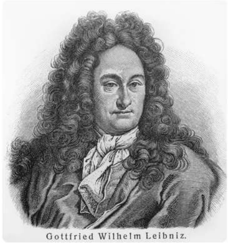
An Unpublished Manuscript of Leibniz's on Duodecimal



by Lloyd Strickland

Manchester Metropolitan University

l.strickland@mmu.ac.uk

In the field of mathematics, the German polymath Gottfried Wilhelm Leibniz (1646–1716)[1] is best known for his independent inventions of the calculus and binary arithmetic. Less well known is that Leibniz also invented the base-16 number system, which he called "sedecimal" (see Strickland and Jones 2022). During his exploration of bases 2 and 16, he had occasion to work a little on other number bases, including the duodecimal. While most of his references to base 12 occur in writings that deal with other topics as well, there does exist one manuscript exclusively devoted to it. An English translation of that manuscript is given at the end of this article, the first time the manuscript has been published in any language. By way of an introduction, and to provide some context, I shall begin with an outline of Leibniz's other engagements with duodecimal, many of which are also unpublished. In contrast to his writings on the binary system, which span hundreds of manuscript pages, Leibniz's treatment of duodecimal is occasional, scattered, and unsystematic. Yet from those it is clear that Leibniz had a sound understanding of duodecimal, and as we shall see, duodecimal may even have had a role in his invention of binary.

The earliest mention of duodecimal in Leibniz's writings occurs in a preface he wrote in 1670 to an edition of the writings of the Italian humanist Marius Nizolius (1498–1576). There he considers the view of those who would have it that truth depends upon the definitions of terms, and definitions of terms in turn upon the human mind (in other words, that truth is arbitrary). To this Leibniz (1969, 128) responds: "In arithmetic, and in other disciplines as well, truths remain the same even if notations are changed, and it does not matter whether a decimal or a duodecimal number system is used." Unfortunately, Leibniz does not reveal where he had learned about duodecimal, though his use of it suggests he thought it was sufficiently well known that educated readers would be able to understand his argument.

Between 1672 and 1676 Leibniz lived in Paris, studying under the tutelage of some of the foremost mathematicians of the day, most notably Christiaan Huygens (1629–1695). As a result of his intense studies he devised the calculus in 1675, and binary arithmetic a few years later. The first references to duodecimal in his mathematical writings date from around this time. For example, in a manuscript entitled "Thesaurus mathematicus" [Mathematical Thesaurus], probably written in either 1678 or 1679, Leibniz works through various topics in arithmetic, geometry, and mechanics; near the end, he outlines how positional notation works in the decimal and then the duodecimal number system:

> From this outline it is clear that only these ten digits are needed: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Those in the first position signify the equivalent number of 1s, namely no 1s, one 1, two 1s, three, four, five, six, seven, eight, nine 1s. Those in the second position signify the equivalent number of *10*s, that is, 1s taken ten times; in the third position, the equivalent number of *100*s, that is, *10*s taken ten times,

[1] *EDITOR'S NOTE:* All numerals in this article default to decimal [d], unless the text explicitly describes use of duodecimal by Leibniz or his contemporaries. Symbols shown for transdecimal digits are those used by the historical figures themselves.

> or the squares of 10; in the fourth position, the equivalent number of *1000*s, that is, *100*s taken ten times, or the cubes of 10, and so on. And in place of 10 one would be able to put any other number, for example, 12. For just as when the base a is 10, the square a^2 signifies 100 and the cube a^3 signifies 1000, so when a is 12, a^2 will be 12 times 12, that is, 144, and a^3 will be 12 times 144. But on this method, instead of the digits mentioned above—0, 1 etc. 9—two new digits would be needed in addition, one which would represent ten, the other which would represent eleven; but [the digits] 10 would signify twelve, and 100 would signify one hundred and forty four. And there are some who prefer to use this method of calculating over the common method, because 12 can be divided by 2, 3, 4, and 6; in addition, a calculation is completed with fewer digits. But the difference is not so great as to be worth abandoning the decimal progression. (LH 35, 1, 25 Bl. 3v)

