education sciences

MDPI

Article

Teacher Development for Equitable Mathematics Classrooms: Reflecting on Experience in the Context of Performativity

Sue Hough and Yvette Solomon *

Faculty of Health and Education, Manchester Metropolitan University, Manchester M15 6GX, UK; s.hough@mmu.ac.uk
* Correspondence: y.solomon@mmu.ac.uk

Abstract: In this article, we chart the development of one of us—Sue Hough—from a teacher who wanted students to understand to one who gained new critical understandings of student thinking, pedagogy, and the very nature of mathematics. We comment on the role of research interventions and learning communities in this development, with a particular focus on Sue's encounter with Realistic Mathematics Education and the connections it makes between informal and formal mathematics through the pedagogy of guided reinvention. Development towards teaching that enables all learners to make sense of mathematics requires fundamental changes in pedagogic practice and a reconceptualisation of progress. Bringing about such radical change relies on one further aspect of Sue's story—the freedom to experiment and learn as a teacher. We note the remoteness of this possibility in a climate of performativity and marketised education, and we discuss the implications of Sue's journey for our pedagogical responsibilities in professional development today.

Keywords: mathematics education; teacher learning; Realistic Mathematics Education; professional development; education policy

Citation: Hough, S.; Solomon, Y. Teacher Development for Equitable Mathematics Classrooms: Reflecting on Experience in the Context of Performativity. *Educ. Sci.* **2023**, *13*, 993. https://doi.org/10.3390/educsci13100993

Academic Editors: Constantinos Xenofontos and Kathleen Nolan

Received: 6 July 2023
Revised: 6 September 2023
Accepted: 25 September 2023
Published: 28 September 2023

1. Introduction and Background

A critical approach to mathematics teacher education involves challenging deeply embedded ideas and approaches on several fronts. In this article, we focus on the everyday challenge of including all students in making sense of mathematics, recognising that the outcome of traditional teaching is frequently the reproduction of privilege rather than understanding. Turning this state of affairs around is a question of thinking about how mathematics instruction is organised (and what mathematics is), how classroom cultures support learning, and how perceptions of students position them as 'able' or 'not able' in mathematics. These three aspects of mathematics education are situationally interrelated—they work together within the social/political system within which teachers and students work and learn. Hence, we are well aware of the difficulties of initiating teacher change in systems in which education values are dominated by performance measurement [1]. This situation is widespread: in the Anglo–American–Australian context, mechanisms of accountability are now normalised to the extent that teachers understand the quality of what they do as solely defined in terms of student test performance [2]. Pressure to perform is even exerted in less test-dominated education cultures such as Norway and Sweden due to international comparisons such as PISA [3], which raise political expectations and feed neoliberal discourses, compounding narrow definitions of good mathematics teaching despite a background of humanist educational ideologies [4]. In this context, the development of teacher action for equity is particularly challenging for us as researchers and teacher educators committed to fundamental change in what is valued in mathematics education. In this paper, we draw on Sue's account of her development from a newly qualified teacher to a teacher educator committed to developing equitable mathematics classrooms in which all students can participate. Building on her account of the particular events, ideas, and interventions that influenced her thinking, and the role of these reflections in the design of teacher education, we consider the implications of her story for the challenges we face in developing professional development towards inclusive mathematics classrooms in the marketised, accountability-driven context in which we now work.

2. Pathways to Equitable Mathematics Classrooms

As we note above, our interpretation of critical mathematics education is one that recognises the role of social systems in the reproduction of a mathematical 'elite' (who may in fact be simply good at following rules but nevertheless possess the examination 'ticket' to a future) and a large underclass of failures (around 30% of 16-year-olds in England fail its high-stakes national examination in mathematics every year [5]). We attribute the problem to the restricted nature of traditional mathematics education, which prioritises a purely formal conceptualisation of mathematics that has little connection to students' informal mathematical actions in the world. In this section, we explain our understanding of what needs to change, drawing on Horn and Garner's [6] work on teacher learning of ambitious and equitable mathematics instruction, and making links to Realistic Mathematics Education as instructional design for making sense of mathematics.

Horn and Garner emphasise the situated nature of the development of classroom spaces where learner conceptions of mathematics are heard, valued, and gradually developed. In particular, they note the need for three shifts in teachers' conceptualisation of teaching in relation to their particular work context, their 'beliefs, values, explanations, definitions, and ideologies' ([6] p. 26), and their resulting pedagogical judgements. The first shift concerns the organisation of instructional activities and a move away from the teacher-centred presentation of procedures towards an emphasis on problem-solving, argumentation, modelling, and proof ([6] p. 27). The second involves a corresponding change in classroom discourse towards whole-class discussion in which teachers remove themselves from the centre, questioning in order to seek information rather than eliciting and evaluating known information. Listening to students and seeking explanations of their thinking is paramount. These moves necessarily generate and maintain a classroom culture of listening and explanation in which students must also invest. The third shift concerns a change in teachers' perceptions of students and of the nature of mathematical competence itself (p. 36). Rejecting hierarchical views of ability and easy reproduction through a focus on calculation, Horn and Garner argue that:

> . . . expanding mathematical competence does not mean watering down mathematics. Indeed, the opposite is true: Expanding mathematical competence means rendering authentic mathematical smartnesses both visible and consequential in classrooms. Looking at mathematics as a field, we see that its great accomplishments have not come about from quick and accurate calculations, but from other kinds of insights, creativity, and intelligence: asking good questions, making astute connections, working systematically, seeing patterns, illustrating representations, and so on ([6] p. 36)

For us, Realistic Mathematics Education (RME, [7]) ticks the boxes of equitable mathematics education in its emphasis on a mathematics which is generated in a 'bottom—up' manner from students' informal actions, rather than being a 'top—down' given presented by a teacher in authority. This is, we argue, an equitable mathematics because it is rooted in realisable and meaningful contexts and most importantly remains so—RME never abandons the learner to meaningless forms. The instructional design underpinning RME thus emphasises the role of emergent mathematics [8]. Students move from models of their informal activity in carefully chosen contexts towards strategies that they can justify in ways that are accessible to themselves and others, thereby developing what Yackel and Cobb [9] describe as intellectual autonomy in mathematics. The shift away from teacher authority towards a mathematics that notices, appreciates, and builds on students' own understandings necessarily creates a different role for the teacher—they must develop the technique of guiding learners [10,11] as they move towards forms that can be articulated and 'taken-as-shared' in the community of the classroom ([9] p. 460). RME thus emphasises a process of mathematisation in which 'acceptable explanation and justifications [have] to involve described actions on mathematical objects rather than procedural instructions' ([9] p. 461).

This emphasis on informal models has implications for the dominant conceptualisation of progress as indicated by the successful application of procedures in standard mathematics topics. RME identifies two ways in which students engage with mathematics; on one level, they solve the contextual problem under consideration ('horizontal mathematisation'), while on the other, they work within the mathematical structure itself by reorganising, finding shortcuts, and recognising the wider applicability of their methods. This 'vertical mathematisation' [12] is indicated when students recognise similarities between their models of a number of related contexts and how these can be generalised to other problems. Thus, they may move from a 'model *of*' a particular situation (for example, a subway sandwich represented as a squared-off bar) to a generalised 'model *for*' solving various types of problems which may span diverse topics [13], such as a double number line or a ratio table. Progress is redefined as the progressive formalisation of models [7] but this does not prioritise the formal method; rather, it prioritises the explicit act of connection between 'model of' and 'model for' that underpins it. This reconceptualisation of mathematics and mathematics learning feeds into inclusion as envisaged by Horn and Garner: RME not only engages all students through its explicit connections to what is imaginable, but it also promotes an inclusive student-centred pedagogy. Most importantly, RME demands a different view of what mathematics is and who can do it, and, therefore, what progress is. Indeed, test results based on procedural knowledge do not reflect, or even demand, the deep understanding built by RME. As Sue's account in Section 5 shows, RME has played a significant part in her development as a teacher and critical teacher educator, and it remains a central plank of our work with teachers now in terms of providing opportunities not only for student access and engagement but also teachers' understanding of mathematics itself.

