual racket. Values of E from such studies have ranged from 10 to 70 GPa for rackets with f_1 of ~100 to 250 Hz [6, 15, 31-35]. Of these studies, the work of Allen et al. [35] was thought to offer the best indication of fibre-polymer composite E, as they used a 3D scanner to obtain the geometry and then defined the wall thickness to align the inertial properties of the model to the actual racket. An E of 15 GPa and a wall thickness of 2 to 3 mm gave f_1 of 137 Hz for a 336 g model, which was similar to the actual racket. Values for ρ used in models of tennis rackets have ranged from 1,750 to 2,150 kg/m^3 [31, 33, 34]. As such, E and ρ of composite tennis racket materials that could be applied to the beam model were estimated to vary from 10 to 30 GPa and 1,500 to 2,200 kg/m^3, respectively [19].

Using typical cross-sectional dimensions (Figure 4), and the estimates for E and ρ, Equation 2 was applied to approximate both wooden and composite tennis rackets as uniform beams (Table 1). The idea was not for the beam to exactly represent the racket in shape nor material, but rather to show how basic properties, like EI, mass and f_1, are dependent on both cross-section shape and material. The depth of the beam corresponds to the frame depth of the racket, with the width of the beam set to 2.5 times the frame width as reported by Taraborrelli et al. [3] (Figure 4). To facilitate efficient characterisation of many different rackets, Taraborrelli et al. [3] estimated frame width as half the difference between the external and internal head width (Figure 1a). The factor of 2.5 times the estimated frame width was selected to give the desired mass of the beam for representative values of racket length and ρ. The beam models presented here are limited by the racket measurements provided by Taraborrelli et al. [3]. Forty grams was added to the mass of the beam to account for additions to the frame, like the string and grip, when calculating f_1 using Equation 1.

4. Exploring the Beam Model

It is possible to change the cross-sectional dimensions of the beam to change EI, while keeping the cross-sectional area and hence volume and mass the same (Tables 1 and 2). For a constant mass of 387 g for wood and 310 g for composite, EI of the beam can be varied across the typical range for such tennis rackets, using typical values for frame depth and width (Figure 4) (Table 1). For instance, a wooden beam with a width of 25 mm and a depth of 24 mm has the same mass (387 g) as one with a width of 40 mm (60% larger) and a depth of 15 mm (38% smaller) (Table 1). The former has an EI 2.5 times larger than the latter, showing the importance of frame depth over width for racket bending stiffness (see Equation 4). Frame width is, however, important for torsional stiffness and to prevent the frame warping and breaking under high string tension, a factor which limits the head size of wooden rackets.

Now considering composites, a beam with a width of 15 mm and a depth of 37 mm has the same mass (310 g) as one with a width of 40 mm (167% larger) and a depth of 11 mm (70% smaller) (Table 1). The former has an EI nine times higher than the latter. These results demonstrate the benefit of using stiff fibre-polymer composites to produce hollow rackets with large cross-sections to give a high I (see Figure 2, and Equation 2). As with wooden rackets, a minimal frame width and wall thickness is needed in composite rackets for structural integrity and torsional strength and stiffness. The ball is also more likely to clip a deeper frame when the player angles the racket face forward during a topspin stroke.

The uniform beam model can be applied to specific rackets, and the central row in the upper half of Table 2 corresponds to the “Maxply Fort” (See Appendix A2). The racket frame width and depth are from Taraborrelli et al. [3] (Figure 4), with the mass, EI and f_1 within 2% of the measured values. As with the wooden racket in Table 1, it is possible to change the width and depth within the typical range (Figure 4) to produce beams with the same mass that vary in EI by about 2.5 times. The central row in the lower half of Table 2 corresponds to the “Original Widebody 280 Hz” represented as a uniform beam. Again, the racket frame width and depth are from Taraborrelli et al. [3], with the mass, EI and f_1 within 2% of measured values. Relative to the values in Table 1, E and frame wall thickness were increased to account for the racket being particularly stiff. Similar to the example in Table 1, it is possible to change the frame width and depth of the racket within the typical range to produce beams with the same mass that vary in EI by ~12 times.

