Pre-service teachers using core philosophical questions to analyze mathematical behavior

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Course overview

In this chapter, I discuss a course in the philosophy of mathematics designed to help future high school mathematics teachers develop an understanding of philosophical questions about mathematics. My aim was to equip these future teachers with philosophical skills for analyzing mathematical behavior. Teachers who can analyze their students' mathematical activity for how concepts are at work can get beyond simple evaluative responses to students – typically assessing their performance as right or wrong – and can begin to explore their students' mathematical thinking. This is a difficult skill to develop, and philosophy offers one way of doing so. Philosophical questions about mathematics open up discussions about why we have the mathematics we have, inviting consideration of how mathematics is embodied in particular material practices.

The pre-service teachers in this course read and discuss primary and secondary literature in the philosophy of mathematics, compose a formal argument in support of a position on a core philosophical question, and design experiments where they collect empirical data from research participants.[1] These experiments usually consists of a set of mathematical tasks or interview questions that focus on some core philosophical question concerned with the nature of mathematical infinity or the role of diagrams in proofs, or some other topic. The pre-service teachers then select 5 to 10 people to engage in the activity - either mathematicians, siblings, peers or others. They record the responses using either video or audio, or in some cases just observation, and also collect any written artifacts. They then analyze the data they've collected, through the lens of their selected core question, and reflect on how their findings inform their understanding of mathematics and mathematics teaching.

Throughout the course, our discussions link core philosophical questions to particular pedagogical approaches to mathematics education. Thus the course mixes traditional philosophy of mathematics with the study of lived experience and everyday embodied habits, approaches more often associated with phenomenology and other continental traditions of philosophy.[2] Such a mixture is extremely challenging, in part because so much of the philosophy of mathematics has historically framed its questions in abstract analytic terms, and in part because this approach demands applying theoretical tools to the study of everyday practice. Corfield (2003) calls this kind of work *descriptive epistemology*

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[1] During the last decade, *Design experiment*s have become the dominant paradigm in educational research methodology. See the seminal paper by Cobb et al (2003) for details. In general, design experiments entail engineering a particular task for one person or a small group of people to engage with, and then observing or videotaping the activity for how it sheds light on learning theories.

[2] For a quick introduction to phenomenology, see http://plato.stanford.edu/entries/phenomenology/ (accessed April 20, 2014).

insofar as it entails interpreting mathematical activity for how it reflects certain philosophical assumptions about the nature of mathematics.

Students at Adelphi University enroll in the course often not knowing anything about philosophy, let alone the philosophy of mathematics. They are usually in their fifth year of a combined bachelor degree in mathematics and master's degree in education, and are just beginning to go into high school mathematics classrooms and practice-teach. Many of them are returning to high school classrooms where procedural drill is the norm for instruction, which runs counter to the inquiry methods they've been learning about in their teaching methods courses as well as the problem-based learning methods they tend to see in their undergraduate mathematics courses at our college. I open the course with a set of statements that Brown (2008) calls the common "image" of mathematics. We go through each statement, and discuss whether they agree or disagree with it. This is often an opportunity to unpack some of the vocabulary within the philosophy of mathematics, and I use it as a way to model traditional philosophical arguments, where initial work must be done to define the terms before arguing one's position. These statements are:

1. Mathematical results are certain
2. Mathematics is objective
3. Proofs are essential
4. Diagrams are psychologically useful, but prove nothing
5. Diagrams can be misleading
6. Mathematics is wedded to classical logic
7. Mathematics is independent of sense experience
8. The history of mathematics is cumulative
9. Computer proofs are merely long and complicated regular proofs
10. Some mathematical problems are unsolvable in principle