3. Teacher Learning in a Climate of Performativity

The explicit move away from procedural mathematics advocated by Horn and Garner and embedded in the principles of RME presents a number of challenges for teachers raised in post-performative times, whose practice is inextricably bound to prioritising test performance [2]. Even teachers who do not subscribe to a procedural approach find that demands for accountability compromise more exploratory aims [14,15]. The emphasis on outcomes measurement consequently reduces the impact of reform-based professional development [16,17]. This situation is exacerbated in England, where much professional development is now government-funded and centralised, and there is a perceived need for consistency across the sector in dictating what is to be learned as received 'best practice' [18,19].

On the contrary, teacher learning is an incremental and context-sensitive process which cannot be blue-printed [20,21]. In a performative climate, moving towards equitable mathematics instruction requires the development of critical noticing [22] which questions discursively fixed conceptions of competence/ability and how these position students, opening the way to different pedagogical actions (see also Horn [23]). Importantly, such actions need to be underpinned by pedagogical reasoning but also supported by what Horn and Garner [6] call pedagogical responsibility—a teacher's sense of what constitutes appropriate teaching in a given context. This comprises not just ethical principles (such as wanting all students to gain the mathematics they need to be able to participate in the world) but also responses to situational constraints. Horn and Garner point to the fact that teachers may feel that they must prepare students for high-stakes assessments, whether or not they think they are legitimate. This means that teachers must navigate their way through the potentially conflicting demands of teaching for deep understanding versus making sure that students can spot what is needed in an exam question. In addition to these genuine dilemmas, we see performative cultures and the marketisation of education as creating additional day-to-day obstacles for teachers. Education in England is increasingly 'delivered' by multi-academy trusts—not-for-profit companies that run groups of schools, often supported by sponsors, including businesses—which are focussed on public 'successes' and branding, fostering business rather than educational priorities [24]. The push towards marketisation is not limited to England; for instance, deregulation is now causing a focus on branding in Norway [25]. Loss of teacher autonomy in this context is a concern [26,27]. Together with the growth of nationalised best practice professional development programmes, this situation has presented us with complex new 'problems of practice' [6] in our work with teachers.

In this article, we are particularly mindful of how, as teacher educators, we need to acknowledge our own pedagogical responsibility in terms of the demands and constraints exerting daily pressures on teachers in the current climate of high-stakes testing and its impact on their students, invoking highly emotional responses for some [28]. Now, more than ever, we need to identify what is important about teacher development for equitable classrooms and how this can be incorporated into our training today.

4. Reflections on Teacher Learning: Telling the Story

The inspiration for these reflections lies in our work over several years researching student inclusion and the impact of RME in classrooms, and our awareness of the considerable pedagogical changes involved in using RME materials. The focus of this Special Issue on critical mathematics teacher education led us to consider the importance of telling a teacher's story in order to capture 'the view from the other side' and embed this reflectively in the context of our work as teacher educators. Thus began a process of digging into Sue's teaching past, collecting 'data' that frequently drew on her masters' degree work in 2003, but also included reflections—sometimes with accompanying student worked examples RME materials, or professional development details—as told to Yvette. Our process in constructing this narrative was for Sue to write about key events in her career, illustrated with her previous work and examples. Yvette read these accounts and asked for more details or explanation; our conversations were often long and completely novel—we had never discussed some of these issues before. A story emerged that seemed important to us in terms of Horn and Garner's [6] account of the shifts required to move towards equitable mathematics classrooms. It had RME at its core, showing its impact on Sue's conceptualisation of mathematics and her classroom practice.

At the same time, we were aware of the context in which we work and its impact on professional development. While the teachers we have worked with often bring stories back to training days of new student engagement and understanding as a result of the work we do with them, an increasing number have also talked about the need to justify their practice to school managers who worry about the impact on their school performance data of lessons in which rote learning and practice are nowhere to be seen. The increasingly performative context of education and professional development outlined in the previous section is important, and it has impacted how the story is told—Sue looks back on her experience of becoming critical with the hindsight of much greater RME knowledge and experience than she set out with, but she also sees this experience through the lens of the challenges she faces as a teacher educator now.

In the next section, Sue takes over the narrative to describe her development over a period of 15 years towards a new understanding of the meaning of 'making sense of maths', identifying critical moments and reflecting on how these raised issues for her regarding what mathematics is, what students bring to the classroom, and what this means for pedagogic practice. Central to her story is the need for a critical response to traditional accounts of mathematics and conceptions of progress, and what this means for professional development today. She draws on a collection of memories and illustrations from her participation in formal professional development, locally organised teacher communities, participation in funded projects, small-scale study of her own classroom, and finally her early experiences as an RME teacher educator. We have chosen to tell Sue's story uninterrupted and in her voice alone.

5. Moving towards Equitable Mathematics Classrooms—Identifying Key Events

5.1. Questioning Traditional Teaching

I began teaching in 1987, using a modern textbook series interspersed with investigations, group work tasks, and activities recommended by the Cockcroft Report [29]. But for the most part, like many other teachers, I settled into models of teaching that mirrored the way I had been taught: exposition of a formal method by the teacher followed by pages of student practice to 'acquire' the method. But planning for teaching lessons often meant that I needed to think more deeply than I had at school, and I learned for the first time what lay behind many of the 'rules'. I vividly recall the moment I realised that multiplying length × width to find the area of a rectangle was actually a way of counting the rows of squares that could be drawn inside it. I found such revelations shocking and felt embarrassed that I had come this far in learning mathematics (I had a bachelor's degree in mathematics) without making such a connection. Why hadn't my teachers told me about this? I became committed to explaining my new-found enlightenment to my classes, keen for them to share this understanding. Nevertheless, in marking my students' tests, it was apparent that the transfer I hoped for had not happened. I first blamed my students—'we had covered the work, they had good notes, they couldn't have bothered to revise the work'. Later, I read Jaworski's [30] critique of transmission teaching and the expectation that knowledge can be simply handed over through good exposition. In blaming my students, I had failed to recognise that successful teaching relies on learners making their own mathematical constructions, and that this is unlikely to happen through 'teacher telling'. This realisation gave me a different perspective: looking back, my own teachers had probably provided justification for the rules they taught me, but hearing their explanations was not the same as developing my own constructions and sense-making. Ironically, here I was, now the teacher, trying to transmit my newly constructed explanations to my own classes—by telling. Something needed to change.

5.2. Learning to Listen to Students

In 1997, I joined the Mathematics Enhancement Project (MEP) [31] as an intervention teacher, working with Year 10 students in their penultimate year of compulsory schooling, which would end with the national GCSE examinations at age 16. MEP arose from England's participation in the international Kassel Project, which highlighted the relatively high performance of several Eastern bloc countries compared to that of the UK. Lesson observation of one of the highest performers, Hungary, revealed that classes in Hungary worked together on problems guided by an 'on the go' teacher with an emphasis on whole-class interaction and student debate. In contrast, teaching in the UK focussed on correct answers rather than the details of the method, with teachers working with students individually rather than with the class as a whole. The subsequent MEP intervention was modelled on the Hungarian system, focussing on whole-class interaction, precise use of language in writing and speaking mathematics, numeracy, applications, and homework as an integral part of the learning journey rather than just an 'add on'.

It was inspiring to watch video of Hungarian students walking to the front of their classrooms, pen in hand, to write their solutions on the board and describe to the class what they had done. The class teacher invariably positioned themselves at the back or to the side of the classroom, asking questions of the student presenting their solution. This practice had a major impact on my own teaching. I began lessons outside my classroom, standing at the door, with students lined up, open book in hand, to show me their homework as they entered the room. As they filed in, I would select around three students to write their homework solutions on the board. Standing towards the back of my classroom, I would invite them to describe to the rest of the class what they had done. I would try to read faces and direct questions to other students to see whether they could follow their classmate's approach in a homework review phase that could take over half an hour.

