Rendiconti di Matematica, Serie VII
Volume 37, Roma (2016), 1 – 41

On the stability and integration of Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$

R. M. ADAMS – R. BIGGS – W. HOLDERBAUM – C. C. REMSING

Abstract: *We consider inhomogeneous quadratic Hamilton-Poisson systems on the Lie-Poisson space* $\mathfrak{so}(3)^*_-$. *There are nine such systems up to affine equivalence. We investigate the stability nature of the equilibria for each of these systems. For a subclass of systems, we find explicit expressions for the integral curves in terms of Jacobi elliptic functions.*

1 – Introduction

Poisson structures appear in very different forms and mathematical contexts such as symplectic manifolds, Lie algebras, singularity theory, and r-matrices. Together with symplectic manifolds, Lie algebras provide the first examples of Poisson manifolds. Namely, the dual of a finite dimensional Lie algebra admits a canonical Poisson structure, called its Lie-Poisson structure (see, *e.g.*, [27, 30]). Lie-Poisson structures arise naturally in a variety of fields of mathematical physics and engineering such as classical dynamical systems, robotics, fluid dynamics, and superconductivity, to name but a few.

On Lie-Poisson spaces, quadratic Hamilton-Poisson systems have been considered by several authors (*e.g.*, [3, 11, 22, 24, 30, 33, 34]), most notably in the context of invariant optimal control and geometric mechanics. Rigid body dynamics appear in many areas of engineering such as underwater vehicles, unmanned air vehicles, robotics, and spacecrafts (*e.g.*, [16, 21, 36]). A systematic treatment of stability and integration of homogeneous systems on $\mathfrak{so}(3)^*_-$ was carried out in [17].

Key Words and Phrases: *Hamilton-Poisson system – Lie-Poisson structure – Lyapunov stability – Energy-Casimir method – Jacobi elliptic function*
A.M.S. Classification: 34D20, 33E05, 22E60, 53D17.

In this paper we consider inhomogeneous quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}(3)^*_-$. There are nine (families of) such systems, under affine equivalence. A system will be referred to as a system of type I if its set of equilibria is a union of lines and planes; otherwise, it will be referred to as a system of type II. For the sake of completeness, a brief treatment of the homogeneous systems is included.

For each system we investigate the (Lyapunov) stability nature of the equilibria. A generalization of the energy-Casimir method and the continuous energy-Casimir method (see [31]) are used to prove stability. Note that for any system on $\mathfrak{so}(3)^*_-$ the origin is a stable equilibrium state. (Indeed, the Casimir function $C(p) = p_1^2+p_2^2+p_3^2$ is a weak Lyapunov function for any Hamiltonian vector field on $\mathfrak{so}(3)^*_-$.) On the other hand, instability usually follows from spectral instability; however, a direct approach is required in some cases. Throughout the paper, we graph the stable and unstable equilibria in blue and red, respectively.

We obtain explicit expressions for the integral curves of systems of type I (but not of type II). In each case we partition the set of initial conditions so as to distinguish between integral curves with different qualitative behaviour. The equations of motion are reduced (using the constants of motion) to a single separable differential equation, which is then transformed into a standard form. An appropriate elliptic integral is used to obtain (after some manipulation) an explicit expression for the integral curve in terms of Jacobi elliptic functions. Mathematica is used to facilitate most of these calculations.

We distinguish between integral curves with different qualitative behaviour by determining when the level surfaces of the Hamiltonian and Casimir are tangent to one another. These surfaces are tangent exactly at equilibria. Hence we get a set of critical values (corresponding to equilibria) for the energy states (h_0, c_0) of the Hamiltonian and Casimir. This set partitions the space of energy states into a number of regions. (Within each region, the corresponding nonconstant integral curves can be continuously deformed into one another.) Each region usually corresponds to different explicit expressions for the integral curves.

For each system we graph the critical energy states. We select some typical values for (h_0, c_0) from each region (as well as some typical critical values) for which we then graph the corresponding level surfaces of the Hamiltonian and Casimir. For convenience, we shall refer to this as a typical configuration. The intersection of these surfaces (*i.e.*, the traces of the corresponding integral curves) and the equilibria are also graphed.

The main motivation for our investigation of the quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)$ comes from our ongoing interest in geometric (optimal) control, particularly on lower-dimensional Lie groups. Similar treatments of quadratic Hamilton-Poisson systems on $\mathfrak{se}(2)^*_-$ and $\mathfrak{se}(1,1)^*_-$ can be found in [3, 4] and [7], respectively. After the completion of this work we learned of several substantial contributions to the (generalized) rigid body dynamics literature from the geometric mechanics perspective. It was Volterra [35] who first found expressions of integral curves (in terms of sigma-functions and exponents). More recently, an explicit integration of Zhukovsky-Volterra gyrostat was obtained by Basak [8] (based on an algebraic parametrization of the invariant curves). The stability nature of the equilibria, as well as bifurcations, have been investigated by several authors ([9, 14, 15, 18, 25]). Frauendiener [20] classified quadratic Hamiltonian systems on the unit sphere under symplectic transformations; Elipe and Lanchares [19] showed that each equivalence class obtained by Fraudiener corresponds to a different type of gyrostat. Nonetheless, we are of the opinion that our alternative investigation, from a Poisson geometry point of view, is more elementary and direct and as such lends a fresh perspective to the topic.

We conclude the paper with some comments and remarks concerning the relationship between invariant optimal control problems and quadratic Hamilton-Poisson systems.

2 – Quadratic Hamilton-Poisson systems

Let $\mathfrak{g}$ be a real Lie algebra. The (minus) *Lie-Poisson structure* on $\mathfrak{g}^*$ is given by $\{F,G\}(p) = -p([\mathbf{d}\,F(p), \mathbf{d}\,G(p)])$. Here $p \in \mathfrak{g}^*$, $F, G \in C^\infty(\mathfrak{g}^*)$, and $\mathbf{d}\,F(p)$, $\mathbf{d}\,G(p) \in \mathfrak{g}^{**}$ are identified with elements of $\mathfrak{g}$. The Lie-Poisson space $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. To each function $H \in C^\infty(\mathfrak{g}^*)$ we associate a *Hamiltonian vector field* $\overrightarrow{H}$ on $\mathfrak{g}^*$ specified by $\overrightarrow{H}[F] = \{F, H\}$. A function $C \in C^\infty(\mathfrak{g}^*)$ is a *Casimir function* provided $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$. A linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is called a *linear Poisson automorphism* if $\{F,G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{g}^*)$. Linear Poisson automorphisms are exactly the dual maps of Lie algebra automorphisms.