Leibniz often repeated the claim that some people had preferred duodecimal to decimal, but it is not clear who he had in mind. Late in life he claimed that a proponent of duodecimal had been identified by the German mathematician Daniel Schwenter (1585–1636): "In the German *Deliciæ mathematicæ*, someone is reported to have given preference to the duodecimal progression, in which eleven digits will be needed, namely 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, δ, ε, where δ is 10 and ε is 11" (LBr 705 Bl. 93r). The reference is probably to Schwenter's *Deliciæ physico-mathematicæ* [*The Charms of Physico-Mathematics*] of 1636, which was posthumously revised and expanded by Georg Philipp Harsdörffer (1607–1658) in 1651 and again in 1653. However, as far as I have been able to tell, in none of those works is there any mention of the duodecimal system, let alone any report of anyone endorsing it.[2] Alternatively, when referring to proponents of duodecimal, Leibniz may have been thinking of Blaise Pascal (1623–1662), who had mentioned duodecimal as an alternative to decimal in an essay presented to the *Académie Parisienne* in 1654 and posthumously published eleven years later. In that essay, Pascal (1665, 42) promised to give a method to determine whether a given number is divisible by any other number, insisting that it would work "not just in our decimal system of numeration (which has been established not as a result of natural necessity, as the common man thinks, but as a result of human custom, and quite foolishly, to be sure), but in a system of numeration based on any progression whatsoever." To illustrate, he applied his method to the duodecimal system. Leibniz was certainly aware of Pascal's essay, and may have taken Pascal's work with duodecimal as an endorsement thereof. He certainly appears to have absorbed what Pascal had to say about duodecimal; in a manuscript written around 1678, Leibniz noted that if the duodecimal system were used, the arithmetic checking method known as casting out nines "could become the proof by casting out elevens" (LH 35, 4, 13 Bl. 21), apparently borrowing the observation from Pascal (1665, 47–48), who had made it some years before.

In the spring of 1680, Leibniz met the Dutch mathematician Johann Jakob Ferguson (1630–1706), and must have mentioned both binary and duodecimal to him, as in the scratch paper upon which both recorded their ideas, Leibniz (1976, 137) wrote out a table showing the values of the decimal numbers 0–8 in binary, and a set of duodecimal digits, with the two extra digits given as `~` and `$`. In August of the same year, Leibniz noted: "It is well known that all fractions can be expressed by an infinite sequence of integers of a certain progression, for example, the decimal, or even the duodecimal,

[2] Schwenter (1636, 117–122) does, however, provide a lengthy list of ways that the numbers 2–30 are embedded or reflected in the physical and spiritual realms. With 12, for example, he notes that there were 12 tribes of Israel and 12 apostles, that there are 12 signs of the zodiac, that the year is divided into 12 months etc.

or the one I prefer overall, the binary" (LH 35, 13, 3 Bl. 33). The remainder of this manuscript is concerned with binary fractions, but given the confidence of his remark here, it is likely that he had already undertaken some investigation of duodecimal fractions. If he had, his work on that remains to be discovered.

If Leibniz himself is to be believed, duodecimal even played a role in his invention of binary. The story Leibniz would tell in the 1690s onwards was that he had hit upon binary as the simplest number system from conscious reflection upon the duodecimal and quaternary number systems. In 1697 he wrote:

> It is apparent that some considered the duodecimal to be more useful while others took pleasure in the Pythagorean tetractys. At some point it occurred to me to consider what would ultimately be revealed if we used the simplest of all [progressions], namely the dyadic or binary. (Strickland and Lewis 2022, 110)

(The "Pythagorean tetractys," by the way, is the quaternary—base 4—number system developed by Erhard Weigel (1673).) We should be cautious of at least some of what Leibniz claims here. Certainly, there is no evidence that he knew of Weigel's quaternary system before 1683, several years after he had invented binary, in which case he could not have been influenced by quaternary. But as we have seen, Leibniz did know about duodecimal at least as far back as 1670, and while the manuscript evidence he left behind does not enable us to verify his later claim that he found his way to binary via duodecimal, it does not enable us to rule it out either.