By now, I was practising being neutral, thanking students for their solutions, commenting on them as 'interesting', but otherwise presenting a blank face. This teaching strategy became a common part of my practice when working with solutions that students brought to the classroom. I see this phase now as one in which I was trying to create a classroom where learners spoke more than me. I would ask students to describe what they had done and to ask questions of each other without needing to go through me. If students did not talk loudly enough, I would repeat their words, but I was conscious to do this verbatim so as to keep the focus on their thinking, not mine. At times, I would sit in a student's seat in a deliberate attempt to lower my presence and status in the room. When working at the board, some students were hesitant and needed support, so I developed less intimidating instruction. 'Explain what you have done' became 'Describe what you did', and 'Compare these two methods' became 'What's the same? What's different?'. All students needed encouragement to realise that their audience was the class, not the teacher. Gradually, student–student interactions increased, so that a student at the front of the class might notice a blank face and ask: 'Did you get that?'. I noticed a shift in mathematical authority in the classroom as students began to scrutinise and question each other's mathematical statements or arguments, no longer looking to me to tell them whether it was correct or not. My teacher role was now one of facilitator, enabling them to think, describe, draw, listen, speak, and question mathematically.

My choice of solutions to showcase was based on mathematical criteria for selecting a solution. I picked out (i) common misconceptions, (ii) standard solutions, and (iii) unusual approaches that might contain a diagram or more long-winded strategy. I was not focussed on whether such answers were correct or not but rather whether they were accessible to other learners. I learned a good deal about the way learners thought mathematically and found novel solutions that bore no relation to the method I had taught, although they made sense to the student when described. There were times when a student might describe what they had done and other students would follow, but I, the teacher, would not. This was a reminder that my way of seeing mathematics might be far away from theirs.

5.3. Recognising the Need for Informal Mathematics

Following my MEP experience in 1998–2002, I joined with a number of teachers in my local area who were interested in developing innovative pedagogies, particularly whole-class interactive teaching, and were working with tutors at Manchester Metropolitan University. We videoed each other and jointly watched and commented on the lessons. This activity helped to develop our awareness of teacher actions such as where we stood, how we presented ourselves as neutral, what sorts of prompts supported student contributions, and what actions helped students reveal their inner mathematical thinking. One member of the group was particularly focussed on developing student-centred approaches and was strongly influenced by Gattegno's claim that "everyone can be a producer rather than a consumer of mathematical knowledge. Mathematics can be owned as a means of mathematizing the universe" ([32] p. 2). Watching one of his lessons, described below, had a profound effect on both the way I began to see learners and how I viewed the school mathematics curriculum:

> Aidan is teaching a bottom set Year 8 group of 10 students. The first question in his quick quiz starter is presented as the multiplication 16×12 written above a rectangle accurately drawn 12 down and 16 across. The side lengths are not labelled, and the outline is drawn on a faintly squared blackboard. The students set to work drawing the rectangle on squared paper, and one enquires if it needs to be 'dead accurate'. Aidan circulates to gain a sense of student approaches. After six minutes, he stops the group and talks to them about how to behave when observing each other working at the board. He reminds them to use manners, be respectful, and remember that they are on a journey to becoming mathematicians.

> He talks about 'us' and 'we', and that we are looking for shortcuts, spotting patterns, making connections, and recognising that we are all at different stages of that journey.
>
> Aidan invites a student to come to the board to show how he found the number of squares. Beginning with the bottom left-hand square, he touch-counts each square, writing '1' in the bottom square through to '12' at the top of the column (see Figure 1). The rest of the group watch. The counting is slow. To me, it feels slightly tedious but apparently not to the class and certainly not to Aidan who comments on the precision with which the student is counting and the hard work that it takes to work in this way. Another student comments that 'you could have just written in 12 because it says it's 12 down the side'. The first student, seemingly not ready to make this connection, continues counting down the second column of squares, writing in numbers as he goes, "13. 14. 15. 16. 17. 18. 19.". Aidan instructs him to stop there and asks 'Where is it going to end? What is the last number he will write in this column?'. He reminds the class not to shout out in order to allow thinking time. After a 30 s pause, Aidan accepts a student's suggestion of 24 with the response 'How did you know it was going to be 24?'. The student refers to 1 and 1 making 2, 2 and 2 making 4. Aidan responds with 'Ok' and moves on to hear from another student. We return to the student at the board and repeat the sequence of counting, stopping part way and hearing other students' rationale for what goes in the last square of the third column. Aidan provides a meta-commentary on their strategies, noting that they are pattern spotting. He carefully selects who comes to the board, based on the strategy they have used. The next student numbers only the bottom square of each row to reveal a total of 192 squares. A third partitions the original rectangle into 4 smaller sized rectangles, but her partitioning does not match the 100/60/20/12 totals she writes in each mini rectangle. Their methods are all based on counting the squares inside although some have developed short-cut ways to do this. None of the class are attempting to multiply 16 $\times$ 12 using a standard algorithm. Eighteen minutes of the lesson is given to sharing strategies, with Aidan commenting on them and directing some students to try others' strategies when it comes to the next lesson.

Observing and analysing this lesson was revelatory for me. It raised more questions than answers and threw up a wealth of contradictions, forcing me to look again at my own practice and indeed my beliefs about what and how my own students were learning. To begin with, the pace of the lesson felt slow, but what did that actually mean? Yes, it was slow in terms of traditional pace metrics such as the number of practice questions a student completes or the amount of different techniques shown to the class. But what if pace is understood as pace of thought or of making connections across representations? Secondly, what about the level of challenge? I had just seen Year 8 students touch-counting columns of squares aloud, slowly, and deliberately. Surely they knew you could just multiply the length by the width of a rectangle to find the total number of squares? Surely they had quicker methods for performing the calculation 16 $\times$ 12? What became apparent to me in this lesson was that no, these students did not realise that counting these squares in columns amounted to the same total as multiplying the length by the width. And no, performing the calculation 16 $\times$ 12 was not easy or straightforward for these learners.

I was forced to question many of my long-held assumptions. These students would first have encountered strategies for finding area using multiplication four years ago in Year 4, and again in Years 5, 6, and 7. Nevertheless, despite all of this curriculum time and experience, they did not connect the multiplication of lengths strategy for finding area with the counting squares version. This was more than simply not remembering—it felt like there was a major gap in conceptual understanding between their concrete, long-winded but accessible strategies and the formal algorithm. If this was true in this case, what other chasms were lurking in my own classroom? Not only that, but what could I do about this?

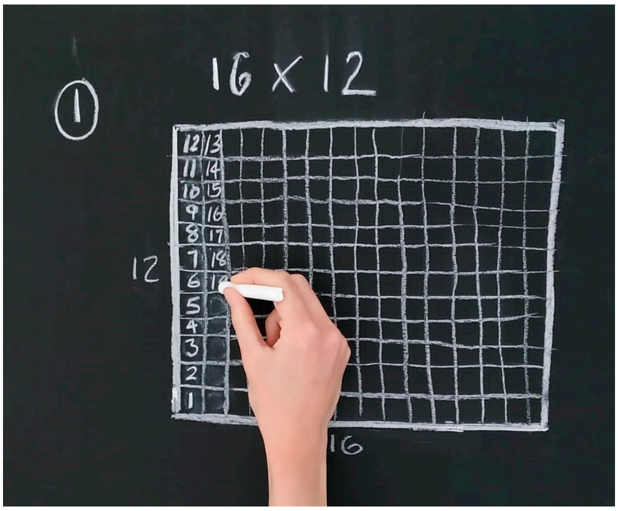

Figure 1. Touch-counting strategy used to find the total number of squares of a 16 × 12 rectangle. Reproduction of classroom board work.

5.4. Developing Teaching That Values Informal Understanding

In response to the revelations in Aidan's lesson, I investigated the topic of area further, initially conducting a study in my own classroom that was designed to improve the ability of lower attaining students to find the areas of various shapes. Informed by my own understanding of area and my awareness of research showing standard student misconceptions, I set out to intervene in students' approaches by teaching them to visualise squares when thinking about area. As I will show below, my later encounter with RME made me radically rethink this approach and the role of informal understanding in mathematical concepts.

