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Macroscopic Axisymmetric Lattice Boltzmann Method (MacAxLAB)

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Abstract

The author has recently proposed a macroscopic lattice Boltzmann method (MacLAB) for the Navier-Stokes equations. The method is formulated for the first time to retain the streaming step but remove the collision step in the two integral steps of the standard lattice Boltzmann method. It relies on one fundamental parameter of lattice size δx in the model. This leads to a revolutionary and precise minimal “Lattice” Boltzmann method, which directly uses physical variables as boundary conditions with less required storage for more accurate and efficient simulations. In this paper, the MacLAB is further developed for solving incompressible axisymmetric flow equations (MacAxLAB). The model is validated through simulations of a 3D unsteady Womersley flow and two 3D steady cylindrical cavity flows. The numerical results have compared with other numerical ones and available analytical solutions; it shows that the MacAxLAB is applicable and accurate for modelling incompressible axisymmetric flow.

Keywords: Macroscopic lattice Boltzmann method, axisymmetric flow, computational fluid dynamics, numerical method.

1. Introduction

The lattice Boltzmann method (LBM) is proposed as a simple model for fluid flows. It is a highly simplified model using a finite number of particles. Since then, it has been developed into a very efficient and flexible alternative numerical method in computational fluid dynamics. The method has been extended to solve many other flow problems. For example, Swift et al.

17 applied the lattice Boltzmann method to simulate nonideal fluids [1]. Spaid
18 and Phelan, Jr. solved the Brinkman equation using the LBM [2]. Zhou
19 developed the LBMs for shallow water flows [3] and groundwater flows [4].

20 The axisymmetric flow case encompasses numerous important flow prob-
21 lems in practice [5–9]. Halliday et al. [10] firstly studied the LBM for ax-
22 isymmetric flows in 2001 through introduction of two source terms into the
23 lattice Boltzmann equation. His method has successfully been applied to a
24 number of axisymmetric flow problems [5, 6, 11]. After realising that the one
25 term in the momentum equation related to radial velocity is missing from
26 the formulation of Halliday et al., which causes large errors for axisymmetric
27 flows with significant radial velocities in non-straight pipes, Lee et al. [12]
28 corrected it and obtained an accurate solution to flows when radial velocities
29 cannot be ignored. In addition, the method of Halliday et al. has been ex-
30 tended to multiphase flow by Premnath and Abrahamand [7] and two phase
31 flow with large density ratio by Shiladitya and Abraham [9]. Following the
32 similar idea to Halliday et al., Reis and Phillips [8, 13] modified the source
33 terms in a slightly different manner without the mistake made by Halliday
34 et al. The main drawbacks of these methods are that the second source
35 term involves more complicated terms than the original equations and the
36 added forces cause inconsistency in the dimension of the lattice Boltzmann
37 equation.

38 In 2009, Guo et al. [14] proposed an axisymmetric lattice Boltzmann
39 model from the continuous Boltzmann equation in cylindrical coordinates
40 for axisymmetric flows with or without swirling in the framework of the
41 lattice Boltzmann approach. Furthermore, Li et al. presented an improved
42 axisymmetric lattice Boltzmann scheme including rotational effect [15]. Both
43 models of Guo et al. and Li et al. are suitable for general axisymmetric ro-
44 tational flows and the added source or force terms to the schemes contains
45 no velocity gradient, which are simpler and easier to use compared to other
46 existing methods. However, these two methods share the same weaknesses:
47 (a) the source or force terms contains more terms than those in the origi-
48 nal governing equations; (b) the expressions for calculating the macroscopic
49 variables like velocities take complex forms instead of the conventional sim-
50 ple sum of the distribution functions due to elimination of the implicitness
51 in the schemes, complicating the algorithm; and (c) each gives its own ex-
52 pression for the viscosity that is different from the standard definition in the
53 lattice Boltzmann dynamics. In addition, Zhou developed an axisymmetric
54 lattice Boltzmann method without swirling [16]. The method has many fea-

55 tures close to the standard lattice Boltzmann approach to the Navier-Stokes
 56 equations, e.g., simple procedure, good numerical stability and standard cal-
 57 culations for macroscopic parameters like velocity, which have been confirmed
 58 in the research by Huang and Lu [17], Li et al. [18], and Tang. et al. [19].
 59 Its only disadvantage is that the introduced force term contains velocity gra-
 60 dients. Later, Zhou [20] improved this method and developed the AxLAB®
 61 to overcome the aforementioned weaknesses for generic axisymmetric flows
 62 involving swirling. Its main features are that the method contains no velocity
 63 gradient and retains all the original desired advantages: (a) the source terms
 64 are the same as the ones additional to the Navier-Stokes equations and (b)
 65 the standard calculations for density and velocities are preserved as those
 66 in the conventional LBM. The main weakness of all these existing methods
 67 is that the physical variables such as velocity and density cannot be used
 68 as boundary conditions without being converted to the corresponding dis-
 69 tribution functions. Also, the no slip boundary condition cannot be exactly
 70 achieved through application of the most popular and efficient bounce-back
 71 scheme. These drawbacks have been removed by Zhou [21] in his newly de-
 72 veloped macroscopic lattice Boltzmann method (MacLAB) for Navier-Stokes
 73 equations for fluid flows. In this paper, the MacLAB is extended to solve the
 74 axisymmetric flow equations.

75 **2. Axisymmetric Flow Equations**

76 The governing equations for the incompressible axisymmetric flows are
 77 continuity and momentum equations. They can be written in a cylindrical
 78 coordinate system as [22]

$$\frac{\partial u_j}{\partial x_j} = -\frac{u_r}{r} \quad (1)$$

79 and

$$\begin{aligned}
 \frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_j)}{\partial x_j} = & -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
 & + \frac{\nu}{r} \left(\frac{\partial u_i}{\partial r} + \frac{\partial u_r}{\partial x_i} \right) - \frac{u_i u_r}{r} - \frac{2\nu u_i}{r^2} \delta_{ir},
 \end{aligned} \quad (2)$$

80 where ρ is the density; p is the pressure; t is the time; ν is the kinematic
 81 viscosity; i is the index standing for r or x ; r and x are the coordinates in

82 radial and axial directions, respectively; u_i is the component of velocity in i
 83 direction; δ_{ij} is the Kronecker delta function defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j; \end{cases} \quad (3)$$

84 and the repeated indexes are the Einstein summation convention, which
 85 means a summation over the space coordinates. Such a convention is used
 86 throughout the paper without further indication.