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5. Applying the Beam Model to Tennis Rackets

We now use the beam models to show why fibre-polymer composites are better than wood for making lightweight, stiff tennis rackets (Figure 5). For the same external dimensions, *I* will be higher for a solid wooden beam than a hollow composite beam (Figure 5a & b). Higher *EI* can, however, be obtained for a hollow composite beam than a solid wooden beam, due to the higher *E* possible with composites (Figure 5a & b). The specific difference in *EI* will depend on the relative stiffness of the materials, with composites typically being stiffer than woods [1, 19, 24]. Other than for beam widths and depths below those typical of wooden rackets (Figure 4), the mass of composite beams were lower than wooden beams due to the lower volume, despite higher ρ (Figure 5c & d). The higher *EI* and lower mass of the composite beams meant they had a higher f_1 across the beam widths and depths associated with tennis rackets (Figure 5c and d). The f_1 of both beams remained almost constant as width increased, as the increase in *EI* was offset by the higher mass (Figure 5d). Similar to width, increasing the wall thickness of the composite beam led to small increases in *EI*, with minimal change in f_1 due to the higher mass (Figure 5e and f). These results clearly show the benefit of designing a fibre-polymer composite tennis racket with a narrow and deep cross-section and thin walls, if the aim is to combine high f_1 with low mass.

We now conclude by comparing *EI* predictions from multiplying our estimates of *E* and *I* (i.e. Figure 5a and b) with *EI* values for the 525 rackets, calculated using Equation 2 (Figure 6). Figure 6a shows that *EI* of the rackets tended to increase with frame depth. The wooden rackets have a narrower range in both frame depth (~16 to 25 mm) and *EI* (~50 to 150 Nm^2) than the composite rackets (~14 to 33 mm, ~50 to 550 Nm^2). As predicted from multiplying estimates of *E* and *I* (Equation 5 and Figure 5a), the composite rackets tended to have higher *EI* than their wooden predecessors, as they are made from stiffer materials (Figure 5a). Where the dataset deviates from the predictions, this is likely due to differences in the specific material properties of each racket, particularly for deeper designs.

The results in Figure 6 lead to three important observations. Firstly, Figure 6b shows no clear relationship between *EI* and frame width for the 525 rackets, in disagreement with the *EI* predictions from multiplying estimates of *E* and *I*. This observation makes sense, as Equations 4 and 5 clearly show that *I* depends more strongly on frame depth than width. Secondly, as predicted, composite rackets with narrow frames tend to have higher *EI* values than wooden rackets with wide frames. This second observation also makes sense, as composite materials tend to be stiffer than wood (i.e. higher *E*). Finally, the results in Figure 6 further reinforce the importance of frame depth over width for making stiff rackets (straight line for *EI* predictions in Figure 6a compared to a curved line in Figure 6b). Again, this final observation is as expected, because for the same material (*E*), *EI* only depends linearly on frame width compared to the third power for frame depth (Equations 4 and 5).

6. Ideas for Student Exploration

Undergraduate students can apply Equation 2 to estimate *EI* of rackets, either by taking measurements themselves or by using published data (e.g. [3], where racket measurements are included in the supplementary material). Students could even make a low-cost device for measuring f_1 of a racket with an accelerometer and microcontroller (e.g. Arduino). Alternatively, students can use a smartphone application (spectrum analyzer e.g. "Spectroid" version 1.1.1 by Carl Reinke) to record the sound of the vibrating racket and determine f_1.

As we have done, students can also apply the uniform beam model to tennis rackets, either by taking measurements themselves or by using published data (e.g. the supplementary material of [3]). They could also critique and discuss the limitations of our approach, including the overly simplified geometry of a uniform beam in comparison to a tennis racket and the use of constants for *E* and ρ, and explore possible improvements for when more precise estimates of bending stiffness are needed. For example, students could measure the frame width and depth along the length of rackets to obtain mean values and develop uniform beams with more representative dimensions. They could also explore other cross-section shapes, like hollow circles or ellipses. Students can use such models to investigate the effect of applying different materials, which could extend to testing materials to obtain *E* and ρ. Students could even 3D print fibre-polymer composite material test samples, and then beams to compare to the models (e.g. Markforged printer). Such experimental work could support students in learning how errors combine through error propagation of the equations and repeated measurements.

7. Conclusions

The bending stiffnesses of 525 diverse tennis rackets were estimated using a uniform beam model. The model was applied to show why fibre-polymer composites are better than wood for making lightweight, stiff tennis rackets. Undergraduate students could measure the dimensions, mass and fundamental frequency of tennis rackets, or other sporting implements, and apply the uniform beam model to estimate the bending stiffness. The students can then use the beam model to predict the effect of changing the cross-sectional shape or material of the implement.