A *quadratic Hamilton-Poisson system* $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is specified by

$$H_{A,\mathcal{Q}} : \mathfrak{g}^* \to \mathbb{R}, \quad p \mapsto p(A) + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$. If $A = 0$, then the system is called *homogeneous*; otherwise, it is called *inhomogeneous*. (When $\mathfrak{g}^*_-$ is fixed, a system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is identified with its Hamiltonian $H_{A,\mathcal{Q}}$.) We say that two quadratic Hamilton-Poisson systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are affinely *equivalent* if the associated vector fields $\overrightarrow{G}$ and $\overrightarrow{H}$ are compatible with an affine isomorphism. That is, two systems are equivalent if there exists an affine isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $T\psi \cdot \overrightarrow{G} = \overrightarrow{H} \circ \psi$. (Here $T\psi$ denotes that tangent map of ψ.)

Lemma 2.1. *The following Hamilton-Poisson systems (on $\mathfrak{g}^*_-$) are equivalent to $H_{A,\mathcal{Q}}$:*

($\mathfrak{E}1$) $H_{A,\mathcal{Q}} \circ \psi$, *where ψ is a linear Poisson automorphism;*
($\mathfrak{E}2$) $H_{A,r\mathcal{Q}}$, *where $r \neq 0$;*
($\mathfrak{E}3$) $H_{A,\mathcal{Q}} + C$, *where C is a Casimir function.*

The three-dimensional orthogonal Lie algebra

$$\mathfrak{so}(3) = \left\{A \in \mathbb{R}^{3\times 3} \,:\, A^\top + A = \mathbf{0}\right\}$$

has standard (ordered) basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutator relations are given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = E_3$. Let (E_1^*, E_2^*, E_3^*) denote the dual of the standard basis. We shall write an element $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^* \in \mathfrak{so}\,(3)^*$ as $\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. The group of linear Poisson automorphisms takes the form

$$\{p \mapsto p\Psi \,:\, \Psi \in \mathbb{R}^{3\times 3},\ \Psi\Psi^\top = \mathbf{1},\, \det \Psi = 1\} \cong \mathsf{SO}(3).$$

Note that $C(p) = p_1^2 + p_2^2 + p_3^2$ is a Casimir function.

Remark 2.2. The Hamiltonian vector fields on $\mathfrak{so}\,(3)^*_-$ are complete as their integral curves evolve on the compact subsets $C^{-1}(c_0)$, $c_0 \geq 0$ (*cf.* [1]).

Remark 2.3. The Hamiltonian vector field associated to a function $H \in C^\infty(\mathfrak{so}\,(3)^*)$ can be expressed as $\vec{H} = \frac{1}{2}\nabla C \times \nabla H$. Hence the (regular) level sets of H and C are tangent exactly at equilibria.

A classification of the quadratic Hamilton-Poisson systems on $\mathfrak{so}\,(3)^*_-$ was obtained in [2]. We shall base our investigation of quadratic systems on this classification. For the sake of completeness, we provide a sketch of the proof.

Theorem 2.4. *Let H be a quadratic Hamilton-Poisson system on $\mathfrak{so}\,(3)^*_-$. If H is homogeneous, then it is equivalent to exactly one of the systems:*

$$H^0(p) = 0 \qquad \text{(type I)}$$

$$H^1(p) = \tfrac{1}{2}p_1^2 \qquad \text{(type I)}$$

$$H^2(p) = p_1^2 + \tfrac{1}{2}p_2^2. \qquad \text{(type I)}$$

If H is inhomogeneous, then it is equivalent to exactly one of the systems:

$$
\begin{aligned}
H^0_{1,\alpha}(p) &= \alpha p_1 && \alpha > 0 && \text{(type I)}\\
H^1_0(p) &= \tfrac{1}{2}p_1^2 && && \text{(type I)}\\
H^1_1(p) &= p_2 + \tfrac{1}{2}p_1^2 && && \text{(type I)}\\
H^1_{2,\alpha}(p) &= p_1 + \alpha p_2 + \tfrac{1}{2}p_1^2 && \alpha > 0 && \text{(type II)}\\
H^2_{1,\alpha}(p) &= \alpha p_1 + p_1^2 + \tfrac{1}{2}p_2^2 && \alpha > 0 && \text{(type I)}\\
H^2_{2,\alpha}(p) &= \alpha p_2 + p_1^2 + \tfrac{1}{2}p_2^2 && \alpha > 0 && \text{(type I)}\\
H^2_{3,\boldsymbol{\alpha}}(p) &= \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \tfrac{1}{2}p_2^2 && \alpha_1, \alpha_2 > 0 && \text{(type II)}\\
H^2_{4,\boldsymbol{\alpha}}(p) &= \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2 && \alpha_1 \geq \alpha_3 > 0 && \text{(type II)}\\
H^2_{5,\boldsymbol{\alpha}}(p) &= \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2 && &&\\
&\qquad \alpha_2 > 0,\ \alpha_1 > |\alpha_3| > 0 \quad \textit{or} \quad \alpha_2 > 0,\ \alpha_1 = \alpha_3 > 0. && && \text{(type II)}
\end{aligned}
$$

Here $\alpha, \alpha_1, \alpha_2, \alpha_3$ parametrize families of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

Remark 2.5. A stronger form of equivalence, namely orthogonal equivalence, has been considered in [33].

Proof (Sketch). We note that the equivalences ($\mathfrak{E}1$)-($\mathfrak{E}3$) are not always sufficient to reduce a system to its normal form. In such cases, we find an explicit affine isomorphism with respect to which the vector fields are compatible.

