Connecting the everyday with the formal: the role of bar models in developing low attainers' mathematical understanding

Sue Hough, Steve Gough and Yvette Solomon

Manchester Metropolitan University

Use of the bar model has gained momentum in England in recent years through the introduction of Singapore maths. Yet bar models originating from the Dutch approach known as Realistic Mathematics Education (RME), such as the fraction bar, the percentage bar and the double number line, have been available since the late 1990s. In this paper, we discuss the use of the bar in an intervention with low-attaining students in which we employed the RME approach. RME bases understanding in the everyday, where the role of the bar is to sustain modelling across multiple contexts, building on students' informal models. We argue that this context-driven 'bottom-up' use of the bar is crucial in supporting progress towards formal mathematics, highlighting important issues to consider in the use of bar modelling, particularly with low attaining students. We suggest a consequent need for caution in use of the Singapore bar as a potential 'top-down' model.

Keywords: Bar model, Realistic Mathematics Education, Singapore bar, Low attainers

Background: different versions of the bar model

The Singapore bar

The success of the Pacific Rim countries in international tests such as PISA (OECD, 2016) has led to the promotion of approaches from Shanghai and Singapore in England, with 'Singapore maths' gaining considerable popularity. Drawing on the work of a number of theorists, but in particular on Bruner's enactive, iconic and symbolic modes of representation (Bruner, 1966), Singapore maths' Concrete, Pictorial, Abstract (CPA) framework focuses on the shift from concrete to abstract via pictorial representations, and features bar modelling as a means of analyzing and solving arithmetic and algebraic word problems. Students are taught to recognize a problem type (part-whole, comparison, before-after) and then apply the bar model procedure to find the solution (Ban Har, 2010). Children in Singapore are taught the method in first or second grade, beginning with the introduction of pictorial representations of quantities using familiar objects such as teddy bears before moving to rectangles in the bar model (Ng & Lee, 2009). Figure 1 shows the comparison model in arithmetic and algebraic forms, used to answer the questions: 'Dunearn Primary School has 280 pupils. Sunshine Primary has 89 pupils more than Dunearn Primary. Excellent Primary has 62 pupils more than Dunearn Primary. How many pupils are there altogether?' (arithmetic) and 'A cow weighs 150kg more than a dog. A goat weighs 130kg less than the

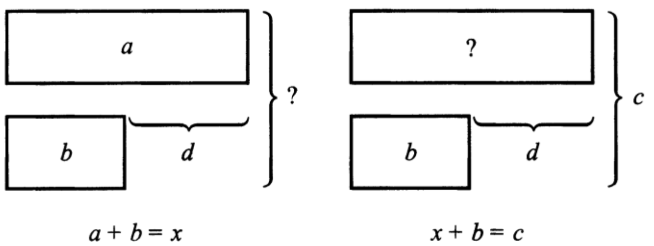

Figure 1 Comparison bar model with arithmetic form on the left, algebraic form on the right. (From Ng & Lee, pp. 287-8)

cow. Altogether the three animals weigh 410kg. What is the mass of the cow?' (algebraic) (Ng & Lee, 2009, p. 286).

Taken up by the influential National Centre for Excellence in Teaching Mathematics (NCETM) in England, bar modelling 'is introduced within the context of part/whole relationships… It exposes the relationships within the structure of the mathematics, which are used to find the unknown elements and thus supports the development of algebraic thinking.' (Griffin, nd). Presented in this way, bar modelling focuses on applying a model once the problem structure has been analysed, or as part of the process of its analysis, with the teacher supplying the model and its associated method.

The RME use of the bar

In RME, bar models – including the fraction bar, the percentage bar, the double number line and the ratio table – also take the role of bridging the gap between the learner's informal understanding of 'reality' on the one hand, and the understanding of more formal systems on the other. However, the connection between reality and formal mathematics differs quite fundamentally, in that models are generated from students' understandings of the context, rather than being offered by the teacher. Mathematical activity emerges from students' informal models, in contrast to the Singapore use of the bar model which focuses from the outset on the structure of the mathematical problem and the relationships between the numbers involved. This emergent characteristic of RME models means that context plays a crucial role in the process of formalization, supporting extended discussion of models which are then generated as mathematisations of reality rather than being presented as tools for solving particular types of problems[1].