87 3. Macroscopic Lattice Boltzmann Method

88 3.1. Lattice Boltzmann equation

89 Zhou's reformulated lattice Boltzmann equation with source or sink and
 90 force terms for axisymmetric flows, AxLAB[®], reads [20]:

$$\begin{aligned} f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \delta t, t + \delta t) = & f_\alpha(\mathbf{x}, t) - \frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) \\ & - Z_r \left[\frac{(2\tau - 1)e_{\alpha r} \delta t}{2\tau r} \right] (f_\alpha - f_\alpha^{eq}) \\ & + w_\alpha \theta \delta t + 3w_\alpha \frac{\delta t}{e^2} e_{\alpha i} F_i, \end{aligned} \quad (4)$$

91 where f_α is the distribution function of particles; f_α^{eq} is the local equilibrium
 92 distribution function; δt is the time step; $\mathbf{x}$ is the space vector i.e., $\mathbf{x} = (r, x)$;
 93 $e = \delta x / \delta t$; δx is the lattice size; Z_r is a constant taking value of 0 for $r = 0$
 94 or 1 for $r \neq 0$; w_α is the weight given by Eq. (7); θ is the source or sink term,

$$\theta = -\frac{\rho u_r}{r}; \quad (5)$$

95 F_i is the force term defined by

$$F_i = -\frac{\rho u_i u_r}{r} - \frac{2\rho \nu u_i}{r^2} \delta_{ir}; \quad (6)$$

96 $e_{\alpha i}$ is the component of $\mathbf{e}_\alpha$, which is the velocity vector of a particle in the α
 97 link. It can be seen from the recovery in Appendix A that the term related
 98 to $1/r$ in the above equation (4) recovers the second term, Λ_{ir}/r , on the
 99 right-hand side of Eq. (A.12) that is zero according to L'Hôpital's rule when
 100 $r = 0$ and hence it does not exist at $r = 0$ [7, 23].

101 If the nine-velocity square lattice (D2Q9) shown in Fig. 1 is used, w_α is
 102 defined as

$$w_\alpha = \begin{cases} \frac{4}{9}, & \alpha = 0, \\ \frac{1}{9}, & \alpha = 1, 3, 5, 7, \\ \frac{1}{36}, & \alpha = 2, 4, 6, 8; \end{cases} \quad (7)$$

103 and $\mathbf{e}_\alpha$ is

$$\mathbf{e}_\alpha = \begin{cases} (0, 0), & \alpha = 0, \\ \lambda_\alpha e \left[\cos \frac{(\alpha-1)\pi}{4}, \sin \frac{(\alpha-1)\pi}{4} \right], & \alpha \neq 0, \end{cases} \quad (8)$$

104 with λ_α ,

$$\lambda_\alpha = \begin{cases} 1, & \alpha = 1, 3, 5, 7, \\ \sqrt{2}, & \alpha = 2, 4, 6, 8. \end{cases} \quad (9)$$

The fluid density ρ and velocity u_i are determined from the distribution

Figure 1: Nine-velocity square lattice (D2Q9).

105 function in the same manner as that in the standard LBM for the Navier-
 106 Stokes equations,
 107

$$\rho = \sum_{\alpha} f_{\alpha}, \quad u_i = \frac{1}{\rho} \sum_{\alpha} e_{\alpha i} f_{\alpha}. \quad (10)$$

108 The local equilibrium distribution function f_{α}^{eq} is

$$f_{\alpha}^{eq} = w_{\alpha} \rho \left(1 + 3 \frac{e_{\alpha i} u_i}{e^2} + \frac{9}{2} \frac{e_{\alpha i} e_{\alpha j} u_i u_j}{e^4} - \frac{3}{2} \frac{u_j u_j}{e^2} \right), \quad (11)$$

109 which can be shown to have the following properties,

$$\rho = \sum_{\alpha} f_{\alpha}^{eq}, \quad u_i = \frac{1}{\rho} \sum_{\alpha} e_{\alpha i} f_{\alpha}^{eq}. \quad (12)$$

110 To avoid determining the density and velocity through Eq. (10) using distri-
111 bution functions, Eq. (4) is rewritten as

$$\begin{aligned} f_{\alpha}(\mathbf{x}, t) &= f_{\alpha}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) - \frac{1}{\tau} [f_{\alpha}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad - f_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t)] \\ &\quad - Z_r \left[\frac{(2\tau - 1)e_{\alpha r} \delta t}{2\tau r} \right] [f_{\alpha}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad - f_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t)] \\ &\quad + w_{\alpha} \delta t \theta + 3w_{\alpha} \frac{\delta t}{e^2} e_{\alpha i} F_i. \end{aligned} \quad (13)$$

112 After substitution of $\tau = 1$ into the above equation following Zhou's idea in
113 MacLAB [21], it can be simplified to

$$\begin{aligned} f_{\alpha}(\mathbf{x}, t) &= f_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) - Z_r \frac{e_{\alpha r} \delta t}{2r} f_{\alpha}^{neq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad + w_{\alpha} \delta t \theta + 3w_{\alpha} \frac{\delta t}{e^2} e_{\alpha i} F_i, \end{aligned} \quad (14)$$

114 in which f_{α}^{neq} is the non-equilibrium distribution function,

$$f_{\alpha}^{neq} = f_{\alpha} - f_{\alpha}^{(eq)}. \quad (15)$$

115 Taking $\sum$ Eq. (14) and $\sum e_{\alpha i}$ Eq. (14) yields

$$\begin{aligned} \sum f_{\alpha}(\mathbf{x}, t) &= \sum f_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad - Z_r \frac{\delta t}{2r} \sum e_{\alpha r} f_{\alpha}^{(neq)}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad + \delta t \sum w_{\alpha} \theta + 3 \frac{\delta t}{e^2} \sum w_{\alpha} e_{\alpha i} F_i \end{aligned} \quad (16)$$

116 and

$$\begin{aligned} \sum e_{\alpha i} f_{\alpha}(\mathbf{x}, t) &= \sum e_{\alpha i} f_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad - Z_r \frac{\delta t}{2r} \sum e_{\alpha r} e_{\alpha i} f_{\alpha}^{(neq)}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad + \delta t \sum w_{\alpha} e_{\alpha i} \theta + 3 \frac{\delta t}{e^2} \sum w_{\alpha} e_{\alpha i} e_{\alpha j} F_j. \end{aligned} \quad (17)$$

117 As $\sum f_\alpha(\mathbf{x}, t) = \rho(\mathbf{x}, t)$ and $\sum e_{\alpha i} f_\alpha(\mathbf{x}, t) = \rho(\mathbf{x}, t) u_i(\mathbf{x}, t)$ due to the require-
 118 ment for the conservation of mass and momentum in the lattice Boltzmann
 119 dynamics, we have

$$\begin{aligned} \rho(\mathbf{x}, t) &= \sum f_\alpha^{eq}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) \\ &- Z_r \frac{\delta t}{2r} \sum e_{\alpha r} f_\alpha^{(neq)}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) \\ &+ \delta t \sum w_\alpha \theta + 3 \frac{\delta t}{e^2} \sum w_\alpha e_{\alpha i} F_i, \end{aligned} \quad (18)$$

