A Multidimensional Fatou Lemma for Conditional Expectations

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Abstract: The classical multidimensional version of Fatou's lemma (Schmeidler [20]) originally obtained for unconditional expectations and the standard non-negative cone in a finite-dimensional linear space is extended to conditional expectations and general closed pointed cones.

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1. Fatou's lemma in several dimensions, the first version of which was obtained by Schmeidler [20], is a powerful measure-theoretic tool initially developed in Mathematical Economics in connection with models of "large" economies with atomless measure spaces of agents; see Aumann [3] and Hildenbrand [16]. In this note we provide two new versions of this lemma: one for unconditional and the other for conditional expectations. Both deal with cones in an n-dimensional linear space $\mathbb{R}^n$ more general than the non-negative orthant $\mathbb{R}^n_+$ as considered in [20]. Our results are motivated by the applications of the theory of von Neumann-Gale [21, 14] dynamical systems to the modeling of financial markets with frictions—transaction costs and portfolio constraints [9, 11, 13, 6, 5].

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and C a pointed closed cone[1] in $\mathbb{R}^n$. We write $a \leq_C b$ if $b - a \in C$. Let $|\cdot|$ be a norm in $\mathbb{R}^n$. The distance measured in terms of the norm $|\cdot|$ between a point a and a set A in $\mathbb{R}^n$ is denoted by $\rho(a, A)$. We will use the standard notation $\mathrm{Ls}(x_k)$ for the set of limit points of the sequence (x_k).

Recall that a sequence of random variables $\beta_k(\omega)$ is called *uniformly integrable* if

$$\lim_{H\to\infty} \sup_k E|\beta_k| \mathbf{1}_{\{|\beta_k|\geq H\}} = 0. \tag{1}$$

Property (1) holds if and only if the following two conditions are satisfied: a) $\sup E|\beta_k| < \infty$; b) $\lim E|\beta_k|\mathbf{1}_{\Gamma_k} = 0$ for any sequence of events Γ_k with $P(\Gamma_k) \to 0$ (see, e.g., Neveu [18]).

Theorem 1. *Let $x_k(\omega)$, $k = 1, 2, ...,$ be a sequence of random vectors in $\mathbb{R}^n$ such that $E|x_k(\omega)| < \infty$ and $Ex_k(\omega) \to y$, where y is some vector in $\mathbb{R}^n$. If the sequence $\rho(x_k(\omega), C)$ is uniformly integrable, then there exist integer-valued random variables*

$$1 < k_1(\omega) < k_2(\omega) < ... \tag{2}$$

and a random vector $x(\omega)$ such that $E|x| < \infty$,

$$\lim_{m\to\infty} x_{k_m(\omega)}(\omega) = x(\omega) \text{ (a.s.)} \tag{3}$$

and

$$Ex(\omega) \leq_C y.$$

Theorem 1 is a version of the multidimensional Fatou lemma in [20] where it is assumed that $C = \mathbb{R}^n_+$ and $x_k(\omega) \in C$, so that $\rho(x_k(\omega), C) = 0$.

[1] A set C in a linear space is called a *cone* if it contains with any its elements x, y any non-negative linear combination $\lambda x + \mu y$ $(\lambda, \mu \geq 0)$ of these elements. The cone C is called *pointed* if the inclusions $x \in C$ and $-x \in C$ imply $x = 0$.

It also extends a result in the paper by Cornet et al. [8], Theorem B, p. 194, in which the function $\rho(x_k(\omega), C)$ is required to be integrably bounded.

3. *Proof of Theorem 1. 1st step.* We have $x_k(\omega) = c_k(\omega) + b_k(\omega)$, where $c_k(\omega) \in C$ and $|b_k(\omega)| = \rho(x_k(\omega), C)$ (a.s.). By assumption, the sequence $|b_k|$ is uniformly integrable, and so $H := \sup E\,|b_k| < \infty$. Since $c_k = x_k - b_k$, we have $E|c_k| \le E|x_k| + H < \infty$, and so the random vectors c_k are integrable. Furthermore, the sequence $Ec_k = Ex_k - Eb_k$ is bounded because $\sup|Eb_k| \le \sup E|b_k| = H$ and the sequence Ex_k is bounded since it converges.