In RME, students are exposed to many contextual situations which can be represented by a 'model of' that particular situation. One of the main reasons for choosing a particular context for students to work with is that when they make drawings to represent that context, they produce a 'model of' the context which the designer knows has potential for developing mathematical thinking. A subway sandwich becomes a rectangular bar when drawn with the ends squared off. Sharing that sandwich fairly, marking cuts on the rectangular representation of the sandwich and labelling the pieces with fractions, leads to a bar model picture - a fraction bar. Other contexts such as shading a rectangular shaped theatre to represent the percentage of seats filled would lead to a percentage bar type of bar model. For some contexts, eg marking bottle stops on a race route, it may be more appropriate to draw a line showing distance on one side and bottle stop positions on the other. This is sometimes described as a 'double number line' but would still be classed as a type of bar model, where the bar has been flattened to look like a line. Essentially, models in RME "should 'behave' in a natural, self-evident way. They should fit with the students' informal strategies – as if they could have been invented by them…" (Van den Heuvel-Panhuizen, 2003, p. 14).

This use of models has implications for the conceptualisation of progress in RME. Whereas the Singapore model is based on Bruner's enactive-iconic-symbolic framework as a progression

[1] It is important to note that in RME 'real' means imaginable (Van den Heuvel-Panhuizen, 2003, pp. 9-10).

heuristic, in RME, progress is indicated when students start to see the similarities in the 'model of' situations, enough to be able to generalise the use of these models and apply them to other problems. At this point they may be in a position to draw and use a bar to represent a situation which is not obviously 'bar like', using a bar as a 'model for' solving a problem. Thus progress is defined in terms of formalisation of models (Van den Heuvel-Panhuizen, 2003), and in particular the progression from 'model of' to 'model for' (Streefland, 1985). In terms of fractions, this can be seen in Figure 2, where the model of sharing the submarine sandwich is linked to a model for the formal comparison of fractions.

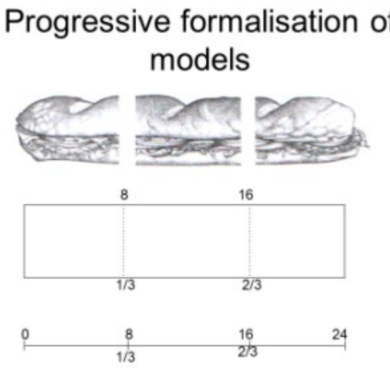

Figure 2 Progression from 'model of' to 'model for'

This shift is also described as vertical mathematisation (Treffers, 1987), where students recognise the mathematical sameness of different problems and are able to choose an appropriate model to solve a problem. Models, in this view, are far more than given strategies for solving problems; they are central to building the understanding needed to deconstruct a problem. In particular, models in RME depend on a 'bottom-up' process in contrast to a 'top-down' approach which offers the model ready-made, to be overlaid onto the problem.

The contrasts between the use of models in RME and the Singapore method raise some questions regarding the current popularity of the Singapore bar – in particular, "what is being modelled, and by who?" The Singapore bar appears to be introduced by the teacher as a *model for* a mathematical problem rather than being generated by the learner from a context which will support the development of a situated *model of* that context which they own and can then build on in vertical mathematisation. In this paper, we explore how a group of low-attaining students worked with the bar in an RME-based intervention which aimed to support them to progress beyond poorly learned algorithms towards a deeper understanding of mathematics. Thus we ask the following research questions:

1. How do students use the bar within an RME context?
2. To what extent do students make progress in terms of moving towards 'model for'?