120 and

$$\begin{aligned} \rho u_i(\mathbf{x}, t) &= \sum e_{\alpha i} f_\alpha^{eq}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) \\ &- Z_r \frac{\delta t}{2r} \sum e_{\alpha r} e_{\alpha i} f_\alpha^{(neq)}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) \\ &+ \delta t \sum w_\alpha e_{\alpha i} \theta + 3 \frac{\delta t}{e^2} \sum w_\alpha e_{\alpha i} e_{\alpha j} F_j. \end{aligned} \quad (19)$$

121 According to the centred scheme [20, 24] both source term θ and force term
 122 F_i can be evaluated at the midpoint between $(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t)$ and $(\mathbf{x}, t)$ as

$$\theta = \theta \left(\mathbf{x} - \frac{1}{2} \mathbf{e}_\alpha \delta t, t - \frac{1}{2} \delta t \right), \quad (20)$$

123 and

$$F_i = F_i \left(\mathbf{x} - \frac{1}{2} \mathbf{e}_\alpha \delta t, t - \frac{1}{2} \delta t \right), \quad (21)$$

124 and f_α^{neq} is estimated by [25]

$$f_\alpha^{neq}(\mathbf{x}, t) = -[f_\alpha^{eq}(\mathbf{x}, t) - f_\alpha^{eq}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t)]. \quad (22)$$

125 It can be seen from Eqs. (18) - (22) that the density and velocity are deter-
 126 mined by the macroscopic physical variables through the local equilibrium
 127 distribution function without calculating the distribution function using E-
 128 q. (4) that is required in Eq. (10) for the density and velocity. These equations
 129 form the macroscopic axisymmetric lattice Boltzmann model (MacAxLAB).
 130 It shows in Appendix A that the fluid viscosity in the absence of collision
 131 step can be naturally taken into account using the particle speed e from

$$e = 6\nu/\delta x, \quad (23)$$

132 instead of $e = \delta x / \delta t$ to calculate the local equilibrium distribution function
 133 f_α^{eq} from Eq. (11). Apparently, after a lattice size δx is chosen, the model
 134 is ready to simulate a flow with a viscosity ν because $(x_j - e_{\alpha j} \delta t)$ stands
 135 for a neighbouring lattice point; f_α^{eq} at time of $(t - \delta t)$ represents its known
 136 quantity at the current time; and the particle speed e is determined from
 137 Eq. (23) for use in computation of f_α^{eq} . In addition, the time step δt is no
 138 longer an independent parameter but is calculated as $\delta t = \delta x / e$, which is
 139 used in simulations of unsteady flows. Consequently, only the lattice size δx
 140 is required in the MacAxLAB for simulation of Axisymmetric flows, bring-
 141 ing the LBM into a precise “Lattice” Boltzmann method. This enables the
 142 model to become an automatic simulator without tuning other simulation
 143 parameters, making it possible and easy to model a large flow system when
 144 a super-fast computer such as a quantum computer becomes available in the
 145 future.

146 The model is unconditionally stable as it shares the same valid condition
 147 as that for f_α^{eq} , or the Mach number $M = U_c / e$ is much smaller than 1,
 148 in which U_c is a characteristic flow speed. The Mach number can also be
 149 expressed as a lattice Reynolds number of $R_{le} = U_c \delta x / \nu$ via Eq. (23). In
 150 practical simulations, it is found that the model is stable if $R_{le} = U_m \delta x / \nu <$
 151 1 where U_m is the maximum flow speed and is used as the characteristic
 152 flow speed. The main features of the described model are that there is no
 153 collision operator and only macroscopic physical variables such as density
 154 and velocity are required, which are directly retained as boundary conditions
 155 with a minimum memory requirement. The simulation procedure is

- 156 (a) Initialise density and velocity,
- 157 (b) Choose the lattice size δx and determine the particle speed e from E-
 158 q. (23),
- 159 (c) Calculate f_α^{eq} from Eq. (11) using density and velocity,
- 160 (d) Update the density and velocity using Eqs. (18) and (19),
- 161 (e) Apply the boundary conditions if necessary, and repeat Step (b) until a
 162 solution is reached.

163 The only limitation of the described model is that, for very small viscosity or
 164 high speed flow, the chosen lattice size after satisfying $R_{le} < 1$ may turn out
 165 to generate very large lattice points (Lattice points, e.g., for one dimension
 166 with length of L is calculated as $N_L = L / \delta x$ and N_L is the lattice points);
 167 if the total lattice points is too big such that the demanding computations
 168 is beyond the power of a current computer, the simulation cannot be carried

169 out. Such difficulties may be solved or relaxed through parallel computing
 170 based on modern computer software and hardware such as GPU processors
 171 and multiple servers, and will largely or completely removed using a future
 172 super-fast computer.

173 3.2. Axisymmetric rotational flows

174 Axisymmetric rotational flows contain an azimuthal velocity u_ϕ , which is
 175 governed by the equation in a cylindrical coordinate system [22],

$$\frac{\partial u_\phi}{\partial t} + \frac{\partial(u_j u_\phi)}{\partial x_j} = \nu \frac{\partial^2 u_\phi}{\partial x_j^2} + \frac{\nu}{r} \frac{\partial u_\phi}{\partial r} - \frac{2u_r u_\phi}{r} - \frac{\nu u_\phi}{r^2}. \quad (24)$$

176 Its further effect on the flow field is taken into account by adding an additional
 177 term to the force F_i in Eq. (6) as

$$F_i = -\frac{\rho u_i u_r}{r} - \frac{2\rho \nu u_i}{r^2} \delta_{ir} + \frac{\rho u_\phi^2}{r} \delta_{ir}. \quad (25)$$

178 Eq. (24) is an advection-diffusion equation and can be solved accurately and
 179 efficiently on a D2Q4, D2Q5 or D2Q9 lattice model [26–28]. In the present
 180 study, the D2Q9 is used and the following lattice Boltzmann equation with
 181 a source or sink term is applied,

$$\begin{aligned} \bar{f}_\alpha(\mathbf{x} + \mathbf{e}_\alpha \delta t, t + \delta t) &= \bar{f}_\alpha(\mathbf{x}, t) - \frac{1}{\bar{\tau}} (\bar{f}_\alpha - \bar{f}_\alpha^{eq}) \\ &- Z_r \left[\frac{(2\bar{\tau} - 1)e_{\alpha r} \delta t}{2\bar{\tau} r} \right] (\bar{f}_\alpha - \bar{f}_\alpha^{eq}) \\ &+ w_\alpha S_\phi \delta t, \end{aligned} \quad (26)$$