Note that the boundedness of Ec_k implies the boundedness of $E|c_k|$ because the random vectors $c_k(\omega)$ take on their values in the closed pointed cone C. Indeed, consider a strictly positive linear functional g on C ($gc > 0$, $0 \ne c \in C$); it exists for each closed pointed cone. For such a functional g, there exists $\gamma > 0$ such that $gc \ge \gamma|c|$ for all $c \in C$. Consequently, $gEc_k = Egc_k \ge \gamma E|c_k|$, which proves the boundedness of $E|c_k|$. This, in turn, implies that $E|x_k|$ is bounded because $\sup E|x_k| \le \sup(E|c_k| + E|b_k|) \le \sup E|c_k| + H < \infty$.

2nd step. Since $\sup E|x_k| < \infty$, we can use the "biting lemma" (e.g. Saadoune and Valadier [19], p. 349) and find a subsequence (x_{k_l}) of (x_k) and measurable sets $\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots$ such that $\bigcap_l \Gamma_l = \emptyset$ and the sequence $x'_{k_l} := x_{k_l}\mathbf{1}_{\Omega\setminus\Gamma_l}$ is uniformly integrable. Put $x''_{k_l} := x_{k_l}\mathbf{1}_{\Gamma_l}$. Clearly $x_{k_l} = x'_{k_l} + x''_{k_l}$.

For each ω the sequence $x'_{k_l}(\omega)$ coincides with $x_{k_l}(\omega)$ from some $l(\omega)$ on because every ω belongs to $\Omega\setminus\Gamma_l$ from some $l(\omega)$ on. Consequently, for all ω we have $\mathrm{Ls}(x'_{k_l}) = \mathrm{Ls}(x_{k_l}) \subseteq \mathrm{Ls}(x_k)$.

The sequences $E|x'_{k_l}|$ and $E|x''_{k_l}|$ are bounded because $|x'_{k_l}| \le |x_{k_l}|$ and $|x''_{k_l}| \le |x_{k_l}|$. By passing to a subsequence, we can assume without loss of generality that $Ex'_{k_l} \to y'$ and $Ex''_{k_l} \to y''$ for some $y', y'' \in \mathbb{R}^n$. Clearly, $y' + y'' = y$ because $Ex'_{k_l} + Ex''_{k_l} = Ex_{k_l} \to y$.

3rd step. Since the sequence (x'_{k_l}) is uniformly integrable and $Ex'_{k_l} \to y'$, by Artstein's theorem [2], Theorem A, there exists an integrable random vector $x(\omega)$ such that $x(\omega) \in \mathrm{Ls}(x'_{k_l}(\omega)) \subseteq \mathrm{Ls}(x_k(\omega))$ (a.s.) and $Ex(\omega) = y'$. We have

$$Ex''_{k_l} = Ex_{k_l}\mathbf{1}_{\Gamma_l} = Ec_{k_l}\mathbf{1}_{\Gamma_l} + Eb_{k_l}\mathbf{1}_{\Gamma_l} \to y'',$$

where $Eb_{k_l}\mathbf{1}_{\Gamma_l} \to 0$ because $P(\Gamma_l) \to 0$ and the sequence b_{k_l} is uniformly integrable. Thus $Ec_{k_l}\mathbf{1}_{\Gamma_l} \to y''$. We have $c_{k_l}(\omega)\mathbf{1}_{\Gamma_l}(\omega) \in C$ because $c_{k_l}(\omega) \in C$ and $0 \in C$. Consequently, $Ec_{k_l}\mathbf{1}_{\Gamma_l} \in C$ as the set C is convex (see, e.g., [1], Appendix II, Lemma 1). Therefore $y'' \in C$ since $Ec_{k_l}\mathbf{1}_{\Gamma_l} \to y''$ and C is closed.