Although Leibniz wrote relatively little about duodecimal, he was clearly aware of its advantages over decimal, and occasionally indicated that, were the decimal system to be dislodged from common usage, it should be replaced either by the duodecimal or sedecimal. In 1694 or 1695 he wrote:

> I think that if anything were to be changed in practice, it would be to use the duodecimal or sedecimal instead of the decimal, for the larger the numbers used by a progression, the more convenient the calculation (Strickland and Lewis 2022, 85).

Leibniz later made the same claim in "Explanation of binary arithmetic," the only one of his many writings on binary published in his own lifetime. There, after outlining binary notation and arithmetic, Leibniz insisted that binary was not intended to replace decimal in everyday usage because the long strings of digits made it impractical, in which case, he said, it is better to stick with decimal because the numbers are not as long. He then stated: "And if we were accustomed to proceed by twelves or sixteens, there would be even more benefit" (Strickland and Lewis 2022, 196). Despite publicly acknowledging the advantages of duodecimal and sedecimal, Leibniz was no vocal advocate of wholesale reform, nor did he make much use of these number systems in all but a small handful of his extensive mathematical writings.

Let us turn, then, to Leibniz's unpublished manuscript on duodecimal. It begins by noting that, in the decimal system, the digital root (that is, the digit sum) of multiples of nine is always nine, e.g. $9 \times 3 = 27$, and $2 + 7 = 9$. Leibniz then generalizes this to any number base, or "progression," supposing that for any base n, the digital root of multiples of $n - 1$ is always $n - 1$. He illustrates this using the duodecimal (or "duodenary") system, showing that the digital root of any multiple of 11 is always 11, or rather, since he uses the Greek letters χ and ϕ for 10 and 11 respectively, the digital root for any multiple of ϕ is always ϕ. To secure the point, Leibniz draws a table of duodecimal numbers in which his "new notation" for duodecimal is shown alongside the "old meaning" (i.e. the decimal equivalents).

Unfortunately, Leibniz's motivation for writing the manuscript is unknown—he gives the impression of simply wanting to record an observation he had made, but does not reveal what inspired him to make the observation in the first place. When did he write the piece? The watermark of the manuscript is found in only two of his other writings, one thought to have been written in 1693, the other positively dated to June 1706, which suggests it was written sometime between those two dates. However, it was filed among Leibniz's mathematical papers of 1695, making it reasonable to think it was written around that time.[3] [4]

References

Harsdörffer, Georg Philipp. 1653. *Delitiæ mathematicæ et physicæ der Mathematischen und Philosophischen Erquickstunden.* Dritter Theil. Nuremberg: Endters.

LBr. = Manuscript held by Gottfried Wilhelm Leibniz Bibliothek – Niedersächsische Landesbibliothek, Hanover, Germany. Cited by shelfmark and Blatt [sheet].

Leibniz, Gottfried Wilhelm. 1969. *Philosophical Papers and Letters*, ed. and trans. Leroy E. Loemker. Dordrecht: D. Reidel.

Leibniz, Gottfried Wilhelm. 1976. Sämtliche Schriften und Briefe. Dritte Reihe, Erster Band, ed. Akademie der Wissenschaften der DDR. Berlin: Akademie-Verlag.

LH = Manuscript held by Gottfried Wilhelm Leibniz Bibliothek – Niedersächsische Landesbibliothek, Hanover, Germany. Cited by shelfmark and Blatt [sheet].

Pascal, Blaise. 1665. *Traité du triangle arithmétique, avec quelques autres petits traitez sur la mesme materière.* Paris: Guillaume Desprez.

Schwenter, Daniel. 1636. *Deliciæ physico-mathematicæ oder Mathematische und philosophische Erquickstunden.* Nuremberg: Dümmler.