182 where $\bar{f}_\alpha$ is the distribution function; $\bar{f}_\alpha^{eq}$ is the local equilibrium distribution
 183 function; $\bar{\tau}$ is the single relaxation time; and S_ϕ is the source or sink term
 184 defined by

$$S_\phi = -\frac{2\rho u_r u_\phi}{r} - \frac{\rho \nu u_\phi}{r^2}. \quad (27)$$

185 There are many expressions for $\bar{f}_\alpha^{eq}$ and a simple one is used here [15],

$$\bar{f}_\alpha^{eq} = w_\alpha \left(1 + \frac{3e_{\alpha j} u_j}{e^2} \right) \rho u_\phi. \quad (28)$$

186 It is easy to show that the above equation has the following properties,

$$\sum_{\alpha} \bar{f}_{\alpha}^{eq} = \rho u_{\phi}, \quad (29)$$

187

$$\sum_{\alpha} e_{\alpha i} \bar{f}_{\alpha}^{eq} = \rho u_i u_{\phi} \quad (30)$$

188 and

$$\sum_{\alpha} e_{\alpha i} e_{\alpha j} \bar{f}_{\alpha}^{eq} = \rho e^2 u_{\phi} \delta_{ij} / 3. \quad (31)$$

189 To formulate a macroscopic lattice Boltzmann method for calculating u_{ϕ} ,
190 Eq. (26) is rewritten as

$$\begin{aligned} \bar{f}_{\alpha}(\mathbf{x}, t) &= \bar{f}_{\alpha}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) - \frac{1}{\bar{\tau}} [(\bar{f}_{\alpha}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad - \bar{f}_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t))] \\ &\quad - Z_r \left[\frac{(2\bar{\tau} - 1) e_{\alpha r} \delta t}{2\bar{\tau} r} \right] \bar{f}_{\alpha}^{neq} + w_{\alpha} S_{\phi} \delta t, \end{aligned} \quad (32)$$

191 in which $\bar{f}_{\alpha}^{neq}$ is the non-equilibrium distribution function,

$$\bar{f}_{\alpha}^{neq} = \bar{f}_{\alpha} - \bar{f}_{\alpha}^{eq}. \quad (33)$$

192 Setting $\bar{\tau} = 1$ and taking $\sum$ Eq. (32) following Zhou's idea [21] lead to

$$\begin{aligned} \sum \bar{f}_{\alpha}(\mathbf{x}, t) &= \sum \bar{f}_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad - Z_r \frac{\delta t}{2r} \sum e_{\alpha r} \bar{f}_{\alpha}^{neq} + \sum w_{\alpha} S_{\phi} \delta t, \end{aligned} \quad (34)$$

193 The mass conservation requires $\sum \bar{f}_{\alpha}(\mathbf{x}, t) = \rho u_{\phi}$, i.e., Eq. (34) can be written
194 as

$$\begin{aligned} \rho(\mathbf{x}, t) u_{\phi}(\mathbf{x}, t) &= \sum \bar{f}_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t) \\ &\quad - Z_r \frac{\delta t}{2r} \sum e_{\alpha r} \bar{f}_{\alpha}^{neq} + \sum w_{\alpha} S_{\phi} \delta t, \end{aligned} \quad (35)$$

195 from which u_{ϕ} is determined after $\bar{f}_{\alpha}^{neq}$ is estimated using [25]

$$\bar{f}_{\alpha}^{neq}(\mathbf{x}, t) = -[\bar{f}_{\alpha}(\mathbf{x}, t) - \bar{f}_{\alpha}^{eq}(\mathbf{x} - \mathbf{e}_{\alpha} \delta t, t - \delta t)]. \quad (36)$$

196 The centred scheme [24] is again used for term S_{ϕ} ,

$$S_{\phi} = S_{\phi} \left(\mathbf{x} - \frac{1}{2} \mathbf{e}_{\alpha} \delta t, t - \frac{1}{2} \delta t \right). \quad (37)$$

197 3.3. Significance

198 The proposed MacAxLAB has three main distinguishable advantages
 199 from existing lattice Boltzmann methods for axisymmetric flows, First of
 200 all, only lattice size δx is required to model flows without tuning other cal-
 201 culation parameters such as time step, reducing computational cost. Then,
 202 physical variables are directly either retained as Dirichlet boundary condi-
 203 tion without additional calculations for boundary lattice points or used as
 204 boundary conditions without being converted to particle distribution func-
 205 tions, saving computational time and increasing accuracy due to avoidance
 206 of errors from the conversion between the variables and particle distribution
 207 functions. Finally, there is no need for computation of particle distribution
 208 functions, saving computer storage and accelerating simulation. As such, the
 209 developed model is more efficient and accurate.

210 4. Numerical simulations

211 4.1. 3D Womersley flow

212 The 3D Womersley flow or a pulsatile flow is an unsteady axisymmetric
 213 flow in a straight pipe. It is driven by a periodic pressure gradient at the
 214 inlet of the pipe and the pressure gradient is normally given as

$$\frac{dp}{dx} = p_0 \cos(\omega t), \quad (38)$$

215 where p_0 is the maximum amplitude of the pressure variation and $\omega = 2\pi/T$
 216 is the angular frequency, in which T is the period. The Reynolds number is
 217 defined as $Re = U_c D / \nu$ with the characteristic velocity U_c given by

$$U_c = \frac{p_0 \alpha^2}{4\omega\rho} = \frac{p_0 R^2}{4\rho\nu}, \quad (39)$$

218 in which $\alpha = R\sqrt{\omega/\nu}$ is the Womersley number, where R is the pipe radius
 219 and D is the diameter. The analytical solution for the velocity component
 220 in axial direction is

$$u_x(r, t) = \text{Re} \left\{ \frac{p_0}{i\omega\rho} \left[1 - \frac{J_0(r\phi/R)}{J_0(\phi)} \right] e^{i\omega t} \right\}, \quad (40)$$

221 where J_0 is the zeroth order Bessel function of the first type; i is the imaginary
 222 unit; $\phi = (-\alpha + i\alpha)/\sqrt{2}$; and Re denotes the real part of a complex number.