4th step. We obtained that $y - y' = y'' \in C$, i.e., $y' \leq_C y$. Furthermore, $Ex(\omega) = y'$, so that $Ex(\omega) \leq_C y$, where $x(\omega) \in$Ls$(x_k(\omega))$ (a.s.). It remains to observe that the inclusion $x(\omega) \in$Ls$(x_k(\omega))$ (a.s.) implies the existence of a sequence $(k_m(\omega))_{m=1}^{\infty}$ of integer-valued random variables such that (3) holds. Indeed, since $x(\omega) \in$Ls$(x_k(\omega))$ (a.s.), for almost all ω there exists a sequence $\kappa = (k_m)_{k=1}^{\infty}$ of natural numbers k_m for which

$$1 < k_1 < k_2 < ... \text{ and } \lim x_{k_m} = x(\omega). \tag{4}$$

Denote by A the set of (ω, κ) satisfying (4). This set is measurable with respect to $\mathcal{F} \times \mathcal{B}(\mathbb{N}^{\infty})$, where $\mathbb{N}^{\infty} := \mathbb{N} \times \mathbb{N} \times ...$ is the product of a countable number of copies of the discrete space $\mathbb{N} := \{1, 2, ...\}$ and $\mathcal{B}(\cdot)$ stands for the Borel σ-algebra. Since $(\mathbb{N}^{\infty}, \mathcal{B}(\mathbb{N}^{\infty}))$ is a standard measurable space[2], we can apply Aumann's measurable selection theorem (see e.g. [1], Appendix I, Corollary 3) and construct a measurable mapping $\kappa(\omega)$ of Ω into $\mathbb{N}^{\infty}$ for which $(\omega, \nu(\omega)) \in A$ for almost all ω. The sequence $\kappa(\omega) = (k_m(\omega))_{m=1}^{\infty}$ of measurable integer-valued random variables satisfies (4) and (3). $\square$

4. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let $C(\omega)$ be a pointed closed convex cone in $\mathbb{R}^n$ depending $\mathcal{G}$-measurably[3] on ω. A random vector $x(\omega)$ is said to be *conditionally integrable* (with respect to the σ-algebra $\mathcal{G}$) if $E[|x(\omega)|\,|\mathcal{G}] < \infty$ (a.s.). A sequence of random variables $\beta_k(\omega)$, $k = 1, 2, ...$, is said to be *uniformly conditionally integrable* if

$$\lim_{H \to \infty} \sup_k E[|\beta_k| \mathbf{1}_{\{|\beta_k| \geq H\}} |\mathcal{G}] = 0 \text{ (a.s.)}. \tag{5}$$

The following result is a version of Theorem 1 for conditional expectations.

Theorem 2. *Let $x_k(\omega)$, $k = 1, 2, ...$, be conditionally integrable random vectors in $\mathbb{R}^n$ and $y(\omega)$ a random vector in $\mathbb{R}^n$ such that*

$$E[x_k(\omega)|\mathcal{G}] \to y(\omega) \text{ (a.s.)}. \tag{6}$$

If the sequence $\rho(x_k(\omega), C(\omega))$ is uniformly conditionally integrable, then there exists a sequence of integer-valued random variables $1 < k_1(\omega) < k_2(\omega) < ...$ and a conditionally integrable random vector $x(\omega)$ such that

$$\lim_{m \to \infty} x_{k_m(\omega)}(\omega) = x(\omega) \text{ (a.s.)}$$

[2] A measurable space is called *standard* if it is isomorphic to a Borel subset of a complete separable metric space with the Borel measurable structure.

[3] A set $C(\omega) \subseteq \mathbb{R}^n$ is said to depend $\mathcal{G}$-measurably on ω if its graph $\{(\omega, c) : c \in C(\omega)\}$ belongs to $\mathcal{G} \times \mathcal{B}(\mathbb{R}^n)$.

and

$$E[x(\omega)|\mathcal{G}] \leq_{C(\omega)} y(\omega) \text{ (a.s.).}$$

In the case when $C(\omega) = \mathbb{R}^n_+$ Theorem 2 was proved in [7], Appendix A, Proposition A.2. For reviews of various results related to multidimensional Fatou lemmas, see Balder and Hess [4] and Hess [15].