Strickland, Lloyd, and Jones, Owain Daniel. 2022. "F Things You (Probably) Didn't Know About Hexadecimal". *The Mathematical Intelligencer*. Available online at: https://link.springer.com/article/10.1007/s00283-022-10206-w.

Strickland, Lloyd, and Lewis, Harry. 2022. *Leibniz on Binary: The Invention of Computer Arithmetic.* Cambridge, Mass.: MIT Press.

Weigel, Erhard. 1673. *Tetractys, Summum tum Arithmeticæ tum Philosophiæ discursivæ compendium, artis magnæ sciendi.* Jena: Meyer.

[3] Work on this paper was graciously supported by the Gerda Henkel Stiftung (award AZ 46/V/21).

[4] *EDITOR'S NOTE:* The following pages include a transcription of the original Latin of Leibniz's manuscript, an English translation on the facing page, and finally an image of the original manuscript itself. The Dozenal Society of America would like to thank Professor Strickland for graciously choosing *The Duodecimal Bulletin* for first publication of this intriguing historical document.

Leibniz: De Progressione Duodenaria (c. 1695)[5]

Novenarii proprietas est, quæ facit ut summa notarum in multiplis ejus novenarium rursus componat:

9 per	1	2	3	4	5	6	7	8	9	10
dat	9	18	27	36	45	54	63	72	81	90

Sed sciendum est hanc proprietatem oriri ex nostro notandi modo qui est arbitrarius; nempe ex eo quod progressione denaria utimur, et post novem redimus ad 1 adjecta 0. Potuissemus vero alia progressione quacunque uti et semper hæc futura esset proprietas numeri ultimi in assumta progressione. Exempli causa, si pro denaria progressione adhiberetur duodenaria, ut a quibusdam curiositatis causa factum est, numerus undenarius simili proprietate gauderet, quod ostendere opera pretium erit. Nempe si duodenaria progressio adhiberetur, numeri decem et undecim proprias acciperent notas, veluti χ pro denario et ϕ pro undenario si placet. Itaque numeri usque ad duodecies duodecim seu centium quadraginta quatuor, ita stabunt:

Notatio novem:	1	2	3	4	5	6	7	8	9	χ	ϕ	10
Significatio antique:	1	2	3	4	5	6	7	8	9	10	11	12
Notatio novem:	11	12	13	14	15	16	17	18	19	1χ	1ϕ	20
Significatio antique:	13	14	15	16	17	18	19	20	21	22	23	24
Notatio novem:	21	22	23	24	25	26	27	28	29	2χ	2ϕ	30
Significatio antique:	25	26	27	28	29	30	31	32	33	34	35	36
Notatio novem:	31	32	33	34	35	36	37	38	39	3χ	3ϕ	40
Significatio antique:	37	38	39	40	41	42	43	44	45	46	47	48
Notatio novem:	41	42	43	44	45	46	47	48	49	4χ	4ϕ	50
Significatio antique:	49	50	51	52	53	54	55	56	57	58	49	60
Notatio novem:	51	52	53	54	55	56	57	58	59	5χ	5ϕ	60
Significatio antique:	61	62	63	64	65	66	67	68	69	70	71	72
Notatio novem:	61	62	63	64	65	66	67	68	69	6χ	6ϕ	70
Significatio antique:	73	74	75	76	77	78	79	80	81	82	83	84
Notatio novem:	71	72	73	74	75	76	77	78	79	7χ	7ϕ	80
Significatio antique:	85	86	87	88	89	90	91	92	93	94	95	96
Notatio novem:	81	82	83	84	85	86	87	88	89	8χ	8ϕ	90
Significatio antique:	97	98	99	100	101	102	103	104	105	106	107	108
Notatio novem:	91	92	93	94	95	96	97	98	99	9χ	9ϕ	$\chi 0$
Significatio antique:	109	110	111	112	113	114	115	116	117	118	119	120
Notatio novem:	$\chi 1$	$\chi 2$	$\chi 3$	$\chi 4$	$\chi 5$	$\chi 6$	$\chi 7$	$\chi 8$	$\chi 9$	$\chi\chi$	$\chi\phi$	$\phi 0$
Significatio antique:	121	122	123	124	125	126	127	128	129	130	131	132
Notatio novem:	$\phi 1$	$\phi 2$	$\phi 3$	$\phi 4$	$\phi 5$	$\phi 6$	$\phi 7$	$\phi 8$	$\phi 9$	$\phi\chi$	$\phi\phi$	100
Significatio antique:	133	134	135	136	137	138	139	140	141	142	143	144