We consider the implications for the widespread adoption of Singapore bar modelling in the English context of pressure to move quickly towards formal mathematics and algorithmic approaches.

Methodology

In England, a large percentage (around 30%) of students fail to gain an acceptable pass grade in mathematics in national examinations (GCSE) at the age of 16 each year, and resit success rates are poor (DFE, 2017). Resit courses tend to be short and focus on recapping methods which students have already failed to learn and retain, hence the larger study of which this is a part investigated the impact of materials based on RME in four GCSE resit classes across three different sites in the North-West of England, during 2014-15. The research team developed and delivered two short modules focusing on number (12 hours) and algebra (9 hours), employing a quasi-experimental design. For reasons of space, we focus on the number module in this paper. Number teaching materials were designed to draw on contexts which supported students in producing situation-dependant bar-type representations and solutions, thus enabling the natural emergence of the bar as a 'model of'. For example, fair sharing problems required students to share out candy strips, pizzas and ribbon, all of which were chosen for their rectangular and hence bar-like properties. Lessons were delivered by three members of the research team, all experienced RME teachers. They followed the RME practice of emphasis on discussion of the context, moving from students' informal (and life-like) representations of the context (eg buying ribbon, mapping water stops on a fun run, sharing a subway sandwich) to discussion of their various strategies for problem solution, and finally to bar models produced by the students.

One of the questions for this intervention is whether students can recognise the potential of the bar model and associated strategies as '*models for*' tackling problems where use of the bar is not suggested by situation-specific imagery (see Van den Heuvel-Panhuizen, 2003, pp 17-29). All students took a short test prior to and at the end of each module. The six problems of the number test were chosen to reflect a range of topics in proportional reasoning, and covered ratio, proportion, finding a percentage of an amount, finding a fraction of an amount, comparing two rates and a reverse percentage calculation. Each question made reference to a context, but unlike in the teaching materials, the contexts were not suggestive of a bar-like representation. Questions were designed to reflect typical GCSE questions on the target subject matter, but students were additionally asked to explain their answers in order to reveal differences in levels of conceptual understanding.

Seventy-five students participated in the intervention classes and 72 in control classes. In the main study, independent evaluators found small but significant gains for the intervention group on the number post-test ($F_{1,93}=4.55$, $p=0.035$, Cohen's $d = 0.26$). They also found a significant correlation between students' improvement from pre- to post-test and the extent to which they used an RME approach (use of the bar or ratio table to solve the problem) ($r = .258$, $n = 86$, $p = .016$) (see Hough et al, 2017, Appendix 8.5). Here, we investigate these findings further through qualitative analysis of the number pre- and post-test scripts in order to identify how students used the bar and its contribution to their learning. Our analysis involved comparing pre-test and post-test solutions by categorising responses given in terms of: (1) use of the bar or ratio table versus standard methods/trial and error; (2) correct labelling and division of bar parts and scaling accuracy; (3) bar strategies employed such as partitioning and adding blocks; (4) error types such as inappropriate halving, incorrect adding, incorrect bar representation; and (5) evidence of progress.

Findings

The extent to which students could vertically mathematise their use of the bar

In post-test scripts, 73 % of intervention students chose to draw a bar-type model for at least one of the questions. This was in stark contrast to pre-test scripts where students' methods referred to operations denoted purely by numbers and symbols. It was extremely rare for students who had not been exposed to RME materials to draw a pictorial model in order to tackle a problem. So, on the surface, it would appear that the intervention led many students to recognise the potential of the bar model as a model *for* tackling a variety of problems, and in that sense to vertically mathematize.

However, it is not enough for students to simply apply the bar model method to the problem. One of the factors which distinguishes RME models is whether the learner perceives the model to be 'emergent' or 'imposed'. Emergent models are described by Gravemeijer and Stephan (2002) as one of the key design heuristics of RME: models are grounded in reality, they make sense to the learner and even when the model progresses to a more formal version, the learner can always fall back on the original meaning in order to make sense – they are 'bottom up' (p. 146). But although the designer may intend engagement with the model to be 'bottom up', this does not necessarily guarantee the model will be experienced in this way. Hence, there may be cases where a learner's experience of working with RME-based models results in the classical 'top down' engagement whereby the model is imposed by the teacher in the traditional sense of demonstrating formal procedures. In the next two sections we look at examples of students using the RME bar model from both a 'bottom up' and 'top down' perspective.