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Appendix

Here are example calculations for the *Dunlop "Maxply Fort"* (Figure 1d). First, ensure a consistent system of units (e.g. kg, m, Hz and Pa): M = 0.414 kg, f_l = 109 Hz, L = 0.684 m, racket thickness = 0.0125 m, b = 0.0313 m, d = 0.0205 m, E = 5500000000 Pa, ρ = 850 kg/m^3.

A1: Applying Equation 2 to calculate EI of the racket from f_l, M and L.

$$EI = f_1^2 \frac{ML^3}{12.681} = \frac{109^2 \cdot 0.414 \cdot 0.683^3}{12.681} = 124\ Nm^2$$

A2: Modelling the racket as a uniform beam.

Step 1: Calculating beam EI from estimates of E and I.

$$I = \frac{bd^3}{12} = \frac{0.0313 \cdot 0.0205^3}{12} = 0.0000000224\ m^4$$
$$EI = 5500000000 \cdot 0.0000000224 = 123\ Nm^2$$

Step 2: Calculating beam M from volume V and ρ.

$$Crosssectionarea(A) = b \cdot d = 0.0313 \cdot 0.0205 = 0.00064\ m^2$$
$$V = A \cdot L = 0.00064 \cdot 0.683 = 0.000438\ m^3$$
$$M = V \cdot \rho = 0.000438 \cdot 0.684 = 0.372\ kg$$

+0.040 kg for additional attachments

$$M = 0.372 + 0.040 = 0.412\ kg$$

Step 3: Applying Equation 1 to calculate f_l of the beam from EI, M and L.

$$f_1 = 3.561 \sqrt{\frac{EI}{ML^3}} = \frac{123}{0.412 \cdot 0.683^3} = 109\ Hz$$

References

[1] Lammer H and Kotze J 2003 Materials and tennis rackets Materials in sports equipment, 1, 222-248
[2] Haake S, Choppin S, Allen T and Goodwill S 2007 The evolution of the tennis racket and its effect on serve speed Tennis science and technology 3 257-271
[3] Taraborrelli L, Grant R, Sullivan M, Choppin S, Spurr J, Haake S and Allen T 2019 Materials have driven the historical development of the Tennis Racket Appl. Sci. 9 4352
[4] Brody H 1985 The moment of inertia of a tennis racket Phys. Teach. 23 213-216
[5] Taraborrelli L, Grant R, Sullivan M, Choppin S, Spurr J, Haake SJ and Allen T 2019 Recommendations for estimating the moments of inertia of a tennis racket Sports Eng. 22 1–9
[6] Allen TB, Haake SJ and Goodwill SR 2011 Effect of tennis racket parameters on a simulated groundstroke J. Sports Sci. 29 311-325
[7] Cross R 2014 Impact of sports balls with striking implements Sports Eng 17 3–22
[8] Mitchell SR, Jones R, King M 2000 Head speed vs. racket inertia in the tennis serve Sports Eng. 3 99–110
[9] Cross R and Bower R 2006 Effects of swing-weight on swing speed and racket power J. Sports Sci. 24 23-30
[10] Schorah D, Choppin S, James D 2015 Effects of moment of inertia on restricted motion swing speed Sport Biom. 14 157–167
[11] Whiteside D, Elliott B, Lay B and Reid M 2013 The effect of racquet swing weight on serve kinematics in elite adolescent female tennis players J Sci Med Sport 17 124–128
[12] Söğüt M 2017 Acute effects of customizing a tennis racket on serve speed Balt. J. Sport Health Sci. 1
[13] Brody H 2000 Player sensitivity to the moments of inertia of a tennis racket Sport Eng. 3 145–148
[14] Cross R 2015 Factors affecting the vibration of tennis racquets Sports Eng. 18 135-147
[15] Allen T, Haake S and Goodwill S 2009 Comparison of a finite element model of a tennis racket to experimental data. Sports Eng 12 87-98
[16] Allen T, Choppin S, Knudson D 2016 A review of tennis racket performance parameters. Sport Eng. 19 1–11
[17] Brody H 1987 Models of tennis racket impacts J. Appl. Biomech. 3 293–296
[18] Banwell G, Roberts J, Halkon BJ, Rothberg S and Mohr S, Understanding the dynamic behaviour of a tennis racket under play conditions, Experimental Mechanics, 54, 527-537, 2014 doi:10.1007/s11340-013-9803-9.
[19] Ashby MF Materials Selection in Mechanical Design, 5th ed.; Butterworth-Heinemann: Oxford, UK, 2016 1–639
[20] Kuebler S Racket having thickened shaft portion U.S. Patent 4,664,380, 12 May 1987
[21] Matthews C 2011 Imeche Engineers' Databook, 4th ed.; John Wiley & Sons: Hoboken
[22] Widing MAB and Moeinzadeh MH 1990 Finite element modeling of a tennis racket with variable string patterns and tensions J. Appl. Biomech. 6 78-91
[23] Fortin-Smith J, Sherwood J, Drane P, Ruggiero E, Campshure B and Kretschmann D 2019 A Finite Element Investigation into the Effect of Slope of Grain on Wood Baseball Bat Durability Appl. Sci. 9 3733
[24] Shenoy MM, Smith LV and Axtell JT 2001 Performance assessment of wood, metal and composite baseball bats Compos. Struct. 52 397-404
[25] Fortin-Smith J, Sherwood J, Drane P and Kretschmann D 2018 Characterization of maple and ash material properties for the finite element modeling of wood baseball bats Appl. Sci. 8 2256
[26] Green David W 1999 Mechanical Properties of Wood/David W. Green, Jerrold E. Winandy, David E. Kretschmann. Wood handbook/David W. Green, Jerrold E. Winandy, David E. Kretschmann.–Madison, WI: US Department of Agrieculture, Fofest Service, Products Laboratory 4-1
[27] Staab, G. H. Laminar Composites, Second.; Butterworth-Heinemann is an imprint of Elsevier: Kidlington, Oxford, 2015.