223 The implementation of the periodic pressure gradient can be achieved by
 224 applying an equivalent periodic body force to the flow [29], i.e., an additional
 225 body force is added to the existing force term F_i ,

$$F_i = -\frac{\rho u_i u_r}{r} - \frac{2\rho\nu u_i}{r^2}\delta_{ir} + p_0 \cos(\omega t)\delta_{ix}. \quad (41)$$

226 In the computation, $\rho = 3$, $p_0 = 0.001$, $D = 40$, $T = 1200$, $\alpha = 8$,
 227 $U_C = 1$, which give $Re = 1200$. 80×40 lattices with $\Delta x = 1$ are used
 228 in the simulation. The periodic boundary conditions are applied to inflow
 229 and outflow boundaries; the exact initial zero velocities are retained along
 230 the pipe walls for exact no-slip boundary conditions without errors as all
 231 computations are carried out on lattice points only within the pipe. The
 232 numerical solutions at different times are obtained after initial running time
 233 of $10T$. The corresponding results for velocity u_x are shown in Figs. 2 and 3,
 234 which are further compared with the analytical solution (40), showing good
 235 agreements. Compared to the previously developed AxLAB®, the present
 236 MacAxLAB produce more accurate results for $n = 0, 1, 7, 8, 9, 10, 15$ and
 237 similarly accurate results for others as shown in Figs. 2 and 3.

238 4.2. Cylindrical cavity flow

239 Steady cylindrical cavity flows have been investigated both experimen-
 240 tally and numerically [14, 15, 30, 31]. The flow problem is known to have
 241 different complex structures depending on combinations of the aspect ratio
 242 $A = H/R$ and the Reynolds number $Re = R^2\Omega/\nu$, where Ω is the the con-
 243 stant angular velocity; and H is the height of the cylinder. For example,
 244 the flow contains a pair of vortex breakdown bubbles when $A = 1.5$ and
 245 $Re = 1290$. Because of its complexity, this case is widely used a bench-
 246 mark test to validate a numerical method such as the 3D lattice Boltzmann
 247 method by Bhaumik and Lakshmisha [31], the axisymmetric lattice Boltz-
 248 mann method by Guo et al. [14], and the axisymmetric lattice Boltzmann
 249 scheme by Li [15]. It is then chosen to test the proposed MacAxLAB. In the
 250 simulations, $R = 1$, $\rho = 1$, and $\Omega = 0.1$. The steady flow state is reached
 251 after certain time steps when the following convergence criterion is satisfied
 252 [14],

$$\frac{\|\mathbf{V}(t) - \mathbf{V}(t - 1000\delta t)\|}{\|\mathbf{V}(t)\|} < 10^{-6}, \quad (42)$$

Figure 2: Comparisons of numerical results with analytical solution when u_x is increasing with time at $t = nT/16$ with $n = 0, 1, 2, 3, 12, 13, 14, 15$.

253 in which $\|\mathbf{V}(t)\| = \sqrt{\sum [u_x^2(x, r) + u_r^2(x, r)]}$. The boundary conditions are

$$\begin{aligned}
 u_x = u_r = u_\phi &= 0, & x &= 0, \\
 u_x = u_r = u_\phi &= 0, & r &= \pm R, \\
 u_x = u_r &= 0, u_\phi = r\Omega, & x &= H,
 \end{aligned} \tag{43}$$

254 which are directly determined once and retained as boundary conditions in
 255 the proposed model as calculations are required only within the cylinder
 256 during simulation. This avoid the additional errors induced by using oth-
 257 er scheme such as bounce back or nonequilibrium extrapolation approach
 258 [32]. The streamlines for the steady solution are plotted in Fig. 4 and clearly
 259 show that a pair of vortex breakdown bubbles is well developed. This is
 260 again in agreement with that observed in the experimental measurements

Figure 3: Comparisons of numerical results with analytical solution when u_x is decreasing with time at $t = nT/16$ with $n = 4, 5, 6, 7, 8, 9, 10, 11$.

[30] and the numerical results [14, 15, 20, 31]. The maximum axial velocity $u_{x,max}$ and its location h_{max} are shown in Table 1 and further compared with the existing experimental data and other numerical results from the AxLAB®, the improved model by Li et al. [15], the 3D LBM and the Navier-Stokes solutions by Bhaumik and Lakshmisha [31]. The relative errors calculated for velocity using $E_u = (u_{x,max} - u'_{x,max})/u'_{x,max}$, and for location using $E_h = (h_{max} - h'_{max})/h'_{max}$, where $u_{x,max}$ stands for numerical result, $u'_{x,max}$ for experimental data, h_{max} and h'_{max} are the locations from computation and measurement in laboratory, respectively. As seen from Table 1, $E_u = 4.71\%$ and $E_h = 5\%$ are among Li's model, the 3D LBM solution and the Navier-Stokes result. This again indicates that the present model can produce accurate solutions, agreeing well with the previous investigations.

Table 1: Comparisons of maximum axial velocities.

Model	$u_{x,max}/u_0$	h_{max}/H	E_u	E_h
MacAxLAB	0.0712	0.147	4.71%	5.00%
AxLAB® [20]	0.0706	0.147	3.82%	5.00%
3D LBM [31]	0.072	0.16	5.88%	14.29%
N-S [31]	0.0665	0.125	2.21%	5.0%
Li's LBM [15]	0.0716	0.147	5.29%	5.00%
Expt. [30]	0.068	0.14	—	—

Note: $R = 1$, $\Omega = 0.1$ and $u_0 = R\Omega$.

273 To demonstrate the potential of the present method in predicting more
274 complex axisymmetric flow, a further case with $A = 2.5$ and $R_e = 2200$ 275 is
simulated. This case has also been investigated in the experiment [30], 276 revealing
that there are two vortex breakdown bubbles. The other calculating 277 parameters
remain the same as those in the first case. The stream lines for 278 the steady
solution are plotted in Fig. 5 and clearly show that two pairs of 279 vortex
breakdown bubbles are well developed. The flow pattern is again in 280 agreement
with that observed in the experimental measurements [30].

281 5. Conclusions

282 A macroscopic axisymmetric lattice Boltzmann method (MacAxLAB) is
283 described for generic axisymmetric flows with or without swirling. No colli-
284 sion operator is involved and the scheme is unconditionally stable. The main
285 features are (i) the Dirichlet boundary condition can exactly be achieved
286 without using other scheme such as bounce-back scheme, the macroscopic
287 variable are directly used for other boundary conditions without recourse to
288 particle distribution functions; (ii) it requires less memory in simulations;
289 and (iii) there is no calculation for data transfer between particle distribu-
290 tions and macroscopic variables. Three numerical examples have shown that
291 the MacAxLAB is simple and accurate, which is suitable for both steady and
292 unsteady axisymmetric rotational flows.

293 Appendix A Recovery of the Axisymmetric Flow Equations

294 In order to recover the axisymmetric flow equations from the MacAxLAB,
295 we take a Taylor expansion to the terms on the right-hand side of Eq. (13),

Figure 4: Streamlines for the case with $A = 1.5$ and $R_e = 1290$, showing a pair of the fully developed vortex breakdown bubbles.