The bar model as a 'bottom up' sense making strategy / model

For many post 16 GCSE resit students, their experience of learning mathematics involves being shown formal procedures which they can make little sense of, and which they are rarely able to accurately reproduce. In the pre-tests, we found many examples of students mis-representing a previously taught procedure including: trying to find 17% of £3300 by writing it as $\frac{3300}{17}$ ×100; working out $\frac{5}{8}$ of £600 by replacing $\frac{5}{8}$ with 0.58; and resorting to incorrect additive strategies for two proportionally related quantities. However, analysis of the post-test scripts revealed several examples of sense-making through use of an RME bar model, as in the following three examples.

Student A's pre-test solution (Figure 3) may be interpreted as an example of a mis-remembered procedure, or an attempt to hand out 2 lots of £140 to one person and 5 lots to the other. In the post-test (Figure 4), in an answer typical of 33% of scripts, she has drawn a bar split into 7 parts, and has used shading to distinguish the portions on £140 allocated to Pat and Julie. Her bar is labelled as a continuum from 0 to £140. She selected a division procedure which she was able to justify in interview because 'there's 7 boxes, it's 140 for the whole thing'. Through creating a bar model representation, the student is now able to correctly identify the required sharing processes.

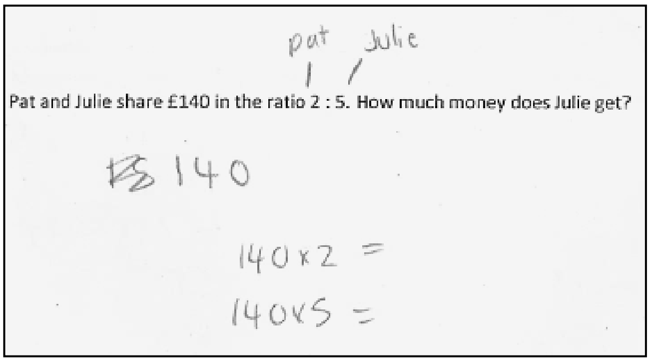

Figure 3 Pre-test Student A

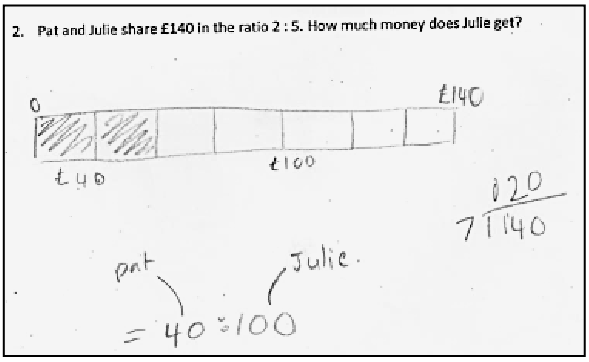

Figure 4 Student A post-test

In the next example, Student B reveals a knowledge of the standard formal procedure used to find a fraction of an amount in her pre-test, but by her own admission she is unable to calculate the division 600 ÷ 8 (Figure 5). In the post-test, and in common with 20% of scripts, she was able to work out an alternative strategy, targeting the values of $\frac{4}{8}$, $\frac{2}{8}$, $\frac{1}{8}$ of £600 and combining these to find $\frac{5}{8}$, as illustrated in Figure 6. By representing the problem using a bar, she was able to engage with informal approaches and see the relationships between the numbers.