Some comments regarding the assumptions of Theorem 2 are in order. Clearly a sequence of random variables $\beta_k(\omega)$ is uniformly conditionally integrable if it is *conditionally integrably bounded,* i.e. $|\beta_k(\omega)| \leq \alpha(\omega)$, where $E[|\alpha(\omega)||\mathcal{G}] < \infty$ (a.s.). The last condition holds, in particular, if $\beta_k(\omega)$ is (unconditionally) integrably bounded: $|\beta_k(\omega)| \leq \alpha(\omega)$ (a.s.) where $E|\alpha(\omega)| < \infty$. It should be noted that uniform integrability does not necessarily imply uniform conditional integrability.

5. *Proof of Theorem 2. 1st step.* Let us regard the sequence of random vectors $x^\infty(\omega) := (x_1(\omega), x_2(\omega), ...)$ as a random element of the standard measurable space $(X^\infty, \mathcal{B}^\infty) := (X, \mathcal{B}) \times (X, \mathcal{B}) \times ...$, where $X := \mathbb{R}^n$ and $\mathcal{B} = \mathcal{B}(X)$ is the Borel σ-algebra on X. Let $\pi(\omega, dx^\infty)$ be the regular conditional distribution of $x^\infty(\omega)$ given the σ-algebra $\mathcal{G}$ (see, e.g., [1], Appendix I, Theorem 1). Denote by x_k^∞ the kth element of the sequence $x^\infty = (x_1, x_2, ...)$ regarded as a function of x^∞. By virtue of (6) and in view of the uniform conditional integrability of $\beta_k(\omega) := \rho(x_k(\omega), C(\omega))$, we have

$$\int \pi(\omega, dx^\infty) x_k^\infty = E[x_k|\mathcal{G}](\omega) \to y(\omega) \quad [x_k = x_k(\omega)], \tag{7}$$

$$\begin{aligned} &\lim_{H\to\infty} \sup_k \int \pi(\omega, dx^\infty) \rho(x_k^\infty, C(\omega)) \mathbf{1}_{\{\rho(x_k^\infty, C(\omega)) \geq H\}} \\ &\qquad = \lim_{H\to\infty} \sup_k E[\beta_k \mathbf{1}_{\{\beta_k \geq H\}}|\mathcal{G}](\omega) = 0 \end{aligned} \tag{8}$$

for all ω belonging to some $\mathcal{G}$-measurable set $\Omega_1 \subseteq \Omega$ of measure one. It follows from (7) and (8) that the assumptions of Theorem 1 are satisfied, and so for each $\omega \in \Omega_1$ there exists a $\mathcal{B}^\infty$-measurable vector function $w^\omega(x^\infty)$ integrable with respect to $\pi(\omega, \cdot)$ and such that

$$\int \pi(\omega, dx^\infty) w^\omega(x^\infty) \leq_{C(\omega)} y(\omega)$$

and

$$w^\omega(x^\infty) \in \mathrm{Ls}\,(x^\infty) \text{ for } \pi(\omega, \cdot)\text{-almost all } x^\infty$$

where $\mathrm{Ls}\,(x^\infty)$ is the set of the limit points of the sequence $x^\infty = (x_1, x_2, ...)$.

2nd step. We will use the following fact. There exists a function $\psi : [0,1] \times X^\infty \to \mathbb{R}^n$ jointly measurable with respect to $\mathcal{B}[0,1] \times \mathcal{B}^\infty$ (where $\mathcal{B}[0,1]$ is the Borel σ-algebra on $[0,1]$) and possessing the following property. For each finite measure μ on $\mathcal{B}^\infty$ and each $\mathcal{B}^\infty$-measurable function $f : X^\infty \to \mathbb{R}^n$, there exists $r \in [0,1]$ such that $\psi(r, x^\infty) = f(x^\infty)$ for μ-almost all $x^\infty \in X^\infty$. This result establishes the existence of a "universal" jointly measurable function parametrizing all equivalence classes of measurable functions $X^\infty \to \mathbb{R}^n$ with respect to all finite measures: any such class contains a representative of the form $\psi(r, \cdot)$, where r is some number in $[0,1]$. The result (extending Natanson [17], Chapter 15, Section 3, Theorem 4) follows from Theorem AI.3 in [12] using the fact that all uncountable standard measurable spaces are isomorphic to the segment $[0,1]$ with the Borel σ-algebra (see e.g. Dynkin and Yushkevich [10], Appendix 2).

