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An Unpublished Manuscript of Leibniz's on Duodecimal

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IN THE FIELD OF MATHEMATICS, the German polymath Gottfried Wilhelm Leibniz (1646–1716)[1] is best known for his independent inventions of the calculus and binary arithmetic. Less well known is that Leibniz also invented the base-16 number system, which he called "sedecimal" (see Strickland and Jones 2022). During his exploration of bases 2 and 16, he had occasion to work a little on other number bases, including the duodecimal. While most of his references to base 12 occur in writings that deal with other topics as well, there does exist one manuscript exclusively devoted to it. An English translation of that manuscript is given at the end of this article, the first time the manuscript has been published in any language. By way of an introduction, and to provide some context, I shall begin with an outline of Leibniz's other engagements with duodecimal, many of which are also unpublished. In contrast to his writings on the binary system, which span hundreds of manuscript pages, Leibniz's treatment of duodecimal is occasional, scattered, and unsystematic. Yet from those it is clear that Leibniz had a sound understanding of duodecimal, and as we shall see, duodecimal may even have had a role in his invention of binary.

The earliest mention of duodecimal in Leibniz's writings occurs in a preface he wrote in 1670 to an edition of the writings of the Italian humanist Marius Nizolius (1498–1576). There he considers the view of those who would have it that truth depends upon the definitions of terms, and definitions of terms in turn upon the human mind (in other words, that truth is arbitrary). To this Leibniz (1969, 128) responds: "In arithmetic, and in other disciplines as well, truths remain the same even if notations are changed, and it does not matter whether a decimal or a duodecimal number system is used." Unfortunately, Leibniz does not reveal where he had learned about duodecimal, though his use of it suggests he thought it was sufficiently well known that educated readers would be able to understand his argument.

Between 1672 and 1676 Leibniz lived in Paris, studying under the tutelage of some of the foremost mathematicians of the day, most notably Christiaan Huygens (1629–1695). As a result of his intense studies he devised the calculus in 1675, and binary arithmetic a few years later. The first references to duodecimal in his mathematical writings date from around this time. For example, in a manuscript entitled "Thesaurus mathematicus" [Mathematical Thesaurus], probably written in either 1678 or 1679, Leibniz works through various topics in arithmetic, geometry, and mechanics; near the end, he outlines how positional notation works in the decimal and then the duodecimal number system:

> From this outline it is clear that only these ten digits are needed: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Those in the first position signify the equivalent number of 1s, namely no 1s, one 1, two 1s, three, four, five, six, seven, eight, nine 1s. Those in the second position signify the equivalent number of *10*s, that is, 1s taken ten times; in the third position, the equivalent number of *100*s, that is, *10*s taken ten times,

or the squares of 10; in the fourth position, the equivalent number of *1000*s, that is, *100*s taken ten times, or the cubes of 10, and so on. And in place of 10 one would be able to put any other number, for example, 12. For just as when the base a is 10, the square a^2 signifies 100 and the cube a^3 signifies 1000, so when a is 12, a^2 will be 12 times 12, that is, 144, and a^3 will be 12 times 144. But on this method, instead of the digits mentioned above—0, 1 etc. 9—two new digits would be needed in addition, one which would represent ten, the other which would represent eleven; but [the digits] 10 would signify twelve, and 100 would signify one hundred and forty four. And there are some who prefer to use this method of calculating over the common method, because 12 can be divided by 2, 3, 4, and 6; in addition, a calculation is completed with fewer digits. But the difference is not so great as to be worth abandoning the decimal progression. (LH 35, 1, 25 Bl. 3v)