5.4.1. Intervening in Student Thinking—The Area Project

In 2003, as part of my master's degree, I conducted a small-scale study with two lower attaining Year 9 groups. I carried out a short pre-test requiring students to find the areas of a rectangle, triangle, parallelogram, and compound shapes, all presented on plain paper, with side lengths labelled. The test also included four non-standard figures presented on squared paper. The average score was 3.5 marks out of 12. A close inspection revealed that students were most successful when finding the areas of shapes presented on squared backgrounds, using strategies of piecing together half squares to make full squares before totalling or by counting and accumulating full and half squares, as shown in Figure 2a,b, respectively.

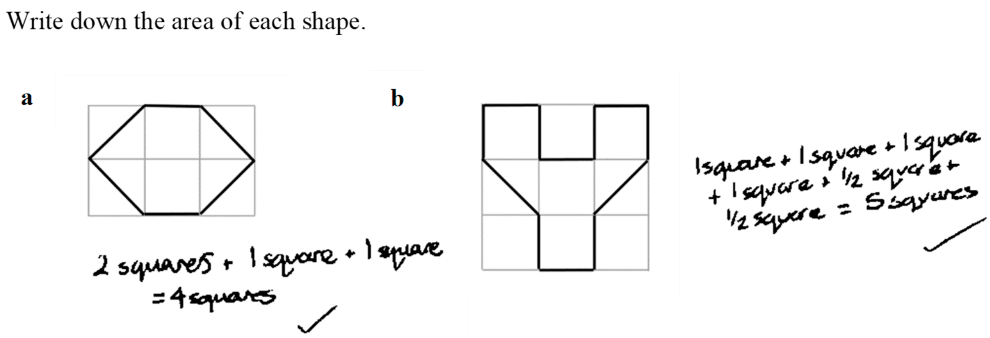

Figure 2. (**a**,**b**) Finding area using a squared background: two different Year 9 student strategies.

Difficulties mainly arose when finding the areas of shapes drawn on a plain background. Some students, as illustrated in Figure 3, consistently found the perimeter rather than the area, while others added the two dimensions given in each question, whatever the shape. Others consistently multiplied the given dimensions, and one repeatedly multiplied the given dimensions and then halved the result. There was no indication in these solutions that students were thinking about the space that could be fitted inside the plain figures. Finding the areas of compound shapes (such as an 'L' shape) proved to be particularly challenging. Several students found the perimeter, while others appeared to be 'doing something with the numbers', be it adding, multiplying, or a combination of both (even when this resulted in unrealistically large number), again apparently unable to connect their calculation with the space-covering aspect of area.

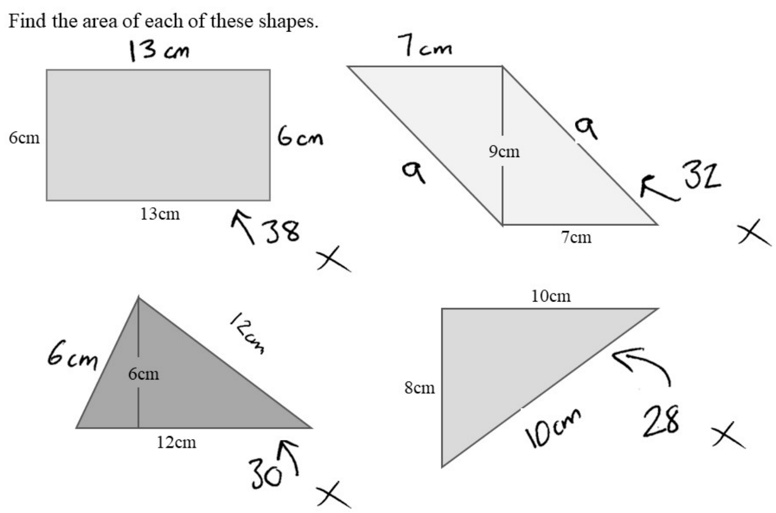

Figure 3. Calculating perimeter rather than area: a common student strategy from the Year 9 pre-test.

It was reassuring to realise that my own observations concurred with the findings of several researchers, particularly their findings on the effect of squared paper. The Assessment of Performance Unit study [33] noted that around one-fifth of the students tested gave perimeter when asked to find area, but that this was less likely to happen if a grid or unit square key was provided. The Chelsea Diagnostic Mathematics Tests study [34] reported similar findings—while only 33–50% of students could correctly evaluate the area of a right-angled triangle presented on a plain background, 78–91% were successful when the triangle was drawn on squared paper. I began to reflect on why my students were experiencing such difficulties. Counting squares is a transparent process closely linked to the informal understanding of area. The concrete nature of this strategy makes sense as a means of comparing one space with another, as a way of quantifying that space, but the traditional trajectory of school mathematics moves students quickly from informal square-counting strategies to multiplicative methods. The disappearance of squares leaves learners with some algebra shorthand and shapes outlined by lines marked with lengths, which give no hint of the small unit squares contained within. I decided to bridge this gap by devising a method that built on counting squares as I needed a way of encouraging students to visualise the squares inside a shape. I rejected the idea of using squared paper or a transparent grid overlay, as these methods had been used before, were not available in an examination, and provided too much of a 'quick fix' in that the student did not have to fully engage in the process or make connections with the side lengths of a shape as corresponding to a row of squares sitting on that length. Instead, I decided to develop their ability to visualise and draw the squares freehand, using the dimensions labelled on the sides of the shapes as a cue for how many unit squares they needed to fit and draw into a row. Having created a row of squares to match the length dimension, students could use the height measurement to partition the vertical dimension and reproduce the required number of rows of squares. Having filled the shape with unit squares, students could then count them to find the area.

I tested the students again on the same questions at the end of my intervention. The images in Figures 4–6 illustrate some of their approaches. For triangles, the preferred method was to draw in the squares to make full rows even though these extended beyond the triangle boundaries, as shown in Figure 4. Students were then able to argue that the triangle used up half of the squares.

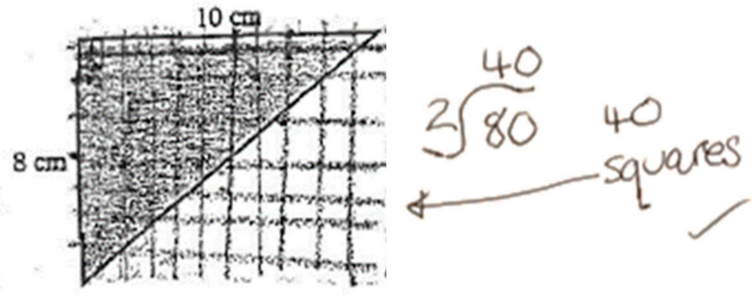

Figure 4. Adapting the drawing in squares method for a right-angled triangle: a student strategy from the Year 9 pre-test.

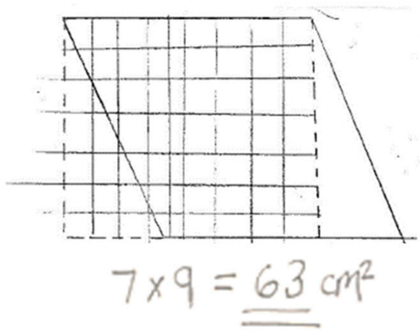

Figure 5. Adapting the drawing in squares method for a parallelogram: a student strategy from the Year 9 post-test.

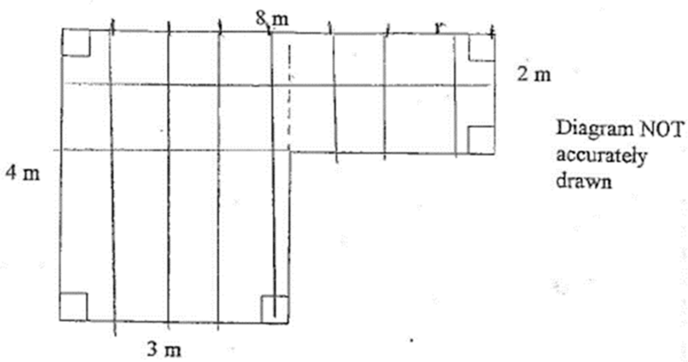

Figure 6. Struggling to fit same-sized squares when the image is not drawn to scale: a student strategy from the Year 9 post-test.