296 $f_\alpha(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t)$ and $f_\alpha^{eq}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t)$, in time and space at point $(\mathbf{x}, t)$,
 297 and have

$$\begin{aligned}
 f_\alpha(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) &= f_\alpha - \delta t \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha \\
 &+ \frac{1}{2} \delta t^2 \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 f_\alpha + \mathcal{O}(\delta t^3) \quad (\text{A.1})
 \end{aligned}$$

298 and

$$f_\alpha^{eq}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) = f_\alpha^{eq} - \delta t \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha^{eq}$$

$$+ \frac{1}{2}\delta t^2 \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 f_{\alpha}^{eq} + \mathcal{O}(\delta t^3). \quad (\text{A.2})$$

299 According to the Chapman-Enskog analysis, f_{α} can be expanded in a series
300 of δt ,

$$f_{\alpha} = f_{\alpha}^{(0)} + \delta t f_{\alpha}^{(1)} + \delta t^2 f_{\alpha}^{(2)} + \mathcal{O}(\delta t^3). \quad (\text{A.3})$$

301 Eqs. (20) and (21) can also be written, via a Taylor expansion, as

$$\theta \left(\mathbf{x} - \frac{1}{2} \mathbf{e}_{\alpha} \delta t, t - \frac{1}{2} \delta t \right) = \theta - \frac{\delta t}{2} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \theta + \mathcal{O}(\delta t^2) \quad (\text{A.4})$$

302 and

$$F_i \left(\mathbf{x} - \frac{1}{2} \mathbf{e}_{\alpha} \delta t, t - \frac{1}{2} \delta t \right) = F_i - \frac{\delta t}{2} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) F_i + \mathcal{O}(\delta t^2) \quad (\text{A.5})$$

303 After substitution of Eqs. (A.1) - (A.5) into Eq. (13), we have the expressions
304 to order δt^0

$$f_{\alpha}^{(0)} = f_{\alpha}^{eq}, \quad (\text{A.6})$$

305 to order δt

$$\left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_{\alpha}^{(0)} = -\frac{f_{\alpha}^{(1)}}{\tau} + w_{\alpha} \theta + 3w_{\alpha} \frac{e_{\alpha i}}{e^2} F_i, \quad (\text{A.7})$$

306 and to order δt^2

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_{\alpha}^{(1)} - \frac{1}{2} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 f_{\alpha}^{(0)} = -\frac{f_{\alpha}^{(2)}}{\tau} \\ & + \frac{1}{\tau} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_{\alpha}^{(1)} - \frac{(2\tau - 1)}{2\tau r} e_{\alpha r} f_{\alpha}^{(1)} \\ & - \frac{w_{\alpha}}{2} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \theta - \frac{3w_{\alpha} e_{\alpha i}}{2e^2} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) F_i. \end{aligned} \quad (\text{A.8})$$

307 Using Eq. (A.7), we can write the above equation as

$$\frac{(2\tau - 1)}{2\tau} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_{\alpha}^{(1)} = -\frac{f_{\alpha}^{(2)}}{\tau} - \frac{(2\tau - 1)}{2\tau r} e_{\alpha r} f_{\alpha}^{(1)}. \quad (\text{A.9})$$

308 From Eq. (A.7) + Eq. (A.9) $\times \delta t$, we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_{\alpha}^{(0)} + \frac{(2\tau - 1)\delta t}{2\tau} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) f_{\alpha}^{(1)} \\ &= -\frac{1}{\tau} (f_{\alpha}^{(1)} + \delta t f_{\alpha}^{(2)}) - \frac{(2\tau - 1)\delta t}{2\tau r} e_{\alpha r} f_{\alpha}^{(1)} \\ &+ w_{\alpha} \theta + \frac{3w_{\alpha}}{e^2} e_{\alpha i} F_i. \end{aligned} \quad (\text{A.10})$$

309 Summation of the above equation over α provides

$$\frac{\partial}{\partial t} \sum_{\alpha} f_{\alpha}^{(0)} + \frac{\partial}{\partial x_j} \sum_{\alpha} e_{\alpha j} f_{\alpha}^{(0)} = \theta. \quad (\text{A.11})$$

310 Using Eq. (A.6) and substitution of Eq. (12) into the above equation result
311 in the continuity equation (1), if the density variation is small enough and
312 can be neglected.

313 Taking $\sum e_{\alpha i} [(A.7) + \delta t \times (A.9)]$ about α yields

$$\frac{\partial}{\partial t} \sum_{\alpha} e_{\alpha i} f_{\alpha}^{(0)} + \frac{\partial \Pi_{ij}^{(0)}}{\partial x_j} = \frac{\partial \Lambda_{ij}}{\partial x_j} + \frac{\Lambda_{ir}}{r} + F_i, \quad (\text{A.12})$$

314 where $\Pi_{ij}^{(0)}$ is the zeroth-order momentum flux tensor given by the following
315 expression,

$$\Pi_{ij}^{(0)} = \sum_{\alpha} e_{\alpha i} e_{\alpha j} f_{\alpha}^{(0)}, \quad (\text{A.13})$$

316

$$\Lambda_{ij} = -\frac{\delta t}{2\tau} (2\tau - 1) \sum_{\alpha} e_{\alpha i} e_{\alpha j} f_{\alpha}^{(1)}, \quad (\text{A.14})$$

317 and

$$\Lambda_{ir} = -\frac{\delta t}{2\tau} (2\tau - 1) \sum_{\alpha} e_{\alpha i} e_{\alpha r} f_{\alpha}^{(1)}. \quad (\text{A.15})$$

318 Evaluating terms in Eq. (A.13) with Eq. (11), we have

$$\Pi_{ij}^{(0)} = p \delta_{ij} + \rho u_i u_j, \quad (\text{A.16})$$

319 where $p = \rho e^2/3$ is the pressure, leading to a sound speed, $C_s = e/\sqrt{3}$.
320 Substitution of the above equation into Eq. (A.12) produces

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \Lambda_{ij}}{\partial x_j} + \frac{\Lambda_{ir}}{r} + F_i. \quad (\text{A.17})$$

321 Applying Eq. (A.7), we can rewrite Eq. (A.14) as

$$\Lambda_{ij} = \Pi_{ij}^{(1)} - \frac{\delta t}{2}(2\tau - 1) \sum_{\alpha} e_{\alpha i} e_{\alpha j} w_{\alpha} \theta, \quad (\text{A.18})$$