Leibniz often repeated the claim that some people had preferred duodecimal to decimal, but it is not clear who he had in mind. Late in life he claimed that a proponent of duodecimal had been identified by the German mathematician Daniel Schwenter (1585–1636): "In the German *Deliciæ mathematicæ*, someone is reported to have given preference to the duodecimal progression, in which eleven digits will be needed, namely 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, δ, ε, where δ is 10 and ε is 11" (LBr 705 Bl. 93r). The reference is probably to Schwenter's *Deliciæ physico-mathematicæ* [*The Charms of Physico-Mathematics*] of 1636, which was posthumously revised and expanded by Georg Philipp Harsdörffer (1607–1658) in 1651 and again in 1653. However, as far as I have been able to tell, in none of those works is there any mention of the duodecimal system, let alone any report of anyone endorsing it.[2] Alternatively, when referring to proponents of duodecimal, Leibniz may have been thinking of Blaise Pascal (1623–1662), who had mentioned duodecimal as an alternative to decimal in an essay presented to the *Académie Parisienne* in 1654 and posthumously published eleven years later. In that essay, Pascal (1665, 42) promised to give a method to determine whether a given number is divisible by any other number, insisting that it would work "not just in our decimal system of numeration (which has been established not as a result of natural necessity, as the common man thinks, but as a result of human custom, and quite foolishly, to be sure), but in a system of numeration based on any progression whatsoever." To illustrate, he applied his method to the duodecimal system. Leibniz was certainly aware of Pascal's essay, and may have taken Pascal's work with duodecimal as an endorsement thereof. He certainly appears to have absorbed what Pascal had to say about duodecimal; in a manuscript written around 1678, Leibniz noted that if the duodecimal system were used, the arithmetic checking method known as casting out nines "could become the proof by casting out elevens" (LH 35, 4, 13 Bl. 21), apparently borrowing the observation from Pascal (1665, 47–48), who had made it some years before.

In the spring of 1680, Leibniz met the Dutch mathematician Johann Jakob Ferguson (1630–1706), and must have mentioned both binary and duodecimal to him, as in the scratch paper upon which both recorded their ideas, Leibniz (1976, 137) wrote out a table showing the values of the decimal numbers 0–8 in binary, and a set of duodecimal digits, with the two extra digits given as ~ and $. In August of the same year, Leibniz noted: "It is well known that all fractions can be expressed by an infinite sequence of integers of a certain progression, for example, the decimal, or even the duodecimal,

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[2]Schwenter (1636, 117–122) does, however, provide a lengthy list of ways that the numbers 2–30 are embedded or reflected in the physical and spiritual realms. With 12, for example, he notes that there were 12 tribes of Israel and 12 apostles, that there are 12 signs of the zodiac, that the year is divided into 12 months etc.

or the one I prefer overall, the binary" (LH 35, 13, 3 Bl. 33). The remainder of this manuscript is concerned with binary fractions, but given the confidence of his remark here, it is likely that he had already undertaken some investigation of duodecimal fractions. If he had, his work on that remains to be discovered.

If Leibniz himself is to be believed, duodecimal even played a role in his invention of binary. The story Leibniz would tell in the 1690s onwards was that he had hit upon binary as the simplest number system from conscious reflection upon the duodecimal and quaternary number systems. In 1697 he wrote:

> It is apparent that some considered the duodecimal to be more useful while others took pleasure in the Pythagorean tetractys. At some point it occurred to me to consider what would ultimately be revealed if we used the simplest of all [progressions], namely the dyadic or binary. (Strickland and Lewis 2022, 110)

(The "Pythagorean tetractys," by the way, is the quaternary—base 4—number system developed by Erhard Weigel (1673).) We should be cautious of at least some of what Leibniz claims here. Certainly, there is no evidence that he knew of Weigel's quaternary system before 1683, several years after he had invented binary, in which case he could not have been influenced by quaternary. But as we have seen, Leibniz did know about duodecimal at least as far back as 1670, and while the manuscript evidence he left behind does not enable us to verify his later claim that he found his way to binary via duodecimal, it does not enable us to rule it out either.

Although Leibniz wrote relatively little about duodecimal, he was clearly aware of its advantages over decimal, and occasionally indicated that, were the decimal system to be dislodged from common usage, it should be replaced either by the duodecimal or sedecimal. In 1694 or 1695 he wrote:

> I think that if anything were to be changed in practice, it would be to use the duodecimal or sedecimal instead of the decimal, for the larger the numbers used by a progression, the more convenient the calculation (Strickland and Lewis 2022, 85).