As we move through the list, I contest any easy collective consensus, and put forward alternative views. I have taught the course for three years, and each time students tend to agree with #1-5 without much deliberation. I believe their agreement with the first five is due, in part, to the fact that these statements seem compelling and commonsensical, and the students haven't been trained to query or interrogate the hidden assumptions of such statements. Their hesitation with #6 reflects their belief that logic is one kind of mathematics, studied in a course on logic, and not related to the rest of their mathematics courses. This strange disconnect seems to be prevalent among the many pre-service teachers I have taught, as well as many high school teachers that I've worked with. As we move to the last four statements, they become less confident about their responses. They often initially don't know what it would mean for the history of mathematics to be or not be cumulative, nor have they ever compared computer proofs with 'regular' proofs in terms of their structure or epistemic claims. Unsolvability "in principle" is not something they have ever encountered. So the list and the kind of dissonance it produces is an effective way to introduce some of the core issues in the philosophy of mathematics. The one statement that they strongly disagree with is #7, and they cite their own learning experiences as evidence of how it is incorrect. I ask them to elaborate, to reveal how this response might be linked to philosophical assumptions about the role of the body and the material world in mathematics. I also introduce into the mix a few references to famous events in the history of mathematics, to help them begin to think about these statements from a more global perspective.

The core questions

Students are exposed to the following set of core questions on the first day of class. These core questions drive many of the discussions and course assignments.

1. Can a diagram function as a mathematical proof?
2. What is the nature of proof? How has mathematical proof changed over history?
3. Is there such a thing as mathematical intuition? Where is it? Is it innate?
4. Is mathematics indispensable to science? Could science work without math?
5. What should the relationship be between logic and mathematics?
6. What is the status of axioms? Are they grounded in reality?
7. Are mathematical propositions *necessarily* true (or false) (rather than culturally or contingently true/false)?
8. Is mathematics a language (a system of symbolic signs and not a part of the physical world)?
9. Can we speak about actual infinity (or just potential infinity)? How has the concept of infinity influenced the development of mathematics?
10. What is the role of the body in doing mathematics? How is mathematical knowledge embodied?
11. Is mathematics discovered or invented? What are the ontological implications of your answer?
12. Is mathematics objective and certain (rather than subjective and open to revision)?

Another essential discussion on the first day of class focuses on the difference between epistemic and ontological concerns. The need to keep these two concerns separate while understanding their relationship helps considerably as they go on to formulate arguments to support their positions on the core questions. Perhaps because these are education students, they seem more at home with epistemological questions (How do we come to know the concept of number?) and are initially baffled by ontological questions (What is number?).[3] I have learned to motivate the latter by suggesting that their future students will ask of mathematics “what does this have to do with anything?” and that they should treat this as a philosophical question rather than offer platitudes or lies that students see through (“You need math in life.”). I suggest that they can unpack the question posed by their own students – the question of relevance – by discussing how this question has fueled the entire history of the philosophy of mathematics. Rather than dismissing these students as bored and not motivated to learn, I suggest that they help these students build their instinctual response (to “How is this relevant?”) into a philosophical engagement with the nature of mathematical knowledge. They can tell their students about the various schools of thought that developed as a means of answering it. They could even, I suggest, introduce one of the core questions into their lessons, as a motivator for all those astute students who have asked this difficult question.

As a way of integrating the history of mathematics with the philosophical questions, pre-service teachers research and present a 10-minute slideshow on a discussion topic each week. Sample topics are: Zeno’s paradoxes, the parallel postulate, fractal geometry, zero. Presentations focus on the history of the topic, and as instructor I introduce links between the topics and the core philosophical questions that structure the course. For instance, a presentation about the parallel

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[3] Epistemological questions concern the nature of knowing, while ontological questions concern the nature of being. Students are trained to parse questions according to these two concerns. For instance, the question “How do we become aware that numbers don’t necessarily have to be numbers of something?" is both epistemological insofar as it asks about how we are “aware” and also ontological in that it asks about the nature of number.

postulate might simply recount attempts to prove it and mention developments of non-Euclidean geometry without linking these developments to our readings about Kant and his claims about the *a priori* synthetic nature of geometry. Further links need to be raised that help the students grasp how this topic is related to questions about the certainty and objectivity of mathematics, and its relation to science. It has always intrigued me that these students, despite being immersed in mathematics, a field known for its careful deductive methods, struggle so much in composing a formal philosophical argument. Many of these students confess to having selected mathematics because they don't enjoy reading and writing. However, I feel strongly that, as future teachers, they need to become excellent communicators, and I treat the course as an opportunity to build that skill as well. I have designed guidelines to help them structure their assignments, and I work with various draft versions of their papers to help them improve this skill.