322 with the first-order momentum flux tensor $\Pi_{ij}^{(1)}$ defined by,

$$\Pi_{ij}^{(1)} = \frac{\delta t}{2}(2\tau - 1) \sum_{\alpha} e_{\alpha i} e_{\alpha j} \left(\frac{\partial}{\partial t} + e_{\alpha k} \frac{\partial}{\partial x_k} \right) f_{\alpha}^{(0)}, \quad (\text{A.19})$$

323 which can also be written using Eq. (A.13) as,

$$\Pi_{ij}^{(1)} = \frac{\delta t}{2}(2\tau - 1) \frac{\partial}{\partial t} \Pi_{ij}^{(0)} + \frac{\delta t}{2}(2\tau - 1) \frac{\partial}{\partial x_k} \sum_{\alpha} e_{\alpha i} e_{\alpha j} e_{\alpha k} f_{\alpha}^{(0)}. \quad (\text{A.20})$$

324 The second term in the above equation can be evaluated with Eq. (11) and
325 Eq. (A.6) as

$$\frac{\partial}{\partial x_k} \sum_{\alpha} e_{\alpha i} e_{\alpha j} e_{\alpha k} f_{\alpha}^{(0)} = \frac{e^2}{3} \frac{\partial}{\partial x_k} (\rho u_i \delta_{jk} + \rho u_j \delta_{ki} + \rho u_k \delta_{ij}). \quad (\text{A.21})$$

326 If we assume that characteristic velocity is U_c , length L_c and time t_c , we have
327 that the term $(\partial/\partial t \Pi_{ij}^{(0)})$ is of order $\rho U_c^2/t_c$ and the term $(\partial/\partial x_k \sum_{\alpha} e_{\alpha i} e_{\alpha j} e_{\alpha k} f_{\alpha}^{(0)})$
328 is of order $\rho e^2 U_c/L_c$, based on which we obtain that the ratio of the former
329 to the latter terms has the order,

$$\begin{aligned} \mathcal{O} \left(\frac{\partial/\partial t \Pi_{ij}^{(0)}}{\partial/\partial x_k \sum_{\alpha} e_{\alpha i} e_{\alpha j} e_{\alpha k} f_{\alpha}^{(0)}} \right) &= \mathcal{O} \left(\frac{\rho U_c^2/t_c}{\rho e^2 U_c/L_c} \right) \\ &= \mathcal{O} \left(\frac{U_c}{e} \right)^2 = \mathcal{O} \left(\frac{U_c}{C_s} \right)^2 = \mathcal{O}(M^2), \end{aligned} \quad (\text{A.22})$$

330 in which $M = U_c/C_s$ is the Mach number. It follows that the first term in
331 Eq. (A.20) is very small compared with the second term and can be neglected
332 if $M \ll 1$ which is consistent with the lattice Boltzmann dynamics; hence
333 Eq. (A.20), after Eq. (A.21) is substituted, becomes

$$\Pi_{ij}^{(1)} = \frac{e^2 \delta t}{6} (2\tau - 1) \frac{\partial}{\partial x_k} (\rho u_i \delta_{jk} + \rho u_j \delta_{ki} + \rho u_k \delta_{ij}), \quad (\text{A.23})$$

334 OR

$$\Pi_{ij}^{(1)} = \nu \left[\frac{\partial(\rho u_i)}{\partial x_j} + \frac{\partial(\rho u_j)}{\partial x_i} + \frac{\partial(\rho u_k)}{\partial x_k} \delta_{ij} \right], \quad (\text{A.24})$$

335 where ν is the kinematic viscosity and defined by

$$\nu = \frac{e^2 \delta t}{6} (2\tau - 1). \quad (\text{A.25})$$

336 Inserting (A.24) into Eq. (A.18) and evaluating the rest terms of the equation
337 lead to

$$\Lambda_{ij} = \nu \left[\frac{\partial(\rho u_i)}{\partial x_j} + \frac{\partial(\rho u_j)}{\partial x_i} + \frac{\partial(\rho u_k)}{\partial x_k} \delta_{ij} \right] - \nu \theta \delta_{ij}. \quad (\text{A.26})$$

338 After applying the continuity equation (1) and Eq. (5) to the above, we
339 obtain

$$\Lambda_{ij} = \nu \left[\frac{\partial(\rho u_i)}{\partial x_j} + \frac{\partial(\rho u_j)}{\partial x_i} \right]. \quad (\text{A.27})$$

340 Similarly, we have

$$\Lambda_{ir} = \nu \left[\frac{\partial(\rho u_i)}{\partial r} + \frac{\partial(\rho u_r)}{\partial x_i} \right]. \quad (\text{A.28})$$

341 Combining Eqs. (6), (A.27) and (A.28) with Eq. (A.17) results in

$$\begin{aligned} \frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j} \left[\frac{\partial(\rho u_i)}{\partial x_j} + \frac{\partial(\rho u_j)}{\partial x_i} \right] \\ &+ \frac{\nu}{r} \left[\frac{\partial(\rho u_i)}{\partial r} + \frac{\partial(\rho u_r)}{\partial x_i} \right] - \frac{\rho u_i u_r}{r} - \frac{2\rho \nu u_i}{r^2} \delta_{ir}. \end{aligned} \quad (\text{A.29})$$

342 Again, if the density variation is assumed to be small enough, the above
343 is just the momentum equation (2). As τ takes a constant, use of $\tau =$
344 1 also recovers the continuity and the axisymmetric flow equations at the
345 second-order accurate as the above derivation shows. In this case, Eq. (A.25)
346 becomes Eq. (23), which determines the particle speed e .

347 Appendix B Recovery of the equation for azimuthal velocity

348 In order to prove that Eq. (24) can be recovered from the lattice Boltz-
349 mann equation (32), we apply the similar Chapman-Enskog analysis to that
350 given in Appendix A and, after taking a Taylor expansion to $\bar{f}_\alpha(\mathbf{x} - \mathbf{e}_\alpha \delta t, t -$

351 δt) and $\bar{f}_\alpha^{eq}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t)$ on the right-hand side of Eq. (32) in time and
 352 space at point $(\mathbf{x}, t)$, we have

$$\begin{aligned} \bar{f}_\alpha(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) &= \bar{f}_\alpha - \delta t \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \bar{f}_\alpha \\ &+ \frac{1}{2} \delta t^2 \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 \bar{f}_\alpha + \mathcal{O}(\delta t^3), \end{aligned} \quad (\text{B.1})$$

353

$$\begin{aligned} \bar{f}_\alpha^{eq}(\mathbf{x} - \mathbf{e}_\alpha \delta t, t - \delta t) &= \bar{f}_\alpha^{eq} - \delta t \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \bar{f}_\alpha^{eq} \\ &+ \frac{1}{2} \delta t^2 \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 \bar{f}_\alpha^{eq} + \mathcal{O}(\delta t^3). \end{aligned} \quad (\text{B.2})$$