IOP Publishing Journal Title
Journal **XX** (XXXX) XXXXXX https://doi.org/XXXX/XXX

[28] Tanaka, K. and Sekizawa, K., 2018. Construction of a Finite Element Model of Golf Clubs and Influence of Shaft Stiffness on Its Dynamic Behavior. In Multidisciplinary Digital Publishing Institute Proceedings (Vol. 2, No. 6, p. 247).

[29] Strangwood M, 2019, Modelling of materials for sports equipment, Edited by Subic Alexander, Materials in sports equipment, 2, pp 3-35 Woodhead Publishing

[30] Society of Automotive Engineers; National Institute for Aviation Research (U.S.). Composite Materials Handbook; SAE International on behalf of CMH-17, a division of Wichita State University: Warrendale, Pa., 2018; Vol. Volume 2, Polymer Matrix Composites, Materials Properties.

[31] Yaodong Gu and Jian She Li 2006 Dynamic simulation in tennis rackets and strings, 24 International Symposium on Biomechanics in Sports

[32] Li L, Yang SH, Hwang CS and Kim YS 2009 Effects of string tension and impact location on tennis playing Journal of Mechanical Science and Technology 23 2990-2997

[33] Banwell G, Mohr S, Rothberg S and Roberts J 2012 Using experimental modal analysis to validate a finite element Procedia Engineering 34 688-693

[34] Li Jiude 2014 Mechanical analysis of tennis racket and ball during impact based on finite element method, BioTechnology: An Indian Journal

[35] Allen T, Hart J, Spurr J, Haake S and Goodwill S 2010 Validated dynamic analysis of real sports equipment using finite element; a case study using tennis rackets Procedia engineering, 2 3275-3280

[36] Allen T, Grant R, Sullivan M, Taraborrelli L, Choppin S, Spurr J and Haake S 2018 Recommendations for measuring tennis racket parameters In Multidisciplinary Digital Publishing Institute Proceedings (Vol. 2, No. 6, p. 263)

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Tables

Table 1. Models of three wooden and three composite rackets represented as a 68 cm long uniform beam, with the central rows corresponding to a "typical" racket of each material.

	Racket frame width (mm)	Width beam (mm)	Depth (mm)	*I* (x10^{-8} m^4)	*EI* (Nm^2)	f_1 (Hz)
Wood	10	25	24	2.9	158	129
E = 5.5 GPa	12	30	20	2.0	110	107
ρ = 850 kg/m^3 M = 387 g*	16	40	15	1.1	62	80
Composite	6	15**	37	3.0	456	242
	11	28	23	1.4	217	169
E = 15 GPa t = 2 mm ρ = 2,100 kg/m^3 M = 310 g*	16	40	11	0.1	51	82

*includes 40 g for strings, grip and other attachments, **below expected limits, based on grip circumferences (100 to 150 mm) from Allen et al. [36].

Table 2. Models of three wooden and three composite rackets represented as uniform beams, with the central rows corresponding to a "Dunlop Maxply Fort" from 1971 and an "Original Widebody 280 Hz" from 1991.