General philosophical themes

The distinction between ontology and epistemology helps us narrow in on students' assumptions about mathematical practices, as we discuss how Platonism and other schools of thought consist of an ontological claim *and* an epistemological claim. In this we follow Bostock (2009). We ask: In what sense can universals (redness or beauty or triangles or numbers) be said to "exist"? This, as Bostock reminds us, is a question about the ontological status of universals. Most students don't quite know how to engage with this question, although they are more than ready to grant universality (generality) to geometric figures or arithmetic entities like numbers. They tend to think of this generality as cross-cultural, and I discuss with them how the question also pertains to the metaphysical. I offer them some choices: If universals do exist, do they exist outside the mind, or simply as mental entities? If they exist outside the mind, are they corporeal or incorporeal? If they exist outside the mind, do they exist in the things that are perceptible by the senses or are they separate (or independent) from them? To further support and scaffold their exploration of these questions, I offer three schools of thought, each with a different answer to these questions, and I ask the students to decide who they most identify with. I am really forcing their hand in this, in that I hope to show them that these three responses do not actually exhaust the possible answers to the ontological question. In the next section, I discuss how new directions in the philosophy of mathematics offer different choices. But the choices first given, drawn from those used by Bostock (2009), are simplifications so that they can begin to engage in debate. As in all such sorting and labeling, we can query whether a particular mathematician or philosopher is a good example of a particular philosophical paradigm, and I am careful to tell the students that they will debate these issues later, after reading more primary texts:

•The *realist* (Plato, Frege, Godel) claims that universals exist outside the mind and are independent of all human thought.
•The *conceptualist* (Descartes, Kant) claims that they exist in the mind and that they are created by the mind. Some claim that we create these universals based on sense perception and some say they are innate and do not require perceptual stimulation.
•The *nominalist* (Hilbert, Field) claims that they do not exist at all. Some claim that the words and symbols we use are mere shorthand for longer ways of expressing the same idea and some claim that statements with such terms are simply untrue in the sense that they refer to nothing.

The assignment of the names to the schools is not perfect, but it works as a starting point. Gold (2013) points out that the way one speaks reveals in part which school one aligns with:

> Simply to say a certain mathematical statement is true involves taking a philosophical position. If you are a formalist, you say, rather than that "this theorem is true," that "it is a theorem within a given axiom system." For a substantial collection of philosophers of mathematics (nominalists, fictionalists), there are no mathematical truths, because there are no mathematical objects for them to be true about (Gold, 2013, p. 153).

One of the difficulties in starting with the main schools of thought, and then trying to tease out the subtle differences and ways in which these philosophers claims are not perfectly aligned with the school, is that the students are not yet ready to delve deeply into these historical subtleties. For instance, it might seem a travesty to put Descartes and Kant together, since Kant pushed past Descartes' claim that mathematical truths are innate, clear, and distinct ideas, so that he might attend to the synthetic nature of mathematical judgment. According to Kant, space and time are the mind's contribution to experience. Space and time are the "form" of experience, a form imposed by us on the raw data of experience. Historians of philosophy usually oppose Descartes (the rationalist) against Locke and Hume (empiricists). Bostock (2009), however, claims that Locke, Hume and Descartes, despite their differences, share the same beliefs about the ontological status of mathematical objects (they are ideas or mental entities), and differ in how they think we acquire these mathematical ideas. One might then associate Kant with this approach as well, since, as Brown (2008) suggests, according to Kant, "Our *a priori* knowledge of geometric truths stems from the fact that space is our own creation."(p.119) Similarly, arithmetic is connected to time and the fact that time is also a form we impose on the world. This conceptualist approach seems to have saturated many of the later treatments of the philosophy of mathematics, seeping into the realist and nominalist camps as well. Brown indicates that Frege (a Platonist) embraced Kant's view on geometry, Hilbert (the formalist or nominalist) embraced Kant's view on arithmetic, and even Russell (the logicist) can be characterized as Kantian. One might also argue that the conceptualist approach has saturated theories of learning, and has become full-fledged in cognitive psychology and its dominant image of learning. This image assumes that learning entails an acquiring of a set of cognitive 'schemas'. Pre-service teachers need to be aware of this history so that they might become empowered to identify and critique the theories of learning that structure the curriculum policy they are meant to adopt in their classrooms.