3rd step. For each $\omega \in \Omega$, consider the set $U(\omega)$ of those $r \in [0,1]$ for which the function $\psi(r, \cdot)$ satisfies

$$\int \pi(\omega, dx^\infty)\psi(r, x^\infty) \leq_{C(\omega)} y(\omega),$$

$$\psi(r, x^\infty) \in \mathrm{Ls}\,(x^\infty) \text{ for } \pi(\omega, \cdot)\text{-almost all } x^\infty. \tag{9}$$

Observe that for $\omega \in \Omega_1$ the set $U(\omega)$ is not empty because it contains an element $r \in [0,1]$ such that $\psi(r, x^\infty) = w^\omega(x^\infty)$ for $\pi(\omega, \cdot)$-almost all x^∞. Further, the set of pairs (ω, r) satisfying $r \in U(\omega)$ is $\mathcal{G} \times \mathcal{B}[0,1]$-measurable because $\pi(\omega, dx^\infty)$ is a conditional distribution given $\mathcal{G}$, the function $\psi(r, x^\infty)$ is $\mathcal{B}[0,1] \times \mathcal{B}^\infty$-measurable, $C(\omega)$ and $y(\omega)$ are $\mathcal{G}$-measurable, and the constraint in (9) can be written as

$$\int \pi(\omega, dx^\infty)F(r, x^\infty) = 1, \tag{10}$$

where $F(r, x^\infty)$ is the indicator function of the set

$$\{(r, x^\infty) : \psi(r, x^\infty) \in \mathrm{Ls}\,(x^\infty)\} \in \mathcal{B}[0,1] \times \mathcal{B}^\infty.$$

The last inclusion follows from the fact that $z \in \mathrm{Ls}\,(x^\infty)$ if and only if for each $M = 1, 2, ...$ and $l = 1, 2, ...$ there exists $k \geq l$ such that $|z - x_k^\infty| < 1/M$.

4th step. By virtue of Aumann's measurable selection theorem (see above), there exists a $\mathcal{G}$-measurable function $r(\omega)$ such that $r(\omega) \in U(\omega)$ (a.s.). Define

$$x(\omega) := \psi(r(\omega), x^\infty(\omega)) \quad [x^\infty(\omega) = (x_1(\omega), x_2(\omega), ...)].$$

Since $\pi(\omega, dx^\infty)$ is the conditional distribution of $x^\infty(\omega)$ given $\mathcal{G}$ and $r(\omega)$ is $\mathcal{G}$-measurable, we have

$$E[x(\omega)|\mathcal{G}] = E[\psi(r(\omega), x^\infty(\omega))|\mathcal{G}] = \int \pi(\omega, dx^\infty)\psi(r(\omega), x^\infty) \leq y(\omega) \text{ (a.s.)}.$$

Furthermore, $x(\omega) \in \text{Ls}\,(x^\infty(\omega))$ (a.s.) because this inclusion is equivalent to the equality $F(r(\omega), x^\infty(\omega)) = 1$ (a.s.) and

$$EF(r(\omega), x^\infty(\omega)) = E\{E[F(r(\omega), x^\infty(\omega))|\mathcal{G}]\} =$$

$$E\int \pi(\omega, dx^\infty)F(r(\omega), x^\infty) = 1$$

by virtue of (9) and (10). Since $x(\omega) \in \text{Ls}\,(x^\infty(\omega))$ (a.s.), by Aumann's theorem, there exist integer-valued random variables $1 < k_1(\omega) < k_2(\omega) < ...$ such that $\lim x_{k_m(\omega)}(\omega) = x(\omega)$ a.s.; this was shown at the end of the proof of Theorem 1. $\square$

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