	Racket frame width (mm)	Width beam (mm)	Depth (mm)	*I* (x10^{-8} m^4)	*EI* (Nm^2)	f_1 (Hz)
Dunlop Maxply Fort	10.0	25.0	25.6	3.5	192	136
L = 68.4 cm *E* = 5.5 GPa	12.5	31.3	20.5	2.2	123	109
ρ = 850 kg/m^3 M = 412 g*	16.0	40.0	16.0	1.4	75	85
Original Widebody	5.6	14.0**	40.0	3.9	656	270
	8.5	21.3**	33.0	3.0	516	240
L = 68.5 cm *E* = 17 GPa t = 2.2 mm ρ = 2,100 kg/m^3 M = 354 g*	17.6	44.0	10.0	0.3	52	76

*includes 40 g for strings, grip and other attachments, **below expected limits, based on value for grip circumference (100 to 150 mm) from Allen et al. [36].

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Figures

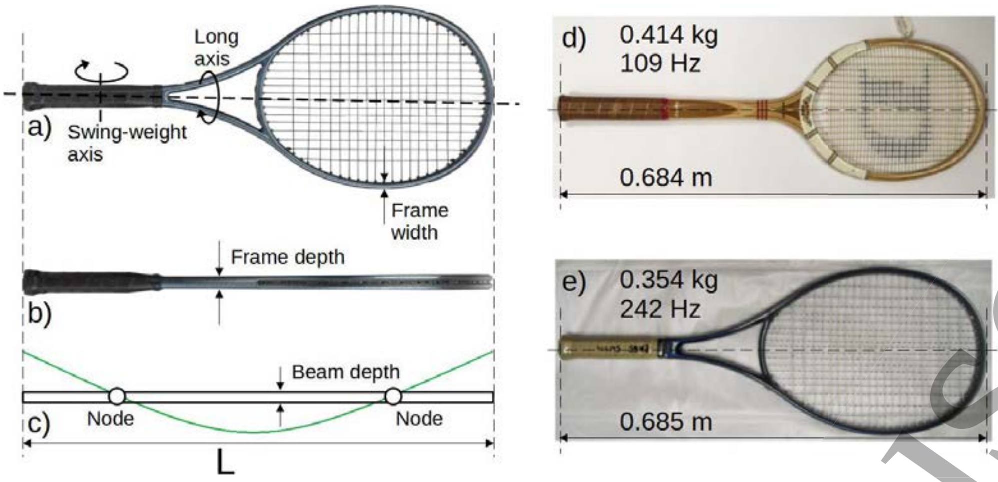

Figure 1. Diagram to show how a tennis racket can be simplified to a uniform beam: a) front view showing both the long axis about which the polar moment of inertia is measured and the axis about which swing-weight is measured and b) side view; c) beam, including vibration nodes and shape of f_1 in green; d) Dunlop "Maxply Fort" and e) Kuebler "Original Widebody 280 Hz" rackets, with values for mass, f_1 and length from [3].

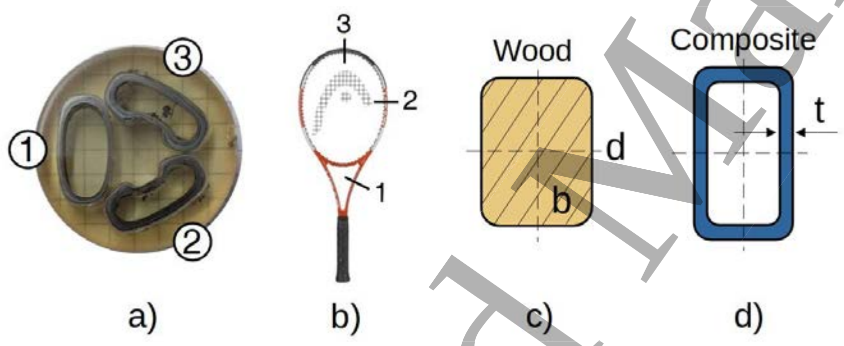

Figure 2. a) Example of cross-sections from a composite frame (HEAD Radical Liquidmetal, image courtesy of HEAD); b) numbers 1, 2 and 3 to indicate the regions of the racket where the cross-sections were taken from. Cross-sections used in the beam models of c) wooden and d) composite frames, showing the width (*b*), depth (*d*) and wall thickness (*t*). The dashed horizontal lines show the neutral axes of the sections.