The pre-service teachers are shown how much of the philosophy of mathematics since the nineteenth century has been contending with the Kantian assertion that mathematical truths are *a priori* and *synthetic*. Kant claimed that if a proposition is thought as (1) necessary and (2) universal then it is an *a priori* judgment. How can that be possible? This is a perennial question in the philosophy of mathematics, the question as to how *pure* mathematics is possible (Hacking, 2013). Hacking claims that one has to look closely at applications of mathematics if one is to address – or contest – this question of purity. By examining the activity of mathematics one begins to see that application is everywhere, whether it be conventional applications of mathematics to the physical sciences, or one set of mathematical concepts applied to another mathematical field, or simply the fact that the statement 2+3=5 seems to say something about the world that is necessarily true. In other words, the distinction between pure and applied is tenuous at best.

Corfield (2003) argues that the philosophy of mathematics has spent far too much time on the foundational ideas of the 1880-1930 period, and neglected the thinking and doing of "real" mathematicians both before and after that period. Corfied believes that a philosophy of mathematics should "concern itself with what leading mathematicians of their day have achieved, how their styles of reasoning evolve, how they justify the course along which they steer their programmes, what constitute obstacles to these programmes, how they come to view a domain as worthy of study and how their ideas shape and are shaped by the concerns of physicists and other scientists."(p.10) He names this approach *descriptive epistemology* and defines it as the "philosophical analysis of the workings of a knowledge-acquiring practice." (p. 233) Imre Lakatos (1976) is often taken as inspiration in this kind of approach to the philosophy of mathematics. He examined the process of meaning-making in mathematics, by studying the historical evolution of concepts and procedures, and offering insight into the form of deliberation that characterized creativity in the work of mathematicians. He was interested less in the so-called foundational issues in mathematics, and more in the empirical and material making of mathematics, an approach he called "critical fallibilism":

> It will take more than the paradoxes and Gödel's results to prompt philosophers to take the empirical aspects of mathematics seriously, and to elaborate a philosophy of critical fallibilism, which takes inspiration not from the so-called foundations but from the growth of mathematical knowledge. (Lakatos, 1978, p. 42)

Hersh (1997) characterizes Lakatos as a philosopher of mathematics who was committed to studying the social and "humanist" aspects of doing mathematics. For Hersh, Lakatos was a humanist because he celebrated the specificity of informal reasoning found in the work of mathematicians, rather than or in addition to the generality of its truth claims. For Lakatos, these examples of informal reasoning are not simply unfinished formal proofs, in which the pertinent axioms and logical rules of inference are suppressed, but rather a significantly different mode of inquiry, a non-axiomatic argument that has its own trajectory and its own becoming.

Despite the significance of this more humanist perspective on the philosophy of mathematics, which values the study of informal and unfinished mathematical activity by experts, we still lack philosophical insight into the experiences of those – students for instance – who, for the most part, do mathematics from an outsider or fringe position. We have yet to grapple philosophically with the embodied, affective and symbolic aspects of these kinds of mathematical encounters. What would it look like if we borrowed Corfield's descriptive epistemology or Lakatos' "critical fallibilism" to study everyday mathematical behavior? This materialist approach, with its emphasis on various kinds of embodied activity (diagramming, gesturing), and its move away from a purely cognitive or brain-based model, lends itself to the study of mathematics teaching and learning in everyday contexts. In the next section, I describe how my course focuses on the role of diagram and gesture in mathematical activity.