Similarly, for a parallelogram, students drew a full row of squares along the top length, as illustrated in Figure 5, and repeated the rows directly underneath to form the rectangle of equivalent area.

However, there were difficulties in acquiring this method. Partitioning a side length into the required amount of squares proved to be challenging for many students, even more so if the side length was an odd number. Some recognised when halving was appropriate and made efficient use of this strategy, but others guessed and checked, adjusting the sizes of their squares to fit as they approached the finish line. Some struggled with the relationship between the number of squares and number of dashes. When faced with dividing a line segment into, say, 12 parts, some students were naturally inclined to put in 12 dashes between the endpoints, creating 13 gaps. The guess and check method of estimating the size of one gap could take time with much rubbing out, while test items not drawn to scale could lead to inconsistencies in the sizes of squares. As illustrated in Figure 6, the student began by marking eight spaces on the top line. As stated in the question, the diagram is not accurately drawn and the eight equal spaces in the top line do not correspond to the 3 spaces on the bottom line, hence the inconsistency.

My intervention appeared to have had an impact, however, with the students introduced to the 'drawing in squares' strategy making post-test gains of 5.6 marks out of 12 compared to +1.3 marks for the traditionally taught 'textbook' group. I concluded that there was definitely something to be gained by teaching lower attaining students to draw in the squares. The method was time-consuming, but it seemed to prompt engagement with the meaning of the dimensions marked on the boundaries of the shape and, in turn, it built up a picture of the squares inside and how these filled the space. Once realised and visible, the students were able to evaluate the amount of squares. Many elected to touch-count each square, a lengthy, primitive strategy, but at least this was a sense-making strategy owned by the student. Some moved on to multiply particular dimensions as a way of determining the amount of squares contained within a particular block, thereby using the formal operation present in all area formulae—that of multiplication. Crucially, this was not because they had been told to do so; rather, it was because it made sense to them as a way of quickening the square-counting process. My encounter with RME later that year underlined the importance of learners constructing for themselves the connections between side lengths realised as squares, through counting squares, and finally to multiplication, and the implications for the role of models in a truly student-centred pedagogy.

5.4.2. Encountering RME: Redefining Progress

My first encounter with RME was later in 2003, when I was recruited as a teacher in Manchester Metropolitan University's pilot study of the Mathematics in Context (MIC) materials, an American version of RME produced under the guidance of Thomas Romberg as a collaboration of the University of Wisconsin in America and The Freudenthal Institute in The Netherlands [35]. I readily abandoned my school's intended scheme of work and began teaching my Year 7 high-attaining group using the *Reallotment* booklet [36], a unit designed to provide students with guided opportunities to work on their ideas of area. The unit was intended to take around four weeks of study. This was September. In mid-December, I reluctantly decided that I had better move on to the next RME module, even though there were still several pages of *Reallotment* left. The experiences of working with that class on an RME-designed curriculum were both exhilarating and shocking, and I was really beginning to question what I believed at the time constituted effective teaching and learning.

In my classroom, unusual events occurred: students willingly volunteered to demonstrate their strategies to the rest of the class; they challenged one another's thinking with newfound confidence and genuine interest; they became active participants in reading text, starting problems without the need to be shown what to do; they instinctively requested materials such as tracing paper and string because they could see how these tools might help them to solve a problem. Ultimately, they saw questions as genuine problems that they could comment on, contribute to, and so have authority over. One student later said, 'It doesn't feel like you are doing maths'. Likewise, for me, the mathematics certainly felt different. My awareness of what was important mathematically changed. I came to recognise the informal strategies of fitting in/reallotting parts of a shape/seeing one shape as a fraction of another as valid in their own right, and I began to recognise that the contexts and associated informal strategies underpinned a learner's developing understanding of area as 'space covering', as in Hughes et al.'s [37] and Foxman et al.'s [38] work. I also realised that tasks such as finding the area of triangles drawn on a 9-pin geoboard [38] may have generated similar strategies, but in the past I would not have valued students' informal thinking, focussing instead on the area of a triangle formula end game.

I recognised now that there was a world of mathematics underneath the area formulae. I learned new ways of thinking about problems through studying a picture, drawing a picture, and applying intuition (as opposed to a formula). I saw imagery that helped bring the space-covering aspects of area into focus. I appreciated the roles of movement (of tracing paper, of parts of shapes) and of matching same-sized areas in developing an understanding of conservation of area. I began approaching mathematics problems at any level with my eyes open wide to imagery, to drawing, and to thinking about alternative representations, using my new-found powers of mathematical common sense. I already had the pedagogy developed through working with Whole Class Interactive teaching communities to support these ways of working, but RME gave me an even better awareness of student thinking and a greater expectation that students could access these problems across the whole attainment range. Through RME, I now had the questions to prompt multiple strategies, provide open access to starting problems, provoke different opinions and ways of seeing, and develop a classroom culture in which these approaches could flourish and take root. However, working with RME brought disturbances too: I had several questions and concerns about progress. How would my students perform in tests? What was behind the design of *Reallotment*? I knew my students were thinking, learning, and becoming empowered, but how did their informal, naturally developed solutions to the contextuallybased area problems link with formulaic approaches? Indeed, what constitutes progress in an RME classroom?

5.4.3. Connecting Informal and Formal

In a standard textbook, formulae appear early and trigger exercises designed to practise their use. Success in these indicates progress. In RME, progress towards formal representations is seen as a longer term aim stretching over weeks, months, and even years. It is not until lesson 7 of *Reallotment* that students are introduced to formulae, and the focus is different: formulae are constructed by the students themselves from carefully chosen images that enable them to reason about how the areas of a bare generic rectangle, triangle, and parallelogram connect to each other. Far from practising how to apply these formulae, the next question offers a slanted triangle and states that sometimes you have to use another strategy to find the area of a triangle. The problems in *Reallotment* appear to be in a random order if we look through the traditional lens of what constitutes progress. We expect to see a build-up of questions, containing harder numbers, or the application of more than one procedure, or the use of symbols as an extension of the use of numbers, in a process known as 'progressive complexity'. In RME, students make progress through progressive *formalisation*, where the level of question difficulty remains the same, but students have the opportunity to move from using informal, intuitive approaches through a variety of pre-formal strategies to using more contracted, efficient and abstract strategies [12]. In Figure 7, a question taken from the *Reallotment* Teachers' Guide asks students to find the price of pieces of wood given that the rectangular piece costs $18. Students are not required to know any area formulae—this is not an application problem. This context and the presentation provoke a range of naturally occurring but conceptually different approaches.

Strategy 1 is to draw in the background squares, figure out the price of one square of the wood, and recognise that the triangle covers three of those squares. tSrategy 2 is to draw a line splitting the original triangle into two smaller right-angled triangles, then see each separate triangle as a half of the corresponding rectangle. Strategy 3 provides a logical reason for seeing the black triangle as half of the original rectangle. In my class, students offered this range of strategies and it was possible to see some students shifting towards more contracted/global strategies, e.g., the area of *any* triangle is half of that of the surrounding rectangle/parallelogram. Through exposure to several context-based problems and by sharing their informal approaches under the guidance of the teacher, learners gradually become 'a little bit more advanced in mathematizing the problems' ([7] p. 15).