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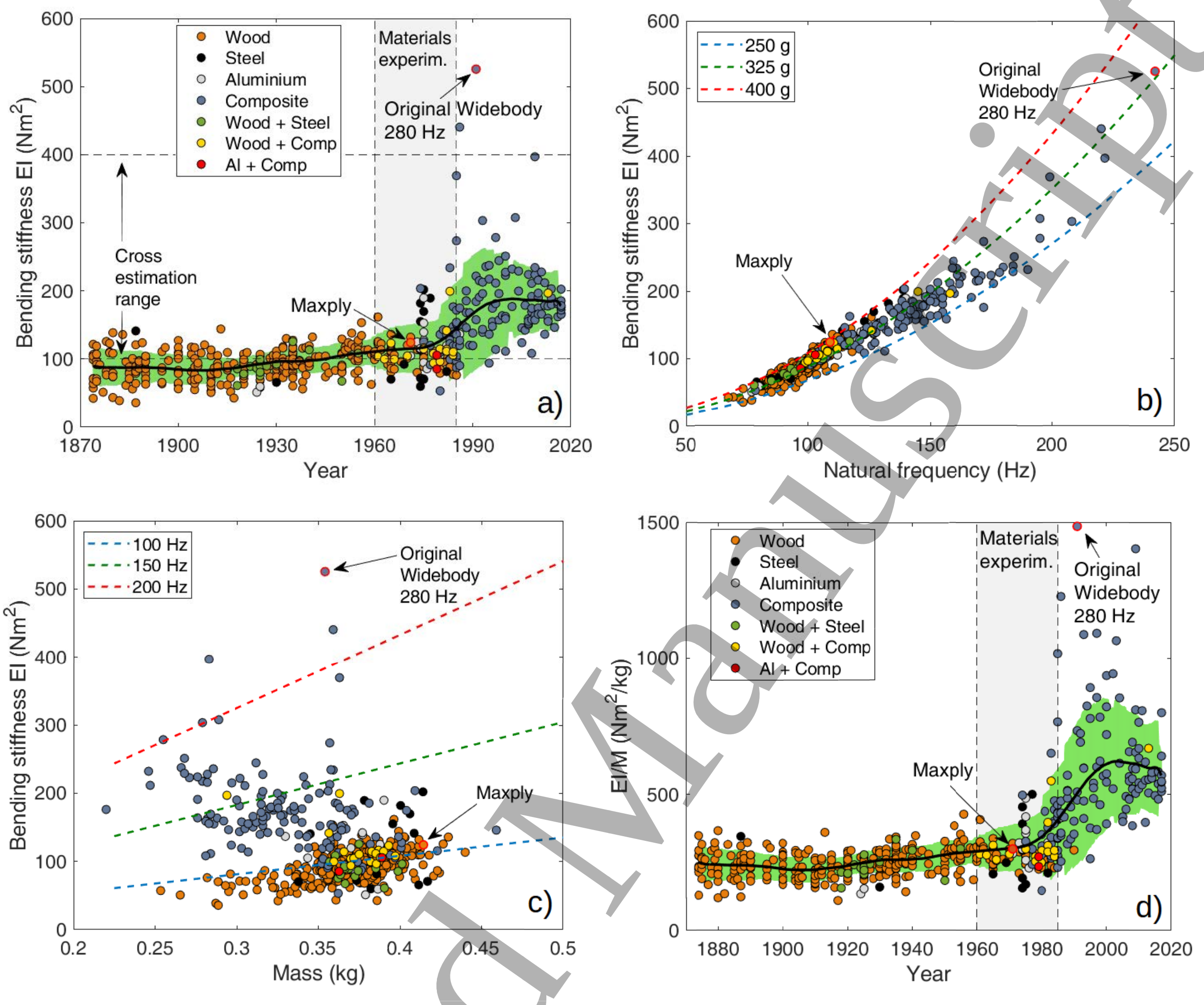

Figure 3. *EI* of 525 rackets from [3], as calculated using Equation 2, vs. a) year, b) f_1 and c) mass, d) EI/M vs. Year. The range in *EI* used by Cross [14] is overlaid in a), the grey region between the vertical dashed lines in a) and d) highlights a period of materials experimentation when shifting from wood to composites, the solid line is the moving average of the data, and the shaded region the SD. The dashed lines in b) show the effect of mass for a fixed f_1, and those in c) show the effect of f_1 for a fixed mass, as calculated using Equation 3.