Diagrams, movement and the mathematical body

Questions about the status of diagrams in proofs are easy for students to connect with, and link directly to the opening readings by and about Plato. Students are drawn to the compelling distinction that Plato draws between the physical world and the realm of mathematics. We discuss the theory of ideal forms, and how Plato was motivated by the gap between the ideas we can conceive and the physical world around us. Some students see in the proposal of an ideal realm a way of reconciling their belief in the universality of mathematics with the messiness of learning, but more often than not they are drawn to a conceptualist approach, perhaps Kantian, whereby mathematics is considered an invention that aligns with the physical universe. Thus they tend to ascribe to the human mind a consciousness or intuition that is capable of bringing together the ideal forms (triangles, numbers, etc) that are unchanging and eternal (the realm of being or essence) with the physical realm (the realm of becoming or change). We discuss how there is a strong dualism (between mind and body) at work in this approach, and how this dualism plays out in different pedagogies. The vast majority of pre-service teachers split mind from body, arguing that we grasp the ideal forms only through mental reflection, while we understand the physical world through the senses, just as Plato might say. Most of the contemporary philosophy of mathematics we read in the course questions the validity of this dualism, and we discuss the main criticisms of Platonism that were formulated centuries ago.

Diagrams figure prominently in this discussion, as they have, since Plato, if not before, bridged the dualism in ways that trouble its claim to a clean distinction (de Freitas, 2012). In small groups, the students are given a set of visual proofs from Nelsen (2000), in some cases without the corresponding algebraic expression or equation, and asked "What does this diagram prove?" I use this question to provoke debate, as it gets to the heart of concerns about what constitutes a legitimate proof in mathematics. Brown supplies a number of interesting examples as well, in particular a set of diagrams that might be considered to be proofs of an infinite pattern. Consider, for instance, the diagram in figure #1.

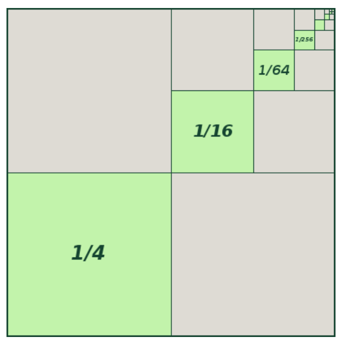

Figure #1: What does this diagram prove?

The students discuss what is entailed in using this diagram to demonstrate that the following equation is true.

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{252}+\ldots=\frac{1}{3}$$

We discuss to what extent the diagram might function as a proof of the statement. When we compare the diagram to a formal proof by induction, the students tend to allow the latter greater mathematical certainty. But what is notable is that over half of the students regularly claim that induction proofs are entirely meaningless to them, and that mathematical induction conveys no certainty at the level of personal belief or conviction. This seems to reinforce the dualism of mind and body, where the diagram stands in for the body, and the formal deductive logic of inductive proof stands in for the disembodied cognitive acts of the mind. My aim is to trouble this distinction so that they might begin to look at the embodied acts of mathematics, in this case the diagram, as exactly where the mathematics is happening.