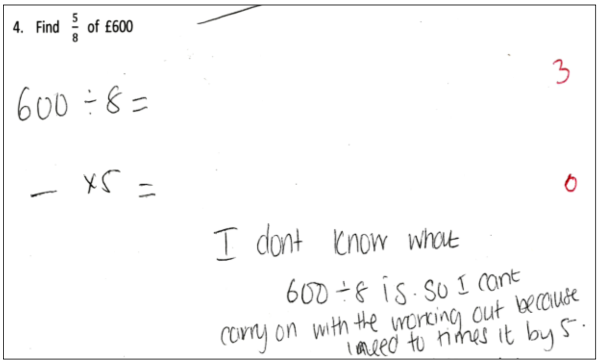

Figure 5 Student B pre-test

The third example concerns a question requiring students to find the original price of a car, when the current price of £6820 is 20% less than the original. None of the control students gained marks on this question in either the pre- or the post-test. By representing the problem on a bar, 18% of intervention students were able to see a straightforward route to the solution as illustrated in Figure 7.

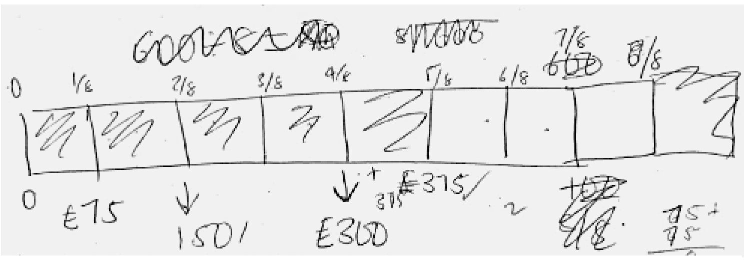
Figure 6 Using a bar model representation to find 5/8 of £600

Reverse percentage problems of this type are notoriously difficult for students to access, not least because the standard method involves setting up an equation in one unknown. By representing the information on a bar, students can be encouraged to fill in what else they know and through this may be able to build up to the required percentage amount.

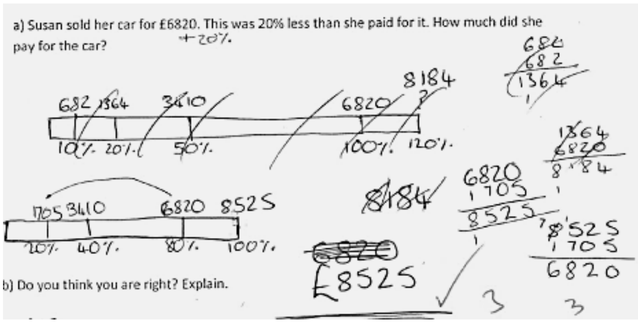

Figure 7 Representing reverse percentage on the bar

The bar model as yet another 'top down' procedure/model

We have seen examples above where the bar model representation enabled students to re-frame what the question was asking for and to link this to mathematical strategies which they could make sense of and therefore use effectively. However, in the next two examples, we see students able to operate with the bar model, but their choice of moves suggests that they are unable to connect the model with meaningful mathematical activity. For example, the student in Figure 8 persisted in going through several cycles of halving and combining chunks to fill in as far as the $\frac{1}{16}$th way point on his bar, but given that the bar is split into 7 pieces, these calculations served no purpose in finding one part. Some students – around 10% of scripts - insisted on always filling in the half- and the quarter- way points of their bars, even when the divisor was an odd amount, suggesting an over-generalisation of the strategy.