Let $H(p) = pA + pQp^\top$, where Q is a symmetric 3×3 matrix. Here $A = a_1E_1 + a_2E_2 + a_3E_3 \in \mathfrak{so}(3)$ is identified with $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\top$. We may assume that Q is positive definite. Given a linear Poisson automorphism $\psi : p \mapsto p\Psi$, we have $(H \circ \psi)(p) = p\Psi A + p\Psi Q \Psi^\top p^\top$. As any symmetric matrix can be diagonalized by an orthogonal matrix (see, *e.g.*, [32]), it follows that there exists a linear Poisson automorphism ψ such that $(H \circ \psi)(p) = p\Psi A + p\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\,p^\top$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Thus $(H \circ \psi)(p) - \lambda_3 C(p) = p\Psi A + p\,\mathrm{diag}(\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0)\,p^\top$ with $\lambda_1 - \lambda_3 \geq \lambda_2 - \lambda_3 \geq 0$. If $\lambda_1 - \lambda_3 = 0$, then (by ($\mathfrak{E}1$) and ($\mathfrak{E}3$)) H is equivalent to an intermediate system $G^0_B(p) = pB$, where $B = \Psi A$. On the other hand, if $\lambda_1 - \lambda_3 > 0$, then $(H \circ \psi)(p) - \lambda_3 C(p) = p\Psi A + (\lambda_1 - \lambda_3)\,p\,\mathrm{diag}\left(1, \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}, 0\right) p^\top$. Thus H is equivalent to $H'(p) = p\Psi A + p_1^2 + \alpha p_2^2$, $\alpha = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}$. If $\alpha = 0$, then $H'(p) = p\Psi A + p_1^2$ and so H' is equivalent to (an intermediate system) $G^1_B(p) = pB + \frac{1}{2}p_1^2$ with $B = \Psi A$. (A similar argument holds when $\alpha = 1$.) On the other hand, suppose $0 < \alpha < 1$. Then the vector fields associated to

$$H'(p) = a_1'p_1 + a_2'p_2 + a_3'p_3 + p_1^2 + \alpha p_2^2 \quad \text{and} \quad G^2_B(p) = b_1p_1 + b_2p_2 + b_3p_3 + p_1^2 + \tfrac{1}{2}p_2^2$$

(here $A' = \Psi A$) are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} -\sqrt{2(1-\alpha)} & 0 & 0 \\ 0 & 2\sqrt{\alpha(1-\alpha)} & 0 \\ 0 & 0 & -\sqrt{2\alpha} \end{bmatrix} + \begin{bmatrix} -\frac{1-2\alpha}{\sqrt{2(1-\alpha)}} a_1' & \frac{1-2\alpha}{2\sqrt{\alpha(1-\alpha)}} a_2' & -\frac{1-2\alpha}{\sqrt{2\alpha}} a_3' \end{bmatrix}$$

provided $b_1 = -\frac{\alpha\sqrt{2(1-\alpha)}}{1-\alpha} a_1'$, $b_2 = \frac{1}{2\sqrt{\alpha(1-\alpha)}} a_2'$, and $b_3 = -\frac{\sqrt{2}(1-\alpha)}{\sqrt{\alpha}} a_3'$.

Suppose that H is homogeneous, *i.e.*, $A = 0$. Then, by the above argument, H is equivalent to $G_0^0 = H^0$, $G_0^1 = H^1$, or $G_0^2 = H^2$. The systems H^1 and H^2 are not equivalent as the set of equilibria for H^1 is the union of a plane and a line whereas the set of equilibria for H^2 is the union of three lines.

The remainder of the proof involves considering each of the intermediate inhomogeneous systems G_B^0, G_B^1, and G_B^2 and using a combination of linear Poisson automorphisms and affine isomorphisms to reduce these systems as much as possible. One then verifies that each representative obtained is distinct and non-equivalent. □

It turns out that any homogeneous system on $\mathfrak{so}(3)^*_-$ is equivalent to a system on $\mathfrak{se}(2)^*_-$, see [12]. The Euclidean Lie algebra

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} = x_1\tilde{E}_1 + x_2\tilde{E}_2 + x_3\tilde{E}_3 \; : \; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

has nonzero commutators $[\tilde{E}_2, \tilde{E}_3] = \tilde{E}_1$ and $[\tilde{E}_3, \tilde{E}_1] = \tilde{E}_2$. The Lie-Poisson space $\mathfrak{se}(2)^*_-$ has Casimir function $\widetilde{C}(\tilde{p}) = \tilde{p}_1^2 + \tilde{p}_2^2$. The systems

$$\begin{aligned} &\left(\mathfrak{so}(3)^*_-, \tfrac{1}{2}p_1^2\right) : && \dot{p}_1 = 0, && \dot{p}_2 = p_1p_3, && \dot{p}_3 = -p_1p_2, \\ &\left(\mathfrak{se}(2)^*_-, \tfrac{1}{2}\tilde{p}_3^2\right) : && \dot{\tilde{p}}_1 = \tilde{p}_2\tilde{p}_3, && \dot{\tilde{p}}_2 = -\tilde{p}_1\tilde{p}_3, && \dot{\tilde{p}}_3 = 0 \end{aligned}$$

are compatible with the linear isomorphism

$$\psi : \mathfrak{so}(3)^* \to \mathfrak{se}(2)^*, \qquad \psi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

On the other hand, the systems

$$\begin{aligned} &\left(\mathfrak{so}(3)^*_-, p_1^2 + \tfrac{1}{2}p_2^2\right) : && \dot{p}_1 = -p_2p_3, && \dot{p}_2 = 2p_1p_3, && \dot{p}_3 = -p_1p_2, \\ &\left(\mathfrak{se}(2)^*_-, \tilde{p}_2^2 + \tilde{p}_3^2\right) : && \dot{\tilde{p}}_1 = 2\tilde{p}_2\tilde{p}_3, && \dot{\tilde{p}}_2 = -2\tilde{p}_1\tilde{p}_3, && \dot{\tilde{p}}_3 = 2\tilde{p}_1\tilde{p}_2 \end{aligned}$$

are compatible with the linear isomorphism

$$\psi : \mathfrak{so}\,(3)^* \to \mathfrak{se}\,(2)^*, \qquad \psi = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}.$$

We shall make use of such an equivalence in the investigation of the system $H^2_{1,\alpha}$ to relate to some results previously obtained.

For quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$, it turns out that the integral curves are often expressible in terms of Jacobi elliptic functions. Given the modulus $k \in [0,1]$, the basic *Jacobi elliptic functions* $\mathrm{sn}(\cdot,k)$, $\mathrm{cn}(\cdot,k)$, and $\mathrm{dn}(\cdot,k)$ can be defined as (see, *e.g.*, [6, 28])

$$\mathrm{sn}(x,k) = \sin\,\mathrm{am}(x,k), \qquad \mathrm{cn}(x,k) = \cos\,\mathrm{am}(x,k),$$
$$\mathrm{dn}(x,k) = \sqrt{1 - k^2 \sin^2 \mathrm{am}(x,k)}$$

where $\mathrm{am}(\cdot,k) = F(\cdot,k)^{-1}$ is the amplitude and $F(\varphi,k) = \int_0^\varphi \frac{dt}{\sqrt{1-k^2\sin^2 t}}\cdot$ The number K is given by $K = F(\frac{\pi}{2},k)$. (The functions $\mathrm{sn}(\cdot,k)$ and $\mathrm{cn}(\cdot,k)$ are $4K$ periodic, whereas $\mathrm{dn}(\cdot,k)$ is $2K$ periodic.) Nine other elliptic functions are defined by taking reciprocals and quotients; in particular, we have $\mathrm{nd}(\cdot,k) = \frac{1}{\mathrm{dn}(\cdot,k)}$, $\mathrm{sd}(\cdot,k) = \frac{\mathrm{sn}(\cdot,k)}{\mathrm{dn}(\cdot,k)}$ and $\mathrm{cd}(\cdot,k) = \frac{\mathrm{cn}(\cdot,k)}{\mathrm{dn}(\cdot,k)}\cdot$

3 – Homogeneous systems

We consider the two homogeneous systems H^1 and H^2 (see Theorem 2.4). The integral curves of the system H^1 can easily be found in terms of elementary functions; it is then a simple matter to determine the stability nature of the equilibria. On the other hand, the integral curves of the system H^2 can be found in terms of basic Jacobi elliptic functions. The stability nature of the equilibria can be determined via the energy-Casimir methods (and the investigation of spectral stability). Proofs will be omitted; somewhat less refined versions of these results were obtained elsewhere (*cf.* [17], see also [3]).

Throughout, we shall parametrize the equilibrium states by $\mu, \nu, \eta \in \mathbb{R}$, $\nu \neq 0$.

3.1 – System H^1

The system $H^1(p) = \frac{1}{2}p_1^2$ has equations of motion

$$\dot p_1 = 0, \qquad \dot p_2 = p_1 p_3, \qquad \dot p_3 = -p_1 p_2.$$

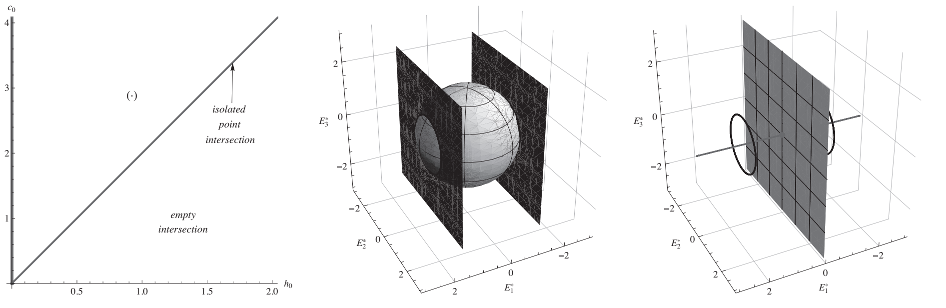


Figure 1: Critical energy states for H^1 and a corresponding typical configuration.

The equilibria are $\mathsf{e}_1^{\mu,\eta} = (0, \mu, \eta)$ and $\mathsf{e}_2^{\mu} = (\mu, 0, 0)$. The states $\mathsf{e}_1^{\mu,\eta} \neq 0$ are unstable whereas the states e_2^{μ} are stable. In Figure 1 we graph the critical energy states (c_0, h_0) and a corresponding typical configuration.

The integral curves of the system are given by

$$\begin{aligned} p_1(t) &= p_1(0) \\ p_2(t) &= p_2(0)\cos(p_1(0)\,t) + p_3(0)\sin(p_1(0)\,t) \\ p_3(t) &= p_3(0)\cos(p_1(0)\,t) - p_2(0)\sin(p_1(0)\,t). \end{aligned}$$

3.2 – System H^2

The system $H^2(p) = p_1^2 + \frac{1}{2}p_2^2$ has equations of motion

$$\dot{p}_1 = -p_2p_3, \qquad \dot{p}_2 = 2p_1p_3, \qquad \dot{p}_3 = -p_1p_2.$$

The equilibria are $\mathsf{e}_1^{\mu} = (\mu, 0, 0)$, $\mathsf{e}_2^{\nu} = (0, \nu, 0)$, and $\mathsf{e}_3^{\nu} = (0, 0, \nu)$.

There are three qualitatively different cases for the intersection of a parabolic cylinder $(H^2)^{-1}(h_0)$ and a sphere $C^{-1}(c_0)$, corresponding to (a) $c_0 < 2h_0$, (b) $c_0 = 2h_0$, and (c) $c_0 > 2h_0$. In Figure 2 we graph the critical energy states (h_0, c_0); in Figure 3 we graph the corresponding typical configurations.

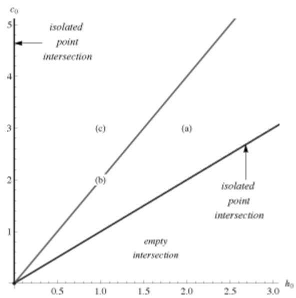
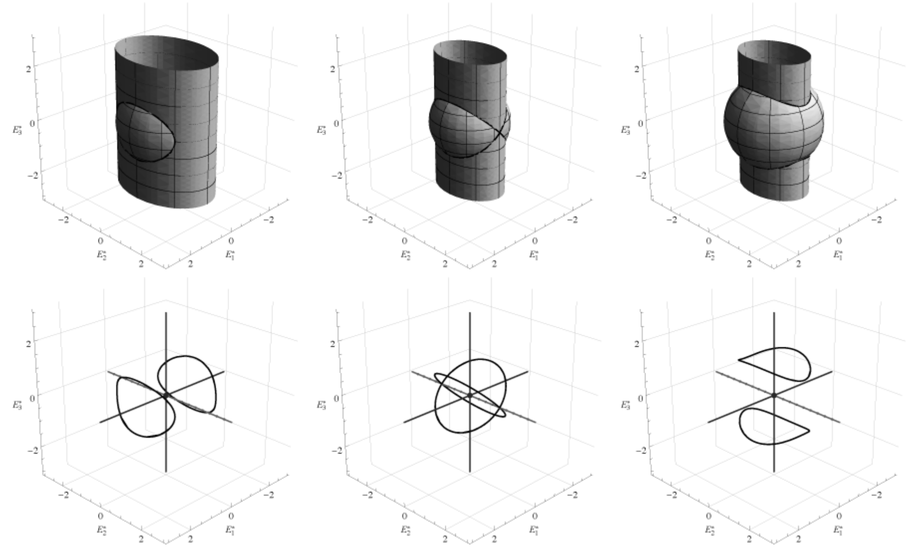


Figure 2: Critical energy states for H^2.