We read excerpts from Plato (Meno, Theatetus, Rupublic) that help ground the core questions in historical contexts. Although they tend to find Socrates overbearing in the Meno, they begin to grasp how the Socratic method emerges from a particular set of philosophical assumptions about the nature of mathematical diagrams and concepts. We compare this method to the kind of questioning sequences they see in their observations in classrooms. For Plato, geometrical knowledge is obtained by pure thought and divorced from sensory observation, which seems to go against what many of the students experience in mathematics classrooms. This is when they become somewhat unhappy with their Platonism. As Brown (2008) explains, Plato considered the diagram as merely a heuristic to help us "access" the pure forms of mathematics. In principle, a good mathematics student would grasp the ideal form of the circle without the need for a diagram, since these were always tainted by the corruptions of the physical world. Plato is rather disparaging of all this talk of diagrams and gestures:

> The science of geometry is in direct contradiction with the language employed by its adepts … their language is most ludicrous … for they speak as if they were doing something and as if all their words were directed toward action .. [they talk] of squaring and applying and adding and the like … whereas in fact the real object of the entire subject is … knowledge … of what eternally exists, not of anything that comes to be this or that at some time and ceases to be. (Plato, cited in Shapiro, 1997, p. 21)
>
> Don't you also know that they use visible forms besides and make their arguments about them, not thinking about them but about those others that they are like? They make the arguments for the sake of the square itself and the diagonal itself, not for the sake of the diagonal they draw, and likewise with the rest. These things themselves that they mold and draw, of which there are shadows and images in water, they now use as images, seeking to see those things themselves, that one can see in no other way that with thought. (Plato, Book VI, 510d, p. 191)

We discuss the consequences of Platonist and conceptualist approaches to mathematics that deny or dismiss the significance of the activity of doing mathematics and prize instead only the mental or cognitive reasoning faculty. We begin to read contemporary theories of embodied cognition that attack this approach philosophically (Lakoff, G. and Núñez, R., 2000; Nemirovsky et al, 2009, 2012a; Radford, 2009; Roth 2010). The students begin to imagine that diagramming (and other embodied activities) are not merely heuristic but rather necessary for thinking mathematically. We discuss what it might mean for thinking to occur in and through this activity rather than independent of it.

The readings in this section of the course range from physiological to more phenomenological approaches, and through these readings we begin to shape our ontological and epistemological discussions around the question of the role of the body in learning mathematics. We begin to consider how an approach to the body as part of the mathematical learning process impacts our philosophy of mathematics. As Núñez et al (1999) state, situated learning and cognition radically shifts the terrain, and directs our attention to both social and physiological concerns:

> We argue that the nature of situated learning and cognition cannot be fully understood by focusing only on social, cultural and contextual factors. One must also take into account the non-arbitrary biological and experiential constraints that shape social activity and language, and through which cognition and learning are realized in a genuine embodied process. The bodily-grounded nature of cognition provides foundations for situatedness, entails a

> reconceptualization of cognition and mathematics itself, and has important consequences for mathematics education. (Núñez et al, 1999, p.45)

More recently, and more specifically, Alibali and Nathan (2012), direct our attention to how gesture and other micro-actions are taken up in mathematical behavior:

> We argue that mathematical cognition is embodied in 2 key senses: It is based in perception and action, and it is grounded in the physical environment. We present evidence for each of these claims drawn from the gestures that teachers and learners produce when they explain mathematical concepts and ideas. We argue that (a) pointing gestures reflect the grounding of cognition in the physical environment, (b) representational gestures manifest mental simulations of action and perception, and (c) some metaphoric gestures reflect body-based conceptual metaphors. Thus, gestures reveal that some aspects of mathematical thinking are embodied. (Alibali & Nathan, 2012, p. 247)

Roth (2010 and later) draws on *material phenomenology* to make similar claims. He studies a young child's tactile handling of a cube in a classroom, where the teacher is teaching about shapes. Roth shows how the child's hands move spontaneously all over the cube, touching and stroking it, without conscious intention or conscious reflection. According to Roth, the child comes to know the cube through his hands in such a way that his knowledge is pre-reflective. He also argues that it is in the hand – rather than or in addition to the brain - that the memory of the cube is *immanent*. In other words, knowing mathematics entails this material gestural encounter, prior to any synthesis or making linguistic or semiotic sense of it. Through touch and the pre-conscious coordination of the hand, eye and other sensory modalities, the child comes to know what a cube is. This is a phenomenological study of learning, focusing on how concepts live in our embodied activity. Other scholars have argued this point with more advanced mathematics as well (de Freitas & Sinclair, 2012). As Nemirovsky and Ferrara (2009) suggest, "thinking is not a process that takes place 'behind' or 'underneath' bodily activity, but is the bodily activity itself". Roth (2010) points out how the phenomenogical tradition helps us think differently about mathematics, moving away from the proposal that knowing rests heavily on mental representations, and towards the proposal that knowing is enacted or folded into activity of various kinds. In his words, this is a theory of learning mathematics that finally shifts away from the Kantian "intellectualist mind" model that dominates learning theories.