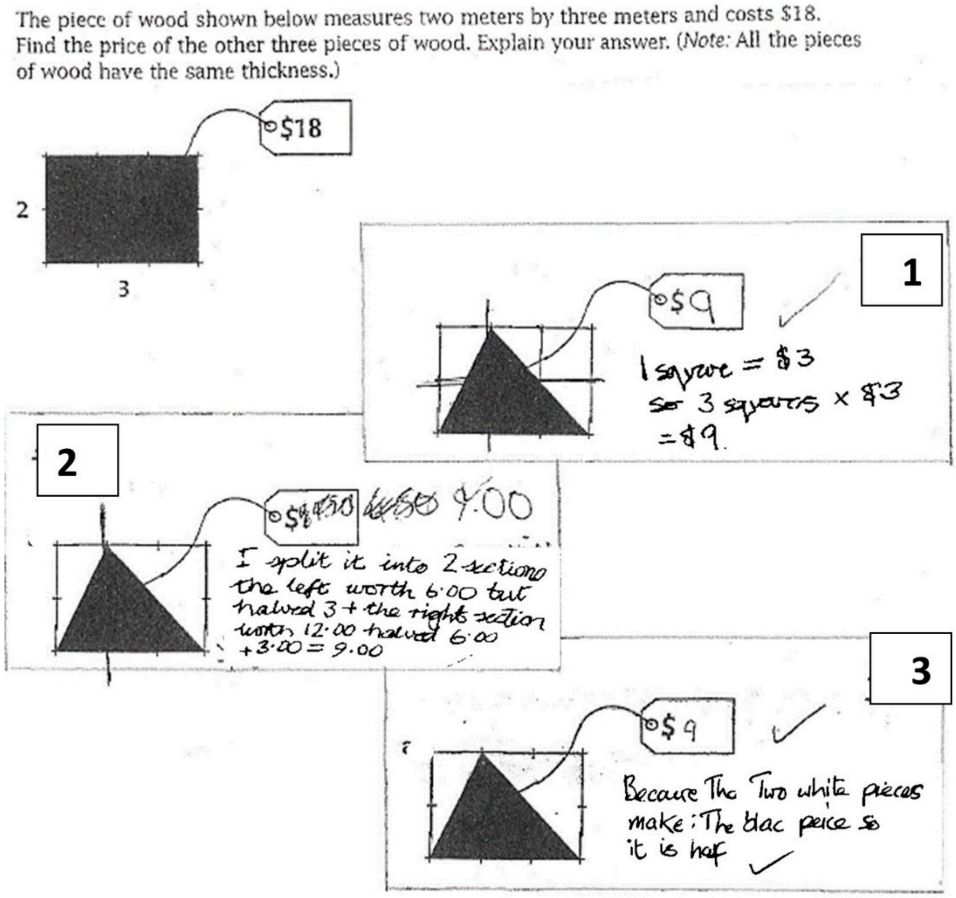

Figure 7. Progressively formal strategies (1 to 3) for answering the wood-pricing problem.

In RME, the formal world of mathematics (in this case, the area formulae) is most definitely available to all learners but the route to it looks very different. It is this shift towards more formal methods and representations that are rooted in context and sense-making that constitutes progress in RME. This understanding enabled me to reflect back on my intervention in students' learning of area. The method of instructing students to draw in the squares could be viewed as a 'top–down' strategy imposed by the teacher rather than generated by the students. But it could also be argued that the original shape (e.g., a rectangle presented on a plain background) is the context and that students are creating their own 'model of' the context by drawing in the squares. While students stayed with drawing in the squares and touch-counting each one they were using informal strategies closely linked to the context and, as such, they could be said to be mathematising horizontally. However, during the course of the intervention and with guidance, it was possible to see some students *vertically* mathematising. These students shortened their approach to drawing in the squares, drawing only the top layer of squares or partitioning down the side of the enclosing rectangle and moving to multiplication rather than touch-counting, as illustrated in Figures 8 and 9.

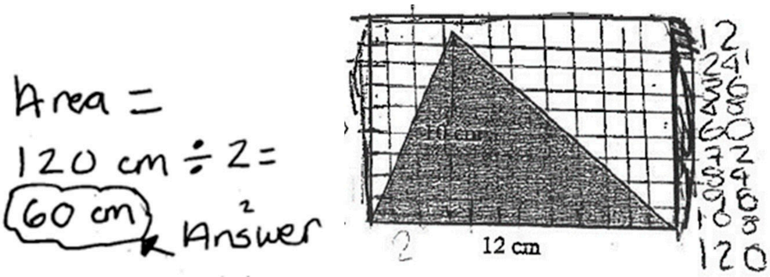

Figure 8. Drawing in squares and counting up in multiples of 12: student solution strategy from the Year 9 post-test.

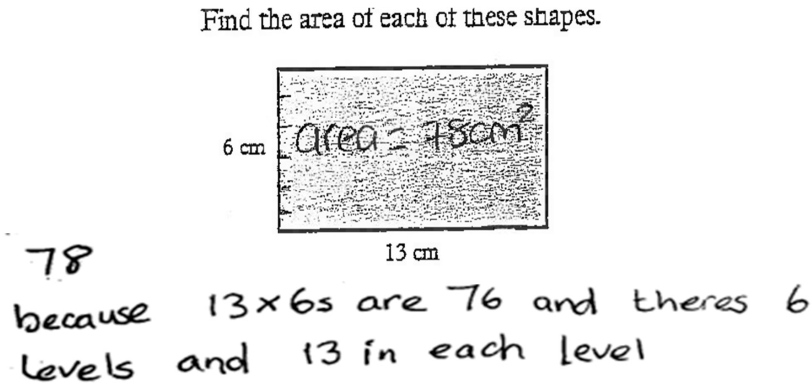

Figure 9. Partitioning the height dimension only and reasoning why multiplication will find the total number of squares: student solution strategy from the Year 9 post-test.

When I devised the drawing in squares approach, I knew that students needed to connect the dimensions of an 'empty' rectangle with the unit squares required to cover the space inside. However, I did not have knowledge of the contexts that could generate strategies which would naturally enable students to engage with the space-covering aspect of area. The contexts in *Reallotment* develop awareness of unit square coverage in subtle ways, such as using the price of a standard tile as a mediating quantity, asking for a comparison of one land area with another, or presenting shapes as 'coloured in' on a squared background visible only around the edges so that the learner engages with ways of seeing and counting the squares inside the shape even though it appears as 'empty'. RME gave me knowledge of informal strategies and helped me to recognise their potential for learner development, supplying the tried and tested contexts that enabled these strategies to emerge rather than being 'delivered' to students.

RME also helped me to become aware of a vast number of informal strategies of which, in 15 years of teaching and as a learner of mathematics myself up to the age of 21, I was not consciously aware. I now saw that these informal models were a legitimate part of mathematics in their own right. Previously, if I had seen learners using long-winded strategies (such as drawing 120 dots and circling them in groups of 8 in order to work out 120 divided by 8), I would have shown them that there was a much quicker way to do it. Now, I recognised the importance of their approach and the need to consider what problems to use in order to shift their thinking slightly, rather than destroy their confidence by offering a formal method that made no sense to them. I could make a long list of informal strategies, including 'matching', 'reallotting', 'comparing', 'creating', and 'using bar models and ratio tables'. These strategies and my awareness of the need to work with what learners bring to the classroom were to become fundamental parts of my work with trainee teachers and professional development programmes over the next 20 years.

5.5. Using My Learning in Teacher Education

My work as a teacher trialling RME in 2003–2004 led to my recruitment as a trainer for other teachers joining the project in the following three years. I was strongly influenced by the key events from my own personal journey outlined above in both designing and leading professional development. I knew from the area project the value of looking closely at student work, even their crossings-out. I knew from using RME in my own classroom how important it was for the teacher to experience the materials as a learner and feel the uncertainty of not knowing what answer or method is required. I was aware from working with whole-class interactive teaching strategies the importance of students sharing their solutions at the board, with the teacher responding in neutral ways, acting as a facilitator, and gradually shifting the authority over the mathematics to their students. Working alongside other teacher trainers, I drew on my experience to design RME-focussed professional development tasks, along with a belief that as trainers we too needed to take on the role of facilitator. We provided teachers with tasks designed to help them notice the difference between their students' responses to traditional teacher-led mathematics tasks and how creative their students could be when tackling an RME problem for themselves freehand. We modelled how RME materials could be used in the classroom, asking teachers what they noticed and inviting them to try. In teacher reflection sessions, we listened, remained neutral without judgement, invited participants to respond to other participant concerns, and encouraged them to negotiate solutions as a group. We tried to provide professional development that replicated RME classrooms, whereby instead of telling our learners what they should see or think or do, we set the task and made space for sharing ideas and reflection.