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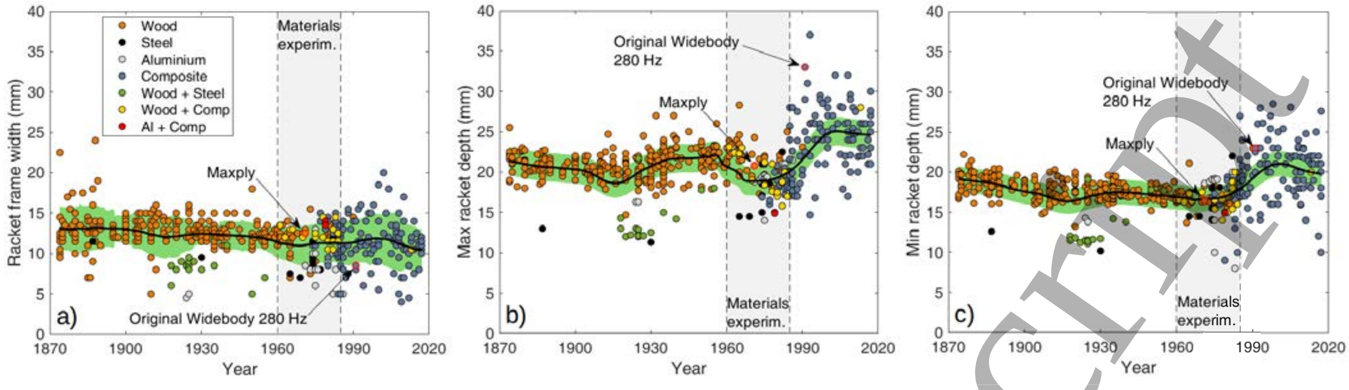

Figure 4. Racket measurements from Taraborrelli et al. [3], a) frame width at the widest point of the head, and b) maximum and c) minimum depth. The grey region between the vertical dashed lines highlights a period of materials experimentation when shifting from wood to composites, the solid lines are the moving average of the data, and the shaded regions the SD.

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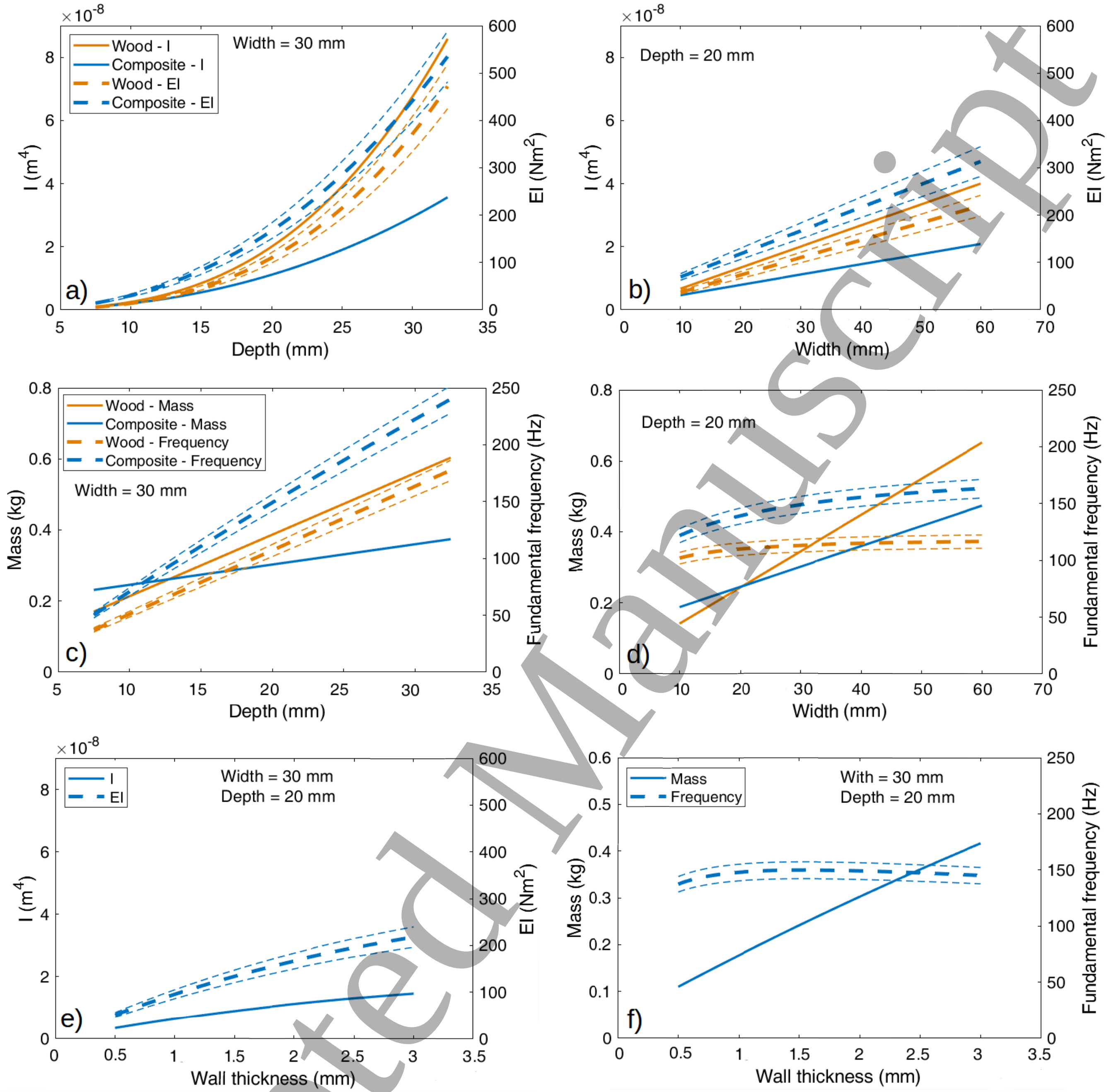