(a) $c_0 < 2h_0$ (b) $c_0 = 2h_0$ (c) $c_0 > 2h_0$

Figure 3: Typical configurations for H^2.

Theorem 3.1. *The equilibrium states have the following behaviour:*

(i) *The states* e_1^μ *are stable.*
(ii) *The states* e_2^ν *are (spectrally) unstable.*
(iii) *The states* e_3^ν *are stable.*

Theorem 3.2. *Let* $p(\cdot)$ *be an integral curve of the system* H^2 *through* $p(0)$*. Let* $h_0 = H^2(p(0))$ *and* $c_0 = C(p(0))$*.*

(a) *If* $0 < c_0 < 2h_0$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{h_0}\,\mathrm{dn}\,(\Omega\, t,\, k)\\
\bar{p}_2(t) &= \sqrt{2}\sqrt{c_0 - h_0}\,\mathrm{sn}\,(\Omega\, t,\, k)\\
\bar{p}_3(t) &= \sigma\sqrt{c_0 - h_0}\,\mathrm{cn}\,(\Omega\, t,\, k)\,.
\end{aligned}$$

Here $\Omega = \sqrt{2h_0}$ *and* $k = \sqrt{\frac{c_0 - h_0}{h_0}}$.

(b) *If* $c_0 = 2h_0 > 0$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma_1, \sigma_2 \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{aligned} \bar{p}_1(t) &= \sigma_1 \sqrt{h_0}\ \operatorname{sech}\left(\sqrt{2h_0}\, t\right) \\ \bar{p}_2(t) &= \sigma_1 \sigma_2 \sqrt{2h_0}\ \tanh\left(\sqrt{2h_0}\, t\right) \\ \bar{p}_3(t) &= \sigma_2 \sqrt{h_0}\ \operatorname{sech}\left(\sqrt{2h_0}\, t\right). \end{aligned}$$

(c) *If* $c_0 > 2h_0 > 0$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{aligned} \bar{p}_1(t) &= \sqrt{h_0}\, \operatorname{cn}(\Omega\, t,\ k) \\ \bar{p}_2(t) &= \sigma \sqrt{2h_0}\, \operatorname{sn}(\Omega\, t,\ k) \\ \bar{p}_3(t) &= \sigma \tfrac{\Omega}{\sqrt{2}}\, \operatorname{dn}(\Omega\, t,\ k). \end{aligned}$$

Here $\Omega = \sqrt{2}\sqrt{c_0 - h_0}$ *and* $k = \sqrt{\frac{h_0}{c_0 - h_0}}$.

4 – Inhomogeneous systems of type I

In this section we consider those inhomogeneous systems whose equilibria are unions of lines and planes (type I). There are five such systems (in fact two systems and three one-parameter families of systems, see Theorem 2.4). Note that the systems which are equivalent to H_0^1 are homogeneous systems in disguise. For each system we obtain explicit expressions for the integral curves: for $H_{1,\alpha}^0$ in terms of elementary functions and for the remaining systems in terms of rational functions of (possibly square roots of) Jacobi elliptic functions. We provide a detailed proof for obtaining the integral curves for one sub-case of the system $H_{2,\alpha}^2$. The integral curves for the remaining systems are obtained in a similar fashion and hence the proofs are omitted.

For each system the stability nature of all equilibria is determined. We provide a detailed proof for the system H_1^1. Similar arguments hold for determining the stability nature of the equilibria of the remaining systems and thus the proofs are omitted, expect where instability does not follow from spectral instability. We note that the system $H_{1,\alpha}^2$ is equivalent to a system on $\mathfrak{se}(2)^*_-$ which has been considered previously in [4]. Again, the equilibria are parametrized by $\mu, \nu, \eta \in \mathbb{R}$, $\nu \neq 0$.

4.1 – System $H_{1,\alpha}^0$

The system $H_{1,\alpha}^0(p) = \alpha p_1$, $\alpha > 0$ has equations of motion

$$\dot{p}_1 = 0, \quad \dot{p}_2 = \alpha p_3, \quad \dot{p}_3 = -\alpha p_2.$$

The equilibria are $\mathsf{e}_1^\mu = (\mu, 0, 0)$; all equilibria are stable. In Figure 4 we graph the critical energy states (h_0, c_0) and a corresponding typical configuration. (The value $\alpha = 1$ was used in Figure 4.)

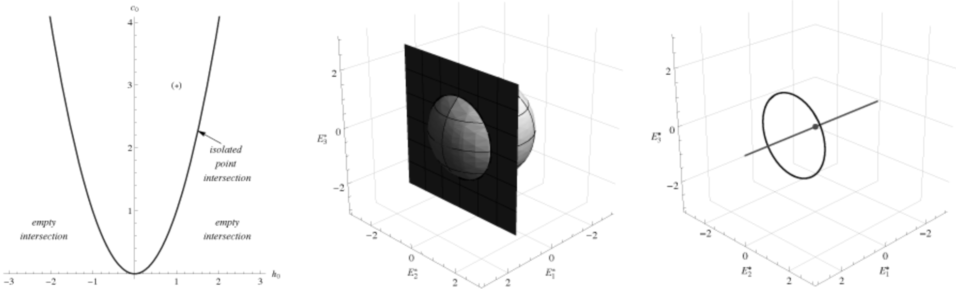


Figure 4: Critical energy states for $H^0_{1,\alpha}$ and a corresponding typical configuration.

The integral curves of this system are given by

$$\begin{aligned} p_1(t) &= p_1(0) \\ p_2(t) &= p_2(0)\cos(\alpha\, t) + p_3(0)\sin(\alpha\, t) \\ p_3(t) &= p_3(0)\cos(\alpha\, t) - p_2(0)\sin(\alpha\, t). \end{aligned}$$

4.2 – System H_1^1

The system $H_1^1(p) = p_2 + \frac{1}{2}p_1^2$ has equations of motion

$$\dot p_1 = -p_3, \qquad \dot p_2 = p_1 p_3, \qquad \dot p_3 = p_1 - p_1 p_2.$$

The equilibria are $\mathsf{e}_1^\mu = (0, \mu, 0)$ and $\mathsf{e}_2^\nu = (\nu, 1, 0)$.