Hinc jam multipli ipsius undenarii:

id est, communa notatione:	11	22	33	44	55	66	77	88	99	110	121	132
nova notatione erunt:	ϕ	1χ	29	38	47	56	65	74	83	92	$\chi 1$	$\phi 0$

ubi etiam summa binarum notarum semper facit undecim.

[5] LH 35, 12 1 Bl. 40r. The original Latin.

Leibniz: On the Duodenary Progression (c. 1695)[6]

It is a property of nines that the sum of the digits in its multiples makes nine again:

9 by	1	2	3	4	5	6	7	8	9	10
gives	9	18	27	36	45	54	63	72	81	90

But it should be known that this property originates from our way of writing, which is arbitrary, namely, from the fact that we use the decimal progression and after nine we return to 1 by adding 0. But we could use any other progression and the aforementioned property would always be the property of the last digit in the progression adopted. For example, if the duodenary progression were used in place of the decimal, as has been done by some people for the sake of curiosity, the number eleven would enjoy a similar property, which it will be worthwhile to show. Of course, if the duodenary progression were to be used, the numbers ten and eleven would have their own digits, such as χ for ten and ϕ for eleven, if you like. Therefore the numbers up to twelve times twelve, that is, one hundred and forty-four, will be as follows:

New notation:	1	2	3	4	5	6	7	8	9	χ	ϕ	10
Old meaning:	1	2	3	4	5	6	7	8	9	10	11	12
New notation:	11	12	13	14	15	16	17	18	19	1χ	1ϕ	20
Old meaning:	13	14	15	16	17	18	19	20	21	22	23	24
New notation:	21	22	23	24	25	26	27	28	29	2χ	2ϕ	30
Old meaning:	25	26	27	28	29	30	31	32	33	34	35	36
New notation:	31	32	33	34	35	36	37	38	39	3χ	3ϕ	40
Old meaning:	37	38	39	40	41	42	43	44	45	46	47	48
New notation:	41	42	43	44	45	46	47	48	49	4χ	4ϕ	50
Old meaning:	49	50	51	52	53	54	55	56	57	58	49	60
New notation:	51	52	53	54	55	56	57	58	59	5χ	5ϕ	60
Old meaning:	61	62	63	64	65	66	67	68	69	70	71	72
New notation:	61	62	63	64	65	66	67	68	69	6χ	6ϕ	70
Old meaning:	73	74	75	76	77	78	79	80	81	82	83	84
New notation:	71	72	73	74	75	76	77	78	79	7χ	7ϕ	80
Old meaning:	85	86	87	88	89	90	91	92	93	94	95	96
New notation:	81	82	83	84	85	86	87	88	89	8χ	8ϕ	90
Old meaning:	97	98	99	100	101	102	103	104	105	106	107	108
New notation:	91	92	93	94	95	96	97	98	99	9χ	9ϕ	$\chi0$
Old meaning:	109	110	111	112	113	114	115	116	117	118	119	120
New notation:	$\chi1$	$\chi2$	$\chi3$	$\chi4$	$\chi5$	$\chi6$	$\chi7$	$\chi8$	$\chi9$	$\chi\chi$	$\chi\phi$	$\phi0$
Old meaning:	121	122	123	124	125	126	127	128	129	130	131	132
New notation:	$\phi1$	$\phi2$	$\phi3$	$\phi4$	$\phi5$	$\phi6$	$\phi7$	$\phi8$	$\phi9$	$\phi\chi$	$\phi\phi$	100
Old meaning:	133	134	135	136	137	138	139	140	141	142	143	144

Hence now the multiples of eleven:

that is, in the common notation:	11	22	33	44	55	66	77	88	99	110	121	132
will be in the new notation:	ϕ	1χ	29	38	47	56	65	74	83	92	$\chi1$	$\phi0$

where also the sum of the two digits always makes eleven.