Leibniz later made the same claim in "Explanation of binary arithmetic," the only one of his many writings on binary published in his own lifetime. There, after outlining binary notation and arithmetic, Leibniz insisted that binary was not intended to replace decimal in everyday usage because the long strings of digits made it impractical, in which case, he said, it is better to stick with decimal because the numbers are not as long. He then stated: "And if we were accustomed to proceed by twelves or sixteens, there would be even more benefit" (Strickland and Lewis 2022, 196). Despite publicly acknowledging the advantages of duodecimal and sedecimal, Leibniz was no vocal advocate of wholesale reform, nor did he make much use of these number systems in all but a small handful of his extensive mathematical writings.

Let us turn, then, to Leibniz's unpublished manuscript on duodecimal. It begins by noting that, in the decimal system, the digital root (that is, the digit sum) of multiples of nine is always nine, e.g. $9 \times 3 = 27$, and $2 + 7 = 9$. Leibniz then generalizes this to any number base, or "progression," supposing that for any base n, the digital root of multiples of $n - 1$ is always $n - 1$. He illustrates this using the duodecimal (or "duodenary") system, showing that the digital root of any multiple of 11 is always 11, or rather, since he uses the Greek letters χ and ϕ for 10 and 11 respectively, the digital root for any multiple of ϕ is always ϕ. To secure the point, Leibniz draws a table of duodecimal numbers in which his "new notation" for duodecimal is shown alongside the "old meaning" (i.e. the decimal equivalents).

Unfortunately, Leibniz's motivation for writing the manuscript is unknown—he gives the impression of simply wanting to record an observation he had made, but does not reveal what inspired him to make the observation in the first place. When did he write the piece? The watermark of the manuscript is found in only two of his other writings, one thought to have been written in 1693, the other positively dated to June 1706, which suggests it was written sometime between those two dates. However, it was filed among Leibniz's mathematical papers of 1695, making it reasonable to think it was written around that time.[3] [4]

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[4] *EDITOR'S NOTE:* The following pages include a transcription of the original Latin of Leibniz's manuscript, an English translation on the facing page, and finally an image of the original manuscript itself. The Dozenal Society of America would like to thank Professor Strickland for graciously choosing *The Duodecimal Bulletin* for first publication of this intriguing historical document.

Leibniz: De Progressione Duodenaria (c. 1695)[5]

Novenarii proprietas est, quæ facit ut summa notarum in multiplis ejus novenarium rursus componat:

9 per	1	2	3	4	5	6	7	8	9	10
dat	9	18	27	36	45	54	63	72	81	90

Sed sciendum est hanc proprietatem oriri ex nostro notandi modo qui est arbitrarius; nempe ex eo quod progressione denaria utimur, et post novem redimus ad 1 adjecta 0. Potuissemus vero alia progressione quacunque uti et semper hæc futura esset proprietas numeri ultimi in assumta progressione. Exempli causa, si pro denaria progressione adhiberetur duodenaria, ut a quibusdam curiositatis causa factum est, numerus undenarius simili proprietate gauderet, quod ostendere opera pretium erit. Nempe si duodenaria progressio adhiberetur, numeri decem et undecim proprias acciperent notas, veluti χ pro denario et ϕ pro undenario si placet. Itaque numeri usque ad duodecies duodecim seu centium quadraginta quatuor, ita stabunt:

Notatio novem:	1	2	3	4	5	6	7	8	9	χ	ϕ	10
Significatio antique:	1	2	3	4	5	6	7	8	9	10	11	12
Notatio novem:	11	12	13	14	15	16	17	18	19	1χ	1ϕ	20
Significatio antique:	13	14	15	16	17	18	19	20	21	22	23	24
Notatio novem:	21	22	23	24	25	26	27	28	29	2χ	2ϕ	30
Significatio antique:	25	26	27	28	29	30	31	32	33	34	35	36
Notatio novem:	31	32	33	34	35	36	37	38	39	3χ	3ϕ	40
Significatio antique:	37	38	39	40	41	42	43	44	45	46	47	48
Notatio novem:	41	42	43	44	45	46	47	48	49	4χ	4ϕ	50
Significatio antique:	49	50	51	52	53	54	55	56	57	58	49	60
Notatio novem:	51	52	53	54	55	56	57	58	59	5χ	5ϕ	60
Significatio antique:	61	62	63	64	65	66	67	68	69	70	71	72
Notatio novem:	61	62	63	64	65	66	67	68	69	6χ	6ϕ	70
Significatio antique:	73	74	75	76	77	78	79	80	81	82	83	84
Notatio novem:	71	72	73	74	75	76	77	78	79	7χ	7ϕ	80
Significatio antique:	85	86	87	88	89	90	91	92	93	94	95	96
Notatio novem:	81	82	83	84	85	86	87	88	89	8χ	8ϕ	90
Significatio antique:	97	98	99	100	101	102	103	104	105	106	107	108
Notatio novem:	91	92	93	94	95	96	97	98	99	9χ	9ϕ	$\chi0$
Significatio antique:	109	110	111	112	113	114	115	116	117	118	119	120
Notatio novem:	$\chi1$	$\chi2$	$\chi3$	$\chi4$	$\chi5$	$\chi6$	$\chi7$	$\chi8$	$\chi9$	$\chi\chi$	$\chi\phi$	$\phi0$
Significatio antique:	121	122	123	124	125	126	127	128	129	130	131	132
Notatio novem:	$\phi1$	$\phi2$	$\phi3$	$\phi4$	$\phi5$	$\phi6$	$\phi7$	$\phi8$	$\phi9$	$\phi\chi$	$\phi\phi$	100
Significatio antique:	133	134	135	136	137	138	139	140	141	142	143	144

Hinc jam multipli ipsius undenarii:

id est, communa notatione:	11	22	33	44	55	66	77	88	99	110	121	132
nova notatione erunt:	ϕ	1χ	29	38	47	56	65	74	83	92	$\chi1$	$\phi0$

ubi etiam summa binarum notarum semper facit undecim.

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[5] LH 35, 12 1 Bl. 40r. The original Latin.

Leibniz: On the Duodenary Progression (c. 1695)[6]

It is a property of nines that the sum of the digits in its multiples makes nine again:

9 BY	1	2	3	4	5	6	7	8	9	10
GIVES	9	18	27	36	45	54	63	72	81	90

But it should be known that this property originates from our way of writing, which is arbitrary, namely, from the fact that we use the decimal progression and after nine we return to 1 by adding 0. But we could use any other progression and the aforementioned property would always be the property of the last digit in the progression adopted. For example, if the duodenary progression were used in place of the decimal, as has been done by some people for the sake of curiosity, the number eleven would enjoy a similar property, which it will be worthwhile to show. Of course, if the duodenary progression were to be used, the numbers ten and eleven would have their own digits, such as χ for ten and ϕ for eleven, if you like. Therefore the numbers up to twelve times twelve, that is, one hundred and forty-four, will be as follows:

NEW NOTATION:	1	2	3	4	5	6	7	8	9	χ	ϕ	10
OLD MEANING:	1	2	3	4	5	6	7	8	9	10	11	12
NEW NOTATION:	11	12	13	14	15	16	17	18	19	1χ	1ϕ	20
OLD MEANING:	13	14	15	16	17	18	19	20	21	22	23	24
NEW NOTATION:	21	22	23	24	25	26	27	28	29	2χ	2ϕ	30
OLD MEANING:	25	26	27	28	29	30	31	32	33	34	35	36
NEW NOTATION:	31	32	33	34	35	36	37	38	39	3χ	3ϕ	40
OLD MEANING:	37	38	39	40	41	42	43	44	45	46	47	48
NEW NOTATION:	41	42	43	44	45	46	47	48	49	4χ	4ϕ	50
OLD MEANING:	49	50	51	52	53	54	55	56	57	58	49	60
NEW NOTATION:	51	52	53	54	55	56	57	58	59	5χ	5ϕ	60
OLD MEANING:	61	62	63	64	65	66	67	68	69	70	71	72
NEW NOTATION:	61	62	63	64	65	66	67	68	69	6χ	6ϕ	70
OLD MEANING:	73	74	75	76	77	78	79	80	81	82	83	84
NEW NOTATION:	71	72	73	74	75	76	77	78	79	7χ	7ϕ	80
OLD MEANING:	85	86	87	88	89	90	91	92	93	94	95	96
NEW NOTATION:	81	82	83	84	85	86	87	88	89	8χ	8ϕ	90
OLD MEANING:	97	98	99	100	101	102	103	104	105	106	107	108
NEW NOTATION:	91	92	93	94	95	96	97	98	99	9χ	9ϕ	$\chi0$
OLD MEANING:	109	110	111	112	113	114	115	116	117	118	119	120
NEW NOTATION:	$\chi1$	$\chi2$	$\chi3$	$\chi4$	$\chi5$	$\chi6$	$\chi7$	$\chi8$	$\chi9$	$\chi\chi$	$\chi\phi$	$\phi0$
OLD MEANING:	121	122	123	124	125	126	127	128	129	130	131	132
NEW NOTATION:	$\phi1$	$\phi2$	$\phi3$	$\phi4$	$\phi5$	$\phi6$	$\phi7$	$\phi8$	$\phi9$	$\phi\chi$	$\phi\phi$	100
OLD MEANING:	133	134	135	136	137	138	139	140	141	142	143	144

Hence now the multiples of eleven:

that is, in the common notation:	11	22	33	44	55	66	77	88	99	110	121	132
will be in the new notation:	ϕ	1χ	29	38	47	56	65	74	83	92	$\chi1$	$\phi0$

where also the sum of the two digits always makes eleven.

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[6] LH 35, 12 1 Bl. 40r. Translated from the Latin.

Novenarij proprietas, qua facit ut summa Notarum
in multiplis ejus novenarium rursus componat

3 per 1 2 3 4 5 6 7 8 9 10
dat 9 18 27 36 45 54 63 72 81 90

Sed sciendum est hanc proprietatem oriri ex nostro notandi
modo qui est arbitrarius. nempe ex eo quod progressione denaria
utimur et post novem redimus ad 1, adiecto 0.
putuissemus alia progressione quacunq uti
et semper hac futura esset proprietas numeri ultimi in
assumpta progressione. Exempli a pro denaria progressione
ad liberetur Duodenaria, ut a quibusdam curiositatis factum
est, numerus undenarius simili proprietate gauderet,
quod ostendere opera pretium erit.

Nempe si duodenaria progressio adhiberetur, numeri decem et undecim
proprias acciperent notas duodecies duodecim seu centum quadraginta quatuor, ita
itaq numeri usq ad

Notatio nova / significatio antiqua
1 2 3 4 5 6 7 8 9 X 11 10
[1 2 3 4 5 6 7 8 9 10 11 12]

Notatio nova / significatio antiqua
11 12 13 14 15 16 17 18 19 1X 11 20
[13 14 15 16 17 18 19 20 21 22 23 24]

Notatio / significatio
21 22 23 24 25 26 27 28 29 2X 2Ɛ 30
31 32 33 34 35 36 37 38 39 3X 3Ɛ 40
41 42 43 44 45 46 47 48 49 4X 4Ɛ 50
51 52 53 54 55 56 57 58 59 5X 5Ɛ 60
61 62 63 64 65 66 67 68 69 6X 6Ɛ 70
71 72 73 74 75 76 77 78 79 7X 7Ɛ 80
81 82 83 84 85 86 87 88 89 8X 8Ɛ 90
91 92 93 94 95 96 97 98 99 9X 9Ɛ X0
X1 X2 X3 X4 X5 X6 X7 X8 X9 XX XƐ Ɛ0
Ɛ1 Ɛ2 Ɛ3 Ɛ4 Ɛ5 Ɛ6 Ɛ7 Ɛ8 Ɛ9 ƐX ƐƐ 100

Hinc jam multipli undenarij
id est communi notatione erunt 11 22 33 44 55 66 77 88 99 110 121
nova notatione erunt 9 1X 2Ɛ 38 47 56 65 74 83 92 X1
ubi etiam summa binarum notarum semper facit undecim