> In Kant's constructivist approach, the knowing subject and the object known are but two abstractions, and a real positive connection between the two does not exist. (Maine de Biran, 1859a,b) The separation between inside and outside, the mind and the body, is inherent in the intellectualist approach whatever the particular brand. (Roth, 2010, p. 9)

My aim in the course is to slowly help pre-service teachers begin to appreciate how this intellectualist and conceptualist model has dominated learning theory in mathematics education, and that contemporary philosophers of mathematics are offering alternative approaches to study the materiality of mathematical thinking. Our discussions of Roth, Nemirovsky and Ferrara, Alibali and Nathan, and Núñez invite the pre-service teachers to consider how philosophical assumptions are at work in the everyday mathematical activity of children as they encounter a physical cube or a symbolic equation. In the next section, I briefly discuss a key assignment of the course.

The application assignment

After the hard work of unpacking the core philosophical questions and gaining some basis in the literature in the philosophy of mathematics, the students end the course with an application assignment, in which they take these philosophical tools and questions, and use them to analyze observed mathematical behavior. I believe that this final assignment aligns with current teacher education concerns that pre-service teachers need to develop skills for analyzing and diagnosing student performance. Future teachers need to learn how to make sense of mathematical behavior for how it conveys student beliefs about particular mathematical concepts (Stevens, 2012). For the application assignment, pre-service teachers individually design their experiments, usually as an interview that includes a set of tasks. Interview questions have included simple opinion questions such as "How would you describe the difference between the infinity of 0,1,2,3,4, … and the infinity of the real numbers?" Tasks have included various mathematical problems such as "Which of the following arguments do you think best proves the following statement? Why?" "Generate as many examples as possible of a polygon" or simply "Talk me through how you would solve this puzzle". Pre-service teachers select these questions and tasks based on their own interests, and hone the wording and the focus with my guidance. Part of my objective is to develop their skill at asking back-up questions that will help their own students become more willing to speculate or make conjectures about mathematics. Once we have a good working set of tasks, and a script for how they will interview their subjects, they proceed to collect the data. Some pre-service teachers are able to implement the experiment in classrooms, where the experiment becomes less an interview and more of a survey, while others select a small group of friends or family or even strangers in a cafeteria. Because of our focus on diagrams, gestures, and embodiment, many of the students design experiments that shed light on how people do mathematics through visual and spatial methods.

In most cases, pre-service teachers learn just how hard it is to generate a task that actually produces ample data to speculate about a participant's thinking. But in some sense, this is the point of the assignment – to force them to pay attention to what the people did and said, and to begin to think about how the mathematical thinking emerges through all this diagramming, gesturing and talking. The analysis, I remind them, can only be tentative and linked to the specific participants under study. But the exercise of interviewing and analyzing, with the aim of studying everyday kinds of mathematical thinking, is an important moment in their training as future teachers. Teachers are more effective when they develop skills at noticing and decoding student gestures, diagrams, and verbal contributions in terms of beliefs about the nature of mathematics. Such skills allow teachers to understand why particular students make sense of mathematical tasks in particular ways. Moreover, the pre-service teachers find that people are genuinely interested in engaging in these tasks and in considering their philosophical questions, and thus they see first-hand how the philosophy of mathematics can be a powerful way of getting their future students to talk and think about mathematics.

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