[6] LH 35, 12 1 Bl. 40r. Translated from the Latin.

Novenarii proprietas est, qua facit ut summa notarum in multiplis ~~[illegible]~~ ejus, novenarium rursus componat

	1	2	3	4	5	6	7	8	9	10
9 per / dat	9	18	27	36	45	54	63	72	81	90

Sed sciendum est hanc proprietatem oriri ex nostro notandi modo qui est arbitrarius, nempe ex eo quod progressione denaria utimur et post novem ~~[illegible]~~, redimus ad 1, adjecto 0. ~~sed~~ uti vero ~~revera quacunque~~ potuissemus alia progressione quacunque et semper haec futura esset proprietas numeri ultimi in ~~[illegible]~~ assumta progressione. Exempli causa si pro denaria progressione adhiberetur duodenaria, ut a quibusdam curiositatis causa factum est, numerus undenarius simili proprietate gauderet, quod ostendere operae pretium erit. Nempe si duodenaria progressio adhiberetur, numeri decem et undecim propias acciperent notas, velut X pro denario, et ꝗ pro undenario si placet, itaque numeri usque ad ~~[illegible]~~ duodecies duodecim seu centum quadraginta quatuor, ita stabunt

Notatio nova	1	2	3	4	5	6	7	8	9	X	ꝗ	10
significatio antiqua	1	2	3	4	5	6	7	8	9	10	11	12
Notatio nova	11	12	13	14	15	16	17	18	19	1X	1ꝗ	20
significatio antiqua	13	14	15	16	17	18	19	20	21	22	23	24
Notatio	21	22	23	24	25	26	27	28	29	2X	2ꝗ	30
significatio	25	26	27	28	29	30	31	32	33	34	35	36
etc.	31	32	33	34	35	36	37	38	39	3X	3ꝗ	40
etc.	37	38	39	40	41	42	43	44	45	46	47	48
	41	42	43	44	45	46	47	48	49	4X	4ꝗ	50
	49	50	51	52	53	54	55	56	57	58	59	60
	51	52	53	54	55	56	57	58	59	5X	5ꝗ	60
	61	62	63	64	65	66	67	68	69	70	71	72
	61	62	63	64	65	66	67	68	69	6X	6ꝗ	70
	73	74	75	76	77	78	79	80	81	82	83	84
	71	72	73	74	75	76	77	78	79	7X	7ꝗ	80
	85	86	87	88	89	90	91	92	93	94	95	96
	81	82	83	84	85	86	87	88	89	8X	8ꝗ	90
	97	98	99	100	101	102	103	104	105	106	107	108
	91	92	93	94	95	96	97	98	99	9X	9ꝗ	X0
	109	110	111	112	113	114	115	116	117	118	119	120
	X1	X2	X3	X4	X5	X6	X7	X8	X9	XX	Xꝗ	ꝗ0
	121	122	123	124	125	126	127	128	129	130	131	132
	ꝗ1	ꝗ2	ꝗ3	ꝗ4	ꝗ5	ꝗ6	ꝗ7	ꝗ8	ꝗ9	ꝗX	ꝗꝗ	100
	133	134	135	136	137	138	139	140	141	142	143	144

Hinc jam multipli ipsius undenarii / id est communi notatione	11	22	33	44	55	66	77	88	99	110	121	132
nova notatione erunt	ꝗ	1X	29	38	47	56	65	74	83	92	X1	ꝗ0

ubi etiam summa binarum notarum semper facit undecim

40