Table 1 lists some of the core aims of our professional development, which emerged over many years of working with teachers on RME-based projects, and provides examples of activities supporting our aims.

Table 1. Our professional development aims and corresponding activities.

Aim	Activity
To develop teacher awareness and knowledge of students' natural informal approaches	Teachers study student solutions to mathematics problems, analysing approaches, looking for connections across methods, linking student answers to how they are taught, ranking solutions from informal to formal
To enable teachers to experience RME both as a teacher and as a learner of mathematics	Trainers model RME lessons in which teachers are positioned as students and trainers provide meta-commentary on their pedagogic decisions. Key strategies: bring learners to the board to showcase a range of their solutions; remain neutral; focus learners on solutions with directions such as 'Say what you see', 'Can you draw something?', 'What's the same, what's different?' [about these solutions].
To develop teachers' appreciation of how RME uses context and models to build a different view of progress	Focus on how a particular model, such as the bar, emerges from many contexts to become a model that learners can apply elsewhere across many topic areas, even to non-contextual, bare number questions
To provide models of teaching that shift the role of teacher from transmission orientation to that of a facilitator	Video observation emphasising noticing and accurate, non-evaluative description. Focus on, e.g., what neutral teacher responses look like and what teachers see as mathematical 'progression'

We knew from my own experience the importance of teachers having time to work with RME in their own classrooms and the freedom to fail. In our early professional development programmes, we set expectations that teachers would spend at least a year using RME materials with one of their classes for 90% of the curriculum time. This was not seen as an issue by the teachers or their schools at the time. In the early 2000s, practice-based classroom research and experimentation was welcomed and admired, bringing kudos to a mathematics department. The fact that this might mean a class had not covered exactly the same curriculum as other classes was considered inconsequential. Experienced teachers were well aware that 'covering' a topic in a traditional way did not translate to many students being able to successfully answer test questions on that topic, and there was plenty of curriculum time for students to encounter topics again and for teachers to identify and work on any gaps in students' knowledge prior to their final examinations. School senior leaders recognised the importance of innovative approaches to mathematics teaching and prioritised it. They sought to employ mathematics curriculum leaders who could deliver innovation and trusted the judgement of these subject experts. However, over the last 15 years, the shift of focus towards accountability and systems of monitoring, measuring, and judging schools and students has undoubtedly stifled the era of experimentation. We discuss this further in the final sections.

6. Discussion: Developing and Delivering Professional Development for Equity in the Current Education Climate

Sue's journey began in the 'progressive' context of the late 1980's when teachers could experiment with new pedagogic approaches, sanctioned by the influential Cockcroft Report [29], which advocated classroom discussion, investigation, and problem-solving in order to develop the full potential of every student. Her participation in research projects, her membership of a community of teachers, and her introduction to RME fed and fostered successive moves in her 'problems of practice' [6] over a number of years in the non-linear, situated, and incremental process of teacher learning described by Clarke and Hollingsworth (2002). As her narrative shows, she developed expertise in ways of teaching that built on students' informal understandings and led to a reconceptualisation of mathematics itself as doing rather than knowing. In terms of Horn and Garner's [6] three shifts, she developed an understanding of instructional design, a repertoire of strategies enabling her to decentre her role in the classroom, and a recognition of the value of every student's contribution. Together, these contributed to the development of pedagogical judgement which went far beyond the acquisition of 'best practice'.

For us as teacher educators, the challenge is now how to 'reinvent' Sue's journey for today. We see a major problem in doing so, which is the increasingly rigid approach to coverage in a curriculum that prioritises formal mathematics. Sue's story highlights the central role of RME in challenging this approach; it not only engages students but also enables them to access mathematics in its fullest sense. Despite Sue's development of a student-centred pedagogy and her awareness that informal approaches were somehow important, it was only her engagement with RME that joined these elements together in terms of a new mathematical understanding that escapes reliance on mere procedures. Using RME materials, and understanding their design, was crucial to making this shift. It is important to note that this learning takes time. In a recent randomised controlled trial of RME in England (2017–2021), we were able to ask teachers to spend at least a quarter of their lesson time working with RME materials. Although this was much less classroom time than in earlier projects, we saw teachers changing in similar ways to Sue's learning—they began to reconceptualise progress and their classroom cultures became more inclusive. But, as we note above, some teachers reported pressure from senior leadership about the coverage of their schemes of learning. This kind of pressure is increasingly evident in our current professional development activities, and it has led us to consider our pedagogical responsibilities towards the teachers we work with. We now know that, in many schools, asking teachers to commit to using RME materials over any length of time is unrealistic and arguably unethical because of the pressure it puts on teachers.

However, our commitment to challenging what we see as deeply inequitable mathematics education has made it difficult to move away from our previous professional development model of supporting teachers to engage with RME materials and pedagogy over a period of one or two years. Although we still model RME lessons in training sessions, we know that teachers may only be allowed to try out one RME lesson rather than a whole module. School managers are insistent that all classes in a year group are taught topics at the same time, with the perception that missing a lesson from the scheme of learning amounts to being behind. Going back to where Sue started, we now focus on providing teachers with ways of working in their classrooms that make small but important inroads into classroom culture, without necessarily having to change the materials they are already using. We have developed 'pedagogies to keep in your back pocket', such as remain neutral, think about where you stand in the room, ask questions about images such as 'Say what you see' and 'What's the same, what's different?', and sharing student solution strategies. Teachers quickly see the potential of such strategies for encouraging student contributions and gaining insight into their mathematical thinking, but without RME materials, the range of methods tends to be less varied or novel. To compensate, we have chosen elements of our materials and attached particular teaching strategies. For example, a photograph of the finish line of a running race with the caption 'Say what you see'. In professional development sessions, we emphasise that the teacher's role is not to explain or say what *they* can see but only to scribe verbatim at the board what students say. This device is helpful because it forces teachers into the role of listening carefully to their learners and of capturing exactly what they say. Teachers see how it can be applied to their own school curriculum, so that traditional questions containing a graph, table, or shape can be stripped down to the image alone, along with the invitation to 'Say what you see'. This alleviates teacher tension about curriculum coverage. They appreciate that investing in 'Say what you see' for photographs of contexts and the resulting shifts in student access and classroom culture have the potential to impact test success.

Like Gore et al., we have worked on making space for teachers to reflect on their classroom experience, opening up 'spaces of freedom … to experience accountability more productively, still with a focus on student outcomes' ([16] p. 454). 'Gap tasks' in between professional development sessions ask teachers to focus on a single strategy that they can support with an RME lesson if they so choose. Sue's story highlights in particular the importance of looking closely at student work, and joint reflection on these tasks with other teachers is a major way of creating space for noticing more about their students' thinking and recognising for themselves the value of the informal mathematics that underlies the formulae. This is not pure RME, but how to address this within our pedagogic responsibilities is our current problem of practice.

7. Implications for Future Research on Professional Development

In this article, we have made space for Sue's narrative in order to highlight her learning in terms of a continual reframing of problems of practice, but it has also been important to situate the narrative in its political context over time. What is unique about Sue's story perhaps is its depth of pedagogical reasoning in challenging accepted versions of progress and the nature of mathematical thinking and learning. The role of these challenges in underpinning pedagogical actions for equity has become even more apparent to us in writing this article as we have thought through the implications for professional development in our marketised world. We suggest that future research should extend Gore et al.'s [16] work by exploring how teachers might develop equitable mathematics classrooms within the constraints of accountability, and the implications for sustainable professional development. We recognise the compromises we are currently forced to make. We're working on it.