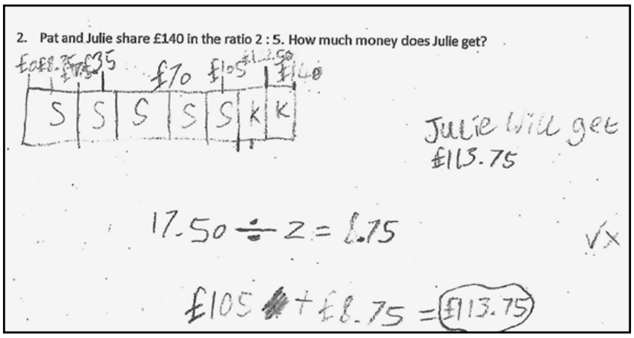

Figure 8 Over-generalisation of the halving strategy

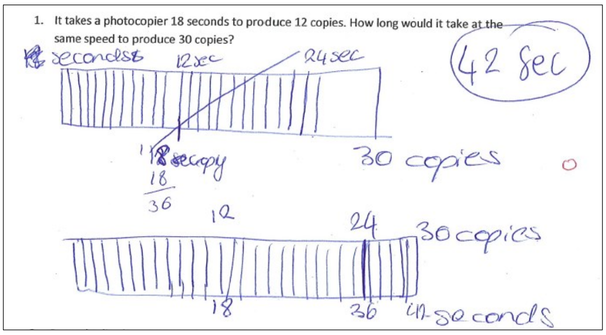

Figure 9 Using the bar in top down fashion

In common with 18% of scripts (38% used a ratio table), the student in Figure 9 used a bar model to represent the situation in which a photocopier takes twice as long to produce twice as many copies, marking up the bar in one second segments and extending it in an attempt to reach a solution. However, the subsequent error of adding one second for every one copy implies that drawing the bar has not afforded this student genuine insight into the underlying structure of this problem. Rather, the bar model has been used as a 'top down' model, as though it is a half–remembered procedure, potentially accounting for errors in 6 out of the 10 scripts using the bar in this question.

Discussion

These findings show that the bar has the potential to enable progress. Drawing a bar-type model prompted students to think differently about the problems, allowing some to engage with informal sense-making strategies, as well as providing a structure within which to organise and record their thinking. Using the models encouraged the students to be flexible and creative. Once they had drawn the model they were free to fill in other quantities as they chose, and this placed much less demand on memory. Many students applied an RME method to several post-test questions, suggesting that they were beginning to recognise the unifying potential of the models as a strategy for answering questions across a range of topics in number, and that they were beginning to vertically mathematise and see how to apply their context-specific models to a range of problems.

However, the transition to using a bar model is not straightforward nor is it a magic fix. Indeed, the issue of inappropriate halving suggests that students can move towards treating the bar as yet another algorithmic method. The intervention was short and as such represented an over-simplification of the RME process, but this served to expose the fragility of these particular students with regard to number sense and their ability to make connections. The complexity of their response to the bar, and the importance of its use for sense-making and not just method raises questions for the Singapore approach to bar-modelling. The significance of this issue is highlighted by Ng and Lee's (2009) analysis of Singapore 5th-graders' solutions to a variety of problems, in which high achievers scored well, but mid-level achievers' strategies often showed erroneous use of the model. Incorrect solutions involved the omission or misrepresentation of crucial information, changing unit generators mid-solution, failure to keep the overall goal in mind, or lack of necessary conceptual knowledge (of fractions, for example) - children were using the method algorithmically, failing to monitor what they were doing. Arguing for the importance of discussion in the classroom about different strategies for solution and the development of meta-cognition in problem representation, Ng and Lee (2009) note that the bar model must be used 'as a problem-solving heuristic that requires children to reflect on how they would accurately represent the information presented in word problems …. This art of representation has to be taught, but it is then the children's responsibility how they choose to use this heuristic effectively' (pp.311-2). Ng and Lee's warning underlines the potential danger in the Singapore bar's reliance on formal conventions in terms of which bar model is required to solve a problem and how the bars should be labelled. From an RME perspective, we would ask what are the contextual mediums that enable struggling students to gain access to these conventions? The model offers learners a way to represent algebraic word problems, but how do students lacking number sense negotiate meaning once the bars can no longer be drawn to scale, or when, as Ng and Lee (2009, p. 308) note, they are challenged by the concept of fraction and represent the relationship between rectangles erroneously? We suggest that these considerations need to be taken into account in the adoption of Singapore bar modelling, where students, not the teacher, need to own the model.

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