There are three qualitatively different cases for the intersection of a parabolic cylinder $(H_1^1)^{-1}(h_0)$ and a sphere $C^{-1}(c_0)$, corresponding to (a) $c_0 < h_0^2$, (b) $c_0 = h_0^2$, and (c) $c_0 > h_0^2$. In Figure 5 we graph the critical energy states (h_0, c_0); in Figure 6 we graph the corresponding typical configurations.

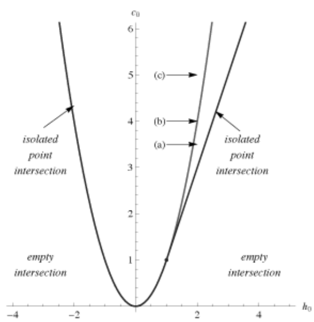
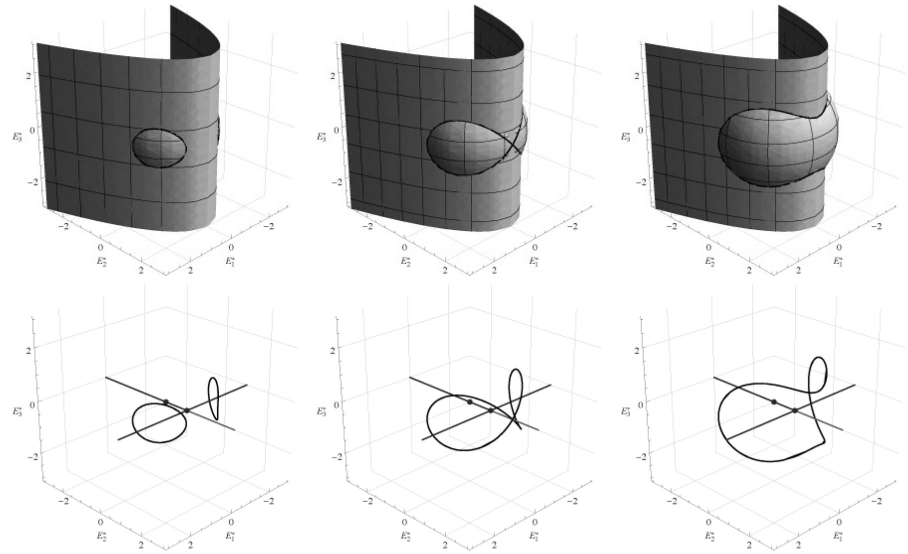


Figure 5: Critical energy states for H_1^1.

THEOREM 4.1. *The equilibrium states have the following behaviour:*

(i) *The states* $\mathsf{e}_1^\mu,\ \mu \leq 1$ *are stable.*
(ii) *The states* $\mathsf{e}_1^\mu,\ \mu > 1$ *are (spectrally) unstable.*
(iii) *The states* e_2^ν *are stable.*



(a) $c_0 < h_0^2$ (b) $c_0 = h_0^2$ (c) $c_0 > h_0^2$

Figure 6: Typical configurations for H_1^1.

PROOF. Let $H_\lambda(p) = \lambda_1 H_1^1(p) + \lambda_2 C(p)$. (i) Suppose $\mu < 1,\ \mu \neq 0$, and let $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2\mu}$. We have $\mathbf{d}\, H_\lambda(\mu, 0, 0) = 0$ and that the Hessian $\mathbf{d}^2\, H_\lambda(\mu, 0, 0) = \operatorname{diag}(\frac{\mu-1}{\mu}, -\frac{1}{\mu}, -\frac{1}{\mu})$ is definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^\mu,\ \mu < 1,\ \mu \neq 0$ are stable. Suppose $\mu = 1$. Then $H(\mathsf{e}_1^1) = C(\mathsf{e}_1^1) = 1$. It is a simple matter to show that $(H_1^1)^{-1}(1) \cap C^{-1}(1) = \{\mathsf{e}_1^1\}$. Thus, by the continuous energy-Casimir method, the state e_1^1 is stable. Likewise, the origin is stable.

(ii) The linearization of the system at e_1^μ has eigenvalues $\lambda_1 = 0,\ \lambda_{2,3} = \pm\sqrt{\mu - 1}$. Thus the states $\mathsf{e}_1^\mu,\ \mu > 1$ are spectrally unstable.

(iii) Let $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. We have $\mathbf{d}\, H_\lambda(\nu, 1, 0) = 0$ and $\mathbf{d}^2\, H_\lambda(\nu, 1, 0) = \operatorname{diag}(0, -1, -1)$ is definite when restricted to $W = \operatorname{span}\{(1, -\nu, 0), (0, 0, 1)\}$. Hence, by the generalized energy-Casimir method, the states e_2^ν are stable. $\square$

Theorem 4.2 ([2]). *Let $p(\cdot)$ be an integral curve of the system H_1^1 through $p(0)$. Let $h_0 = H_1^1(p(0))$ and $c_0 = C(p(0))$.*

(a) *If $c_0 < h_0^2$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t+t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{2\delta}\, \frac{1 + k \operatorname{sn}(\Omega\, t,\, k)}{\operatorname{dn}(\Omega\, t,\, k)} \\
\bar{p}_2(t) &= h_0 + \delta - \frac{2\delta}{1 - k \operatorname{sn}(\Omega\, t,\, k)} \\
\bar{p}_3(t) &= -\sigma k \Omega \sqrt{2\delta}\, \frac{\operatorname{cn}(\Omega\, t,\, k)}{1 - k \operatorname{sn}(\Omega\, t,\, k)}.
\end{aligned}$$

Here $\Omega = \sqrt{h_0 - 1 + \delta}$, $k = \sqrt{\frac{h_0 - 1 - \delta}{h_0 - 1 + \delta}}$, and $\delta = \sqrt{h_0^2 - c_0}$.