354 Eq. (37) is also written, via a Taylor expansion, as

$$\begin{aligned} S_\phi \left(\mathbf{x} - \frac{1}{2} \mathbf{e}_\alpha \delta t, t - \frac{1}{2} \delta t \right) &= S_\phi(\mathbf{x}, t) \\ &- \frac{1}{2} \delta t \left(\frac{\partial}{\partial t} + e_{\alpha i} \frac{\partial}{\partial x_i} \right) S_\phi(\mathbf{x}, t) + \mathcal{O}(\delta t^2). \end{aligned} \quad (\text{B.3})$$

355 From the Chapman-Enskog expansion, we have

$$\bar{f}_\alpha = \bar{f}_\alpha^{(0)} + \delta t \bar{f}_\alpha^{(1)} + \delta t^2 \bar{f}_\alpha^{(2)} + \mathcal{O}(\delta t^3). \quad (\text{B.4})$$

356 After substitution of Eqs. (B.1), (B.2), (B.3) and (B.4) into Eq. (32), the
 357 equation is to order δt^0

$$\bar{f}_\alpha^{(0)} = \bar{f}_\alpha^{eq}, \quad (\text{B.5})$$

358 to order δt

$$\left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \bar{f}_\alpha^{(0)} = -\frac{\bar{f}_\alpha^{(1)}}{\tau} + w_\alpha S_\phi, \quad (\text{B.6})$$

359 and to order δt^2

$$\begin{aligned} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \bar{f}_\alpha^{(1)} - \frac{1}{2} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 \bar{f}_\alpha^{(0)} &= -\frac{\bar{f}_\alpha^{(2)}}{\tau} \\ &+ \frac{1}{\tau} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \bar{f}_\alpha^{(1)} - \frac{(2\tau - 1)}{2\tau r} e_{\alpha r} \bar{f}_\alpha^{(1)} \\ &- \frac{w_\alpha}{2} \left(\frac{\partial}{\partial t} + e_{\alpha i} \frac{\partial}{\partial x_i} \right) S_\phi. \end{aligned} \quad (\text{B.7})$$

360 Substitution of Eq. (B.6) into the above equation gives

$$\frac{(2\tau - 1)}{2\tau} \left(\frac{\partial}{\partial t} + e_{\alpha i} \frac{\partial}{\partial x_i} \right) \bar{f}_\alpha^{(1)} = -\frac{\bar{f}_\alpha^{(2)}}{\tau} - \frac{(2\tau - 1)}{2\tau r} e_{\alpha r} \bar{f}_\alpha^{(1)}. \quad (\text{B.8})$$

361 Taking $\sum$ [Eq. (B.6) + $\delta t \times$ Eq. (B.8)] yields

$$\frac{\partial}{\partial t} \sum_\alpha \bar{f}_\alpha^{(0)} + \frac{\partial}{\partial x_i} \sum_\alpha e_{\alpha i} \bar{f}_\alpha^{(0)} = \frac{\partial \Gamma_i}{\partial x_i} + \frac{\Gamma_r}{r} + S_\phi, \quad (\text{B.9})$$

362 where

$$\Gamma_i = -\frac{(2\tau - 1)\delta t}{2\tau} \sum_\alpha e_{\alpha i} \bar{f}_\alpha^{(1)}, \quad (\text{B.10})$$

363 and

$$\Gamma_r = -\frac{(2\tau - 1)\delta t}{2\tau} \sum_\alpha e_{\alpha r} \bar{f}_\alpha^{(1)}. \quad (\text{B.11})$$

364 Inserting Eq. (B.6) into Eq. (B.10) leads to

$$\Gamma_i = \frac{\delta t}{2} (2\tau - 1) \sum_\alpha e_{\alpha i} \left(\frac{\partial}{\partial t} + e_{\alpha j} \frac{\partial}{\partial x_j} \right) \bar{f}_\alpha^{(0)}, \quad (\text{B.12})$$

365 or

$$\Gamma_i = \frac{\delta t}{2} (2\tau - 1) \left(\frac{\partial}{\partial t} \sum_\alpha e_{\alpha i} \bar{f}_\alpha^{(0)} + \frac{\partial}{\partial x_j} \sum_\alpha e_{\alpha i} e_{\alpha j} \bar{f}_\alpha^{(0)} \right). \quad (\text{B.13})$$

366 The order analysis indicates that $\partial/\partial t \sum_\alpha e_{\alpha i} \bar{f}_\alpha^{(0)}$ has order of $\rho U_c^2/t_c$ from
 367 Eq. (30), and $\partial/\partial x_j \sum_\alpha e_{\alpha i} e_{\alpha j} \bar{f}_\alpha^{(0)}$ has order of $\rho e^2 U_c/L_c$ from Eq. (31), based
 368 on which the ratio of the former to the latter has the order of

$$\begin{aligned} \mathcal{O} \left(\frac{\partial/\partial t \sum_\alpha e_{\alpha i} \bar{f}_\alpha^{(0)}}{\partial/\partial x_j \sum_\alpha e_{\alpha i} e_{\alpha j} \bar{f}_\alpha^{(0)}} \right) &= \mathcal{O} \left(\frac{\rho U_c^2/t_c}{\rho e^2 U_c/L_c} \right) \\ &= \mathcal{O} \left(\frac{U_c}{e} \right)^2 = \mathcal{O} \left(\frac{U_c}{C_s} \right)^2 = \mathcal{O}(M^2). \end{aligned} \quad (\text{B.14})$$

369 This suggests that the first term in Eq. (B.13) is much smaller compared
 370 to the second and can be dropped if $M \ll 1$, which again conform to the
 371 lattice Boltzmann method; hence Eq. (B.13) can be approximated by

$$\Gamma_i = \frac{\delta t}{2} (2\tau - 1) \frac{\partial}{\partial x_j} \sum_\alpha e_{\alpha i} e_{\alpha j} \bar{f}_\alpha^{(0)}. \quad (\text{B.15})$$

372 Applying Eq. (31) into above, we have

$$\Gamma_i = \nu \frac{\partial(\rho u_\phi)}{\partial x_i}. \quad (\text{B.16})$$

373 Similarly, we have

$$\Gamma_r = \nu \frac{\partial(\rho u_\phi)}{\partial r}. \quad (\text{B.17})$$

374 Substitution of Eqs. (29), (30), (B.16) and (B.17) into Eq. (B.9) results in
 375 the governing equation (24) if the density variation is assumed to be small
 376 enough.

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Figure 5: Streamlines for the case with $A = 2.5$ and $R_e = 2200$, showing two pairs of the fully developed vortex breakdown bubbles.