Two Lost Operations of Arithmetic: Duplation and Mediation

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How many basic operations of arithmetic are there? Four, right? That is certainly the impression one gets from reading the mathematics textbooks of the last few hundred years. But wind the clock back further than that and you find mathematicians who thought differently. Take, for example, duplation and mediation, that is, the operations of successive doubling and successive halving respectively. In medieval Europe, duplation and mediation were treated as distinct arithmetical operations in their own right, warranting discussion separate from that concerning the operations of multiplication and division. Indeed, according to pedagogical treatises, duplation and mediation were to be learned before multiplication and division. The source of this view was the English monk Johannes de Sacrobosco (c.1195–c.1256), whose *Tractatus de arte numerandi* [Treatise on the Art of Reckoning]—generally known as his *Algorismus*—of c.1225, explained that 'There are nine species of this art [of reckoning], namely: numeration, addition, subtraction, mediation, duplation, multiplication, division, progression, and extraction of roots' [1].

Sacrobosco's aim was to promote Hindu-Arabic methods of arithmetic in Europe, and his identification of nine basic operations drew on various Hindu and Arabic sources, which recognize operations such as duplation and mediation as basic. It is not hard to see why: many ancient forms of reckoning, such as those found in Egypt and elsewhere across Africa and the Middle East, use methods of multiplication and division that require successive doubling or halving. For example, in the ancient Egyptian method one computes 11×25 by repeatedly doubling 1 (yielding 1, 2, 4, 8; stopping at 8 because $16 > 11$), repeatedly doubling 25 (25, 50, 100, 200), and then adding up $25 + 50 + 200$ to get the result 275, those terms having been selected since $1 + 2 + 8 = 11$. With such reckoning methods, the need to master duplation and mediation is clear. But Sacrobosco mentions none of that in his treatise, which focuses not on *why* one should learn those operations but only on *how* they should be performed. With no rationale provided, Sacrobosco gives the impression that duplation and mediation are little more than a gentle way of introducing students to the standard European methods of multiplication and division.

Sacrobosco's treatise became the blueprint for European mathematical texts for hundreds of years thereafter, with many medieval authors cleaving fast to his identification of nine basic arithmetical operations. But there were regional variations: in 1494, the Italian mathematician Luca Pacioli (c.1447–1517) omitted duplation and mediation from the list of basic operations on the grounds that 'duplation is implicit in multiplication, and mediation in division' [2]. This view was shared by the Dutch mathematician Gemma Frisius (1508–1555), who in 1540 reduced the number of basic operations to four. Frisius was in fact scathing about including duplation and mediation:

> Some people are wont to identify duplation and mediation as species [of operation] distinct from multiplication and division. But I don't know what came over those stupid people, since both the definition and the operation is the same [in each case]. For to double is to multiply by two and to halve is to divide by two. But if these are distinct operations [from multiplication and division] then there will be an infinite number of them, for we will accept triplation, quadruplation etc. as species [3].

But not everyone got Frisius's memo, and in Germany, Sacrobosco's view held firm. In a popular and oft-reprinted work on arithmetic, *Rechnung auff der Linihen und Federn* [Reckoning with Lines and Quills], the German mathematician Adam Ries (or Riese, 1492–1559) discussed the same nine operations Sacrobosco had, each under its own separate heading. So important was it to learn duplation and mediation that Ries even showed how they could be performed by moving a coin around on a counting board. Ries' book went through over a hundred editions, and continued to be used as a textbook in Germany until the middle of the seventeenth century [4]. Curiously, however, Ries' characterizations of duplation and mediation offered no grounds for treating them as distinct operations in their own right: 'Duplation means two-fold', Ries states, and 'is nothing other than multiplying by two', while 'mediation means making half, and is nothing other than splitting a number into two equal parts' [5].

Those who retained duplation and mediation among the basic arithmetical operations did so more out of deference to the European tradition started by Sacrobosco than because of any perceived theoretical or practical need. Like Ries, they provided no grounds for supposing duplation and mediation were in any way distinct from multiplication and division, and struggled to come up with practical examples where one would need the specific skills of doubling or halving. A French textbook from 1535 illustrates mediation with the unlikely question of how much would be paid to a man offering to sell his robe for the price of 43,690 francs and 8 gros reduced by half twenty-four times over [6]. The value of such a question lies more in its ability to promote familiarity with the complexities of the currency, in which there were 12 gros to the franc, 16 deniers to the gros, and 2 mailles to the denier, than it does in identifying any real world application of the operation of mediation. (The answer, by the way, is 1 maille.)

What of Britain? By the time books on arithmetic came to be written in English, in the mid-sixteenth century, there was no appetite among British mathematicians to include duplation and mediation among the basic operations of arithmetic. In *The Ground of Artes* (1543), the Welsh mathematician Robert Recorde followed Pacioli in identifying seven basic operations of arithmetic: numeration, addition, subtraction, multiplication, division, progression, and extraction of roots. Recorde declined to include duplation and mediation on the grounds that they are contained under multiplication and division respectively [7]. Two decades later, Humfrey Baker managed to ignore duplation and mediation altogether in *The Welspring of Sciences* (1564), at least until he came to fractions, where he 'treateth of duplation, triplation, and quadruplation of all broken numbers' [8]. Posthumous editions of Recorde's book, expanded by John Dee, followed suit, adding brief discussions on the mediation and duplation of fractions. Dee noted, for example, that if one wanted to double $\frac{5}{12}$ it could be done either by doubling the numerator, to make $\frac{10}{12}$, or by halving the denominator, to make $\frac{5}{6}$ [9]. Boosted by numerous reprints, Recorde and Baker between them cornered the market in British textbooks on arithmetic until the end of the seventeenth century, helping to keep duplation and mediation from falling into total obscurity, even if they were by now gathering dust at the bottom of the arithmetician's toolbox.

As European reckoning methods did not make use of successive doubling or halving, it was inevitable that duplation and mediation would fall out of favour. By the turn of the eighteenth century, they were routinely omitted from lists of the basic operations of arithmetic throughout Europe. Even the terms 'duplation' and 'mediation' themselves were rapidly falling out of use, and by the second half of the eighteenth century were about as common as they are today. From this, one might conclude that, almost five hundred years after they had been introduced in Europe, duplation and mediation finally disappeared, almost as if they had eventually cancelled each other out.

But while the seventeenth century may have marked the end of duplation and mediation as distinct operations in *decimal*, it marked their beginning as such in *binary*. In the late 1670s, the German polymath Gottfried Wilhelm Leibniz (1646–1716) invented the binary number system, in which, he quickly realized, multiplying a nonnegative value by two involves nothing more than shifting all of the digits of the value one place to the left, discarding the leftmost digit, and adding a 0 on the right [10]. Hence:

0	0	1	1	0	1	1	0	54
0	1	1	0	1	1	0	0	108

Similarly, to divide a nonnegative value by two, simply shift each of the digits one place to the right, discard the rightmost digit, and add a 0 on the left:

0	0	0	0	1	1	0	0	12
0	0	0	0	0	1	1	0	6

These operations, known today as ‘bit shifting’, are commonplace in assembly programming; consequently high-level languages such as Java, Python, C, and C++ have specific commands for them. Duplation and mediation have thus survived as distinct operations in their own right, at least in binary.

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