(b) *If $c_0 = h_0^2$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t+t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= 2\sigma\sqrt{h_0 - 1}\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right) \\
\bar{p}_2(t) &= h_0 - 2(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)^2 \\
\bar{p}_3(t) &= 2\sigma(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)\ \tanh\left(\sqrt{h_0 - 1}\, t\right).
\end{aligned}$$

(c) *If $c_0 > h_0^2$, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sqrt{2(h_0 + \delta - 1)}\, \operatorname{cn}(\Omega\, t,\, k) \\
\bar{p}_2(t) &= h_0 - (h_0 + \delta - 1)\, \operatorname{cn}(\Omega\, t,\, k)^2 \\
\bar{p}_3(t) &= \sqrt{2\delta(h_0 + \delta - 1)}\, \operatorname{dn}(\Omega\, t,\, k)\ \operatorname{sn}(\Omega\, t,\, k)\,.
\end{aligned}$$

Here $\Omega = \sqrt{\delta}$, $k = \sqrt{\frac{h_0 + \delta - 1}{2\delta}}$, and $\delta = \sqrt{1 + c_0 - 2h_0}$.

4.3 – System $H^2_{1,\alpha}$

The system $H^2_{1,\alpha}(p) = \alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$, $\alpha > 0$ has equations of motion

$$\dot{p}_1 = -p_2 p_3, \quad \dot{p}_2 = (\alpha + 2p_1)p_3, \quad \dot{p}_3 = -(\alpha + p_1)p_2.$$

The equilibria are $\mathsf{e}_1^\mu = (\mu, 0, 0)$, $\mathsf{e}_2^\nu = (-\alpha, \nu, 0)$, and $\mathsf{e}_3^\nu = (-\frac{\alpha}{2}, 0, \nu)$.

The system $(\mathfrak{so}(3)^*_-, H^2_{1,\alpha})$ is equivalent to the system $(\mathfrak{se}(2)^*_-, \widetilde{H}_\alpha)$, where $\widetilde{H}_\alpha(\tilde{p}) = \tilde{p}_1 + \frac{1}{\alpha^2}\tilde{p}_2^2 + \frac{1}{2}\tilde{p}_3^2$. Stability and integration of $\widetilde{H}_\alpha$ were treated in [4].

Explicitly, the systems $(\mathfrak{se}(2)^*_-, \widetilde{H}_\alpha)$ and $(\mathfrak{so}(3)^*_-, H^2_{1,\alpha})$ are compatible with the affine isomorphism $\psi : \mathfrak{se}(2)^* \to \mathfrak{so}(3)^*$ given by

$$\tilde{p} \mapsto \tilde{p} \begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{\alpha} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} -\frac{\alpha}{2} & 0 & 0 \end{bmatrix}. \tag{4.1}$$

Hence, any integral curve of $(\mathfrak{so}\,(3)^*_-, H^2_{1,\alpha})$ is just the image under ψ of an integral curve of $(\mathfrak{se}\,(2)^*_-, \widetilde{H}_\alpha)$. The expressions for the integral curves split into a number of cases. (Some divisions are based on qualitative grounds, whereas others where retrospectively made to facilitate integration.) An index of the conditions defining these cases appears in Table 1. In Figure 7 we graph the critical energy states $(h_0,\, c_0)$; in Figure 8 we graph the corresponding typical configurations. (The value $\alpha = \frac{3}{2}$ was used in both these figures.)

Conditions ($\omega_\pm = 2h_0 + \alpha(\alpha \pm \sqrt{\alpha^2 + 4h_0})$)			Index
$h_0 \leq 0$	$2c_0 > \omega_+$	$\alpha^2 + h_0 > c_0 + \sqrt{c_0^2 + h_0^2 - c_0\,(\alpha^2 + 2h_0)}$	1a(i)
		$\alpha^2 + h_0 = c_0 + \sqrt{c_0^2 + h_0^2 - c_0\,(\alpha^2 + 2h_0)}$	1a(ii)
		$\alpha^2 + h_0 < c_0 + \sqrt{c_0^2 + h_0^2 - c_0\,(\alpha^2 + 2h_0)}$	1a(iii)
	$2c_0 = \omega_+$	$c_0 < \alpha^2 + h_0$	1b(i)
		$c_0 = \alpha^2 + h_0$	1b(ii)
	$\omega_- < 2c_0 < \omega_+$		1c
$h_0 > 0$	$c_0 > \alpha^2 + 2h_0$		2a
	$c_0 = \alpha^2 + 2h_0$		2b
	$c_0 < \alpha^2 + 2h_0$	$2c_0 > \omega_+$	2c(i)
		$2c_0 = \omega_+$	2c(ii)
		$\omega_- < 2c_0 < \omega_+$	2c(iii)

Table 1: Index of cases for integral curves of $H^2_{1,\alpha}$.

We give a proof detailing how the expressions for the integral curves on $\mathfrak{so}\,(3)^*_-$ are obtained from those on $\mathfrak{se}\,(2)^*_-$ only for case 1a(i). (The remaining cases follow a similar argument and thus the proofs are omitted.)

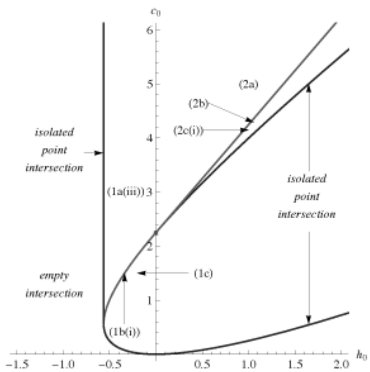


Figure 7: Critical energy states for $H^2_{1,\alpha}$.

THEOREM 4.3. *The equilibrium states have the following behaviour:*

(i) *The states* e_1^μ, $\mu \in (-\infty, -\alpha) \cup [-\frac{\alpha}{2}, \infty)$ *are stable.*
(ii) *The states* e_1^μ, $-\alpha < \mu < -\frac{\alpha}{2}$ *are (spectrally) unstable.*
(iii) *The state* $\mathsf{e}_1^{-\alpha}$ *is unstable.*
(iv) *The states* e_2^ν *are (spectrally) unstable.*
(v) *The states* e_3^ν *are stable.*

PROOF. (iii) Consider the equilibrium state $\mathsf{e}_1^{-\alpha}$. We have that

$$p(t) = \left(\frac{-\alpha^3 t^2}{2 + t^2\alpha^2}, \frac{-2\alpha^2 t}{2 + t^2\alpha^2}, \frac{-2\alpha}{2 + t^2\alpha^2} \right)$$

is an integral curve of the system $H^2_{1,\alpha}$ such that $\lim_{t\to-\infty} p(t) = \mathsf{e}_1^{-\alpha}$. Let $\mathcal{B}_\varepsilon$ be the open ball of radius $\varepsilon = \alpha$ centred at the point $\mathsf{e}_1^{-\alpha}$. For any neighbourhood $V \subset \mathcal{B}_\varepsilon$ of $\mathsf{e}_1^{-\alpha}$ there exists $t_0 < 0$ such that $p(t_0) \in V$. Furthermore $\|p(0) - \mathsf{e}_1^{-\alpha}\| = \sqrt{2}\alpha > \varepsilon$, *i.e.*, $p(0) \notin \mathcal{B}_\varepsilon$. Hence the state $\mathsf{e}_1^{-\alpha}$ is unstable. $\square$