Author Contributions: Conceptualization, S.H. and Y.S.; methodology, S.H. and Y.S.; formal analysis, S.H. and Y.S.; investigation, S.H.; data curation, S.H.; writing—original draft preparation, S.H. and Y.S.; writing—review and editing, S.H. and Y.S.; visualization, S.H. and Y.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Biesta, G. Good education in an age of measurement: On the need to reconnect with the question of purpose in education. *Educ. Assess. Eval. Acc.* **2009**, *21*, 33–46. [CrossRef]
2. Holloway, J.; Brass, J. Making accountable teachers: The terrors and pleasures of performativity. *J. Educ. Policy* **2018**, *33*, 361–382. [CrossRef]
3. OECD. Mathematics Performance (PISA) (Indicator). 2023. Available online: https://data.oecd.org/pisa/mathematics-performance-pisa.htm (accessed on 1 July 2023).
4. Frostenson, M.; Englund, H. Teachers, performative techniques and professional values: How performativity becomes humanistic through interplay mechanisms. *Camb. J. Educ.* **2020**, *50*, 695–710. [CrossRef]
5. DfE. *Key Stage 4 Performance, 2019 (Revised)*; Department for Education: London, UK, 2020. Available online: https://www.gov.uk/government/statistics/key-stage-4-performance-2019-revised (accessed on 1 July 2023).
6. Horn, I.; Garner, B. *Teacher Learning of Ambitious and Equitable Mathematics Instruction*; Routledge: New York, NY, USA, 2022.
7. Van den Heuvel-Panhuizen, M. The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educ. Stud. Math.* **2003**, *54*, 9–35. [CrossRef]
8. Gravemeijer, K.; Stephan, M. Emergent Models as an instructional design heuristic. In *Symbolizing, Modeling and Tool Use in Mathematics Education*; Gravemeijer, K., Lehrer, R., Van Oers, B., Verschaffel, L., Eds.; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2002; pp. 145–169.
9. Yackel, E.; Cobb, P. Sociomathematical norms, argumentation, and autonomy in mathematics. *J. Res. Math. Educ.* **1996**, *27*, 458–477. [CrossRef]
10. Stephan, M.; Underwood-Gregg, D.; Yackel, E. Guided reinvention: What is it and how do teachers learn this teaching approach? In *Transforming Mathematics Instruction*; Li, Y., Silver, E.A., Li, S., Eds.; Springer: Dordrecht, The Netherlands, 2014; pp. 37–57.
11. Solomon, Y.; Hough, S.; Gough, S. The role of appropriation in guided reinvention: Establishing and preserving devolved authority with low-attaining students. *Educ. Stud. Math.* **2021**, *106*, 171–188. [CrossRef]
12. Treffers, A. *Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction: The Wiskobas Project*; Reidel: Dordrecht, The Netherlands, 1987.
13. Streefland, L. Wiskunde als activiteit en de realiteit als bron [Mathematics as an activity and reality as a source]. *Nieuwe Wiskrant* **1985**, *5*, 60–67.
14. Appel, M. Performativity and the demise of the teaching profession: The need for rebalancing in Australia. *Asia-Pac. J. Teach. Educ.* **2020**, *48*, 301–315. [CrossRef]
15. Ball, S.J. The teacher's soul and the terrors of performativity. *J. Educ. Policy* **2003**, *18*, 215–228. [CrossRef]
16. Gore, J.; Rickards, B.; Fray, L. From performative to professional accountability: Re-imagining 'the field of judgment' through teacher professional development. *J. Educ. Policy* **2023**, *38*, 452–473. [CrossRef]
17. Wake, G.D.; Burkhardt, H. Understanding the European policy landscape and its impact on change in mathematics and science pedagogies. *ZDM* **2013**, *45*, 851–861. [CrossRef]
18. Boylan, B.; Adams, G. Market mirages and the state's role in professional learning: The case of English mathematics education. *J. Educ. Policy* **2023**, *21*, 33–46. [CrossRef]
19. Ellis, V.; Mansell, W.; Steadman, S. A new political economy of teacher development: England's Teaching and Leadership Innovation Fund. *J. Educ. Policy* **2021**, *36*, 605–623. [CrossRef]
20. Clarke, D.; Hollingsworth, H. Elaborating a model of teacher professional growth. *Teach. Teach. Educ.* **2002**, *18*, 947–967. [CrossRef]
21. Goldsmith, L.; Doerr, H.; Lewis, C. Mathematics teachers' learning: A conceptual framework and synthesis of research. *J. Math. Teach. Educ.* **2014**, *17*, 5–36. [CrossRef]
22. Louie, N.L. Culture and ideology in mathematics teacher noticing. *Educ. Stud. Math.* **2018**, *97*, 55–69. [CrossRef]
23. Horn, I.S. Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *J. Learn. Sci.* **2007**, *16*, 37–79.
24. Ryan-Atkin, H.; Rowley, H. When the MAT moves in: Implications for legitimacy in terms of governance and local agency. In *Inside the English Education Lab: Critical Qualitative and Ethnographic Perspectives on the Academies Experiment*; Kulz, C., Morrin, K., McGinity, R., Eds.; University of Manchester Press: Manchester, UK, 2022; pp. 60–85.
25. Dahle, D.Y. Brand on the run? Marketization, market position, and branding in upper secondary schools. In *Public Branding and Marketing*; Zavattaro, S.M., Ed.; Springer Nature Switzerland: Cham, Switzerland, 2021; pp. 175–195.
26. Faris, L. A primary school head teacher's experience of pressure to join a multi-academy trust. *Lond. Rev. Educ.* **2022**, *20*, 44. [CrossRef]
27. Male, T. The rise and rise of academy trusts: Continuing changes to the state-funded school system in England. *Sch. Leadersh. Manag.* **2022**, *42*, 313–333. [CrossRef]

28. Chen, G.A.; Marshall, S.A.; Horn, I.S. 'How do I choose?': Mathematics teachers' sensemaking about pedagogical responsibility. *Pedagog. Cult. Soc.* **2021**, *29*, 379–396. [CrossRef]
29. Cockcroft, W.H. *Mathematics Counts. Report of the Committee of Inquiry into the Teaching of Mathematics in Schools under the Chairmanship of Dr WH Cockcroft*; Her Majesty's Stationery Office: London, UK, 1982. Available online: https://www.stem.org.uk/elibrary/resource/29976 (accessed on 1 July 2023).
30. Jaworski, B. *Investigating Mathematics Teaching: A Constructivist Enquiry*; Routledge: London, UK, 1994.
31. Burghes, D. *Mathematics Enhancement Programme: The First 3 Years*; University of Exeter, Centre for Innovation in Mathematics Teaching: Exeter, UK, 2000. Available online: https://www.cimt.org.uk/projects/mep/intrep00.pdf (accessed on 1 July 2023).
32. Gattegno, C. *The Common Sense of Teaching Mathematics*; Educational Solutions Worldwide: Toronto, ON, Canada, 2010.
33. Foxman, D.; Ruddock, G.; Joffe, L.; Mason, K.; Mitchell, P.; Sexton, B. *A Review of Monitoring in Mathematics 1978 to 1982: Part 1 and Part 2*; Assessment of Performance Unit, Department of Education and Science (Now Located at the School Examinations and Assessment Council): London, UK, 1985.
34. Hart, K.; Brown, M.; Kerslake, D.; Kuchemann, D.; Ruddock, G. *Chelsea Diagnostic Mathematics Tests Teacher's Guide*; NFER-Nelson: Slough, UK, 1985.
35. Romberg, T.; Shafer, M. Implications and Conclusions (Mathematics in Context Monograph 8). 2005. Available online: http://micimpact.wceruw.org/working_papers/Monograph%208%20Final.pdf (accessed on 1 July 2023).
36. Gravemeijer, K.; Pligge, M.A.; Clarke, B. Reallotment. In *Mathematics in Context*; Wisconsin Center for Education Research & Freudenthal Institute, Ed.; Encyclopædia Britannica, Inc.: Chicago, IL, USA, 2003.
37. Hughes, E.R.; Rogers, J.; Hughes, E.R. *Conceptual Powers of Children: An Approach through Mathematics and Science: A Report from the Schools Council Project on the Development of Scientific and Mathematical Concepts (7–11)*; Macmillan: Basingstoke, UK, 1979.
38. Johnston-Wilder, S.; Mason, J. (Eds.) *Developing Thinking in Geometry*; Sage: London, UK, 2005.