Figure 5. Beam models to illustrate the effect of the depth, width and wall thickness for wood (E = 5.5 GPa, ρ = 8,750 kg/m^3) and fibre-polymer composites (E = 15 GPa, ρ = 2,100 kg/m^3, wall thickness = 2 mm). Variation of I and EI: a) with depth for a width of 30 mm; b) with width for a fixed depth of 20 mm; c) variation of mass and f_1; c) with depth for a width of 30 mm; d) with width for a depth of 20 mm; variation of e) I and EI and f) mass and f_1, for fixed width and depth. Thin dashed lines show ± 10% change in E.

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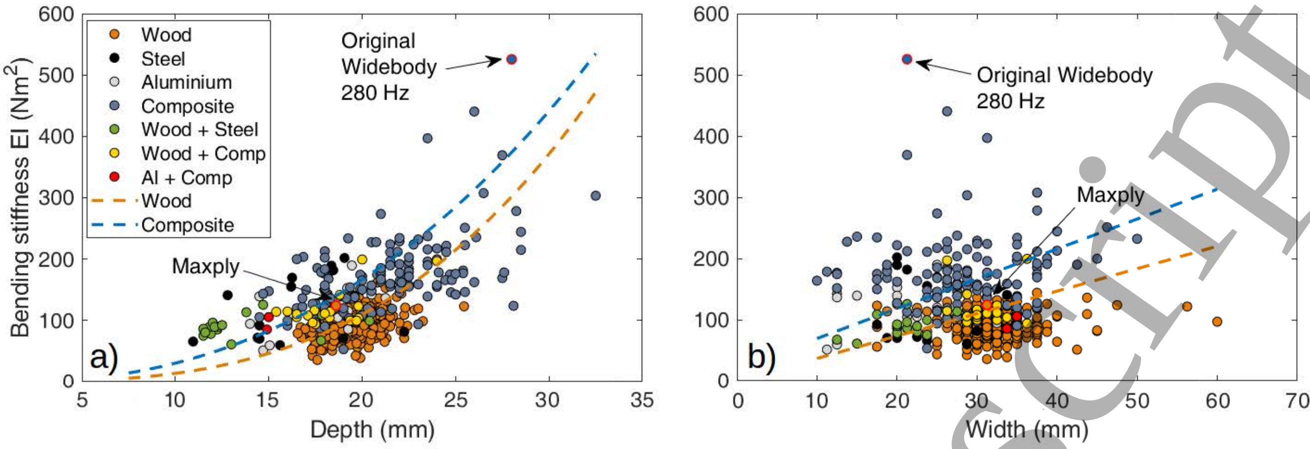

Figure 6. *EI* of 525 rackets from Taraborrelli et al. [3], as calculated using Equation 2, versus a) mean depth (Figure 3b, c) and b) width. *EI* predictions from multiplying estimates for *E* and *I* for wood (*E* = 5.5 GPa) and fibre-polymer composites (*E* = 15 GPa, wall thickness = 2 mm) are overlaid as dashed lines for comparison, using either a fixed width of 30 mm or a fixed depth of 20 mm. For b), the frame width of the racket measurements (Figure 1a, 3a) were multiplied by 2.5 to align them with the predictions on the x-axis.