NOTE 4.4. In Theorems 4.5–4.13 we shall find it convenient to use $\eta_0 = \sqrt{\alpha^2 + 4h_0}$ instead of h_0; also, we shall make use of the following notation

$$\delta = \tfrac{1}{4}\sqrt{(\alpha^2 - 4c_0)^2 - 2(\alpha^2 + 4c_0)\eta_0^2 + \eta_0^4} \quad \text{and} \quad \rho_\pm = \tfrac{1}{\sqrt{2}}\sqrt{4c_0 - \alpha^2 - \eta_0^2 \pm 4\delta}.$$

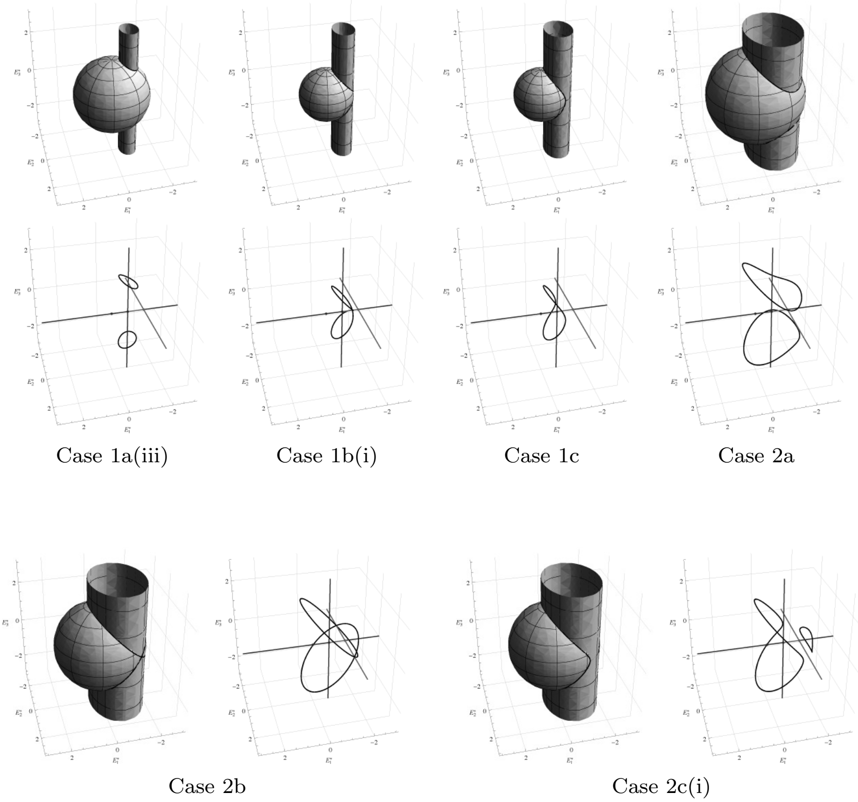


Figure 8: Typical configurations for $H^2_{1,\alpha}$.

Theorem 4.5 (case 1a(i)). *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case* 1a(i) *are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\bar{p}_1(t) = -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2}\, \frac{\rho_- - \rho_+ \operatorname{sn}(\Omega\, t,\, k)}{\rho_+ - \rho_- \operatorname{sn}(\Omega\, t,\, k)}$$

$$\bar{p}_2(t) = \sigma\eta_0\sqrt{2\delta}\, \frac{\operatorname{cn}(\Omega\, t,\, k)}{\rho_+ - \rho_- \operatorname{sn}(\Omega\, t,\, k)}$$

$$\bar{p}_3(t) = -\tfrac{2\sigma\delta}{k'}\, \frac{\operatorname{dn}(\Omega\, t,\, k)}{\rho_+ - \rho_- \operatorname{sn}(\Omega\, t,\, k)}.$$

Here

$$\Omega = \frac{1}{2}\sqrt{4c_0 - \alpha^2 - 3\eta_0^2 + 4\delta},$$

$$k = \sqrt{\frac{\alpha^2 + 4\delta - 4c_0 + 3\eta_0^2}{\alpha^2 - 4\delta - 4c_0 + 3\eta_0^2}}, \quad \textit{and} \quad k' = 2\sqrt{\frac{2\delta}{4c_0 - \alpha^2 - 3\eta_0^2 + 4\delta}}.$$

PROOF. Any integral curve of $H^2_{1,\alpha}$ is the image under ψ of an integral curve of $\tilde{H}_\alpha$. In [4], explicit expressions for all integral curves of $\tilde{H}_\alpha$ are determined; there are a number of cases (corresponding to different explicit expressions). The expression for the integral curve $\tilde{p}(\cdot)$ of $\tilde{H}_\alpha$ through a point $\tilde{p}(0) \in \mathfrak{se}(2)^*$ involves the constants $\tilde{h}_0 = \widetilde{H}_\alpha(\tilde{p}(0))$ and $\tilde{c}_0 = \widetilde{C}(\tilde{p}(0))$. The various cases are expressed in terms of inequalities in $\tilde{h}_0$ and $\tilde{c}_0$. We wish to find the image $\psi(\tilde{p}(\cdot))$ of each such integral curve and to express $\tilde{c}_0$ and $\tilde{h}_0$ in terms of the constants $h_0 = H^2_{1,\alpha}(\psi(\tilde{p}(0)))$ and $c_0 = C(\psi(\tilde{p}(0)))$. Moreover, we wish to find the corresponding conditions for the various cases on $\mathfrak{so}(3)^*$ in terms of inequalities in h_0 and c_0.

Let $\tilde{p} \in \mathfrak{se}(2)^*$ and let

$$\tilde{h}_0 = \widetilde{H}_\alpha(\tilde{p}) = \tilde{p}_1 + \tfrac{1}{\alpha^2}\tilde{p}_2^2 + \tfrac{1}{2}\tilde{p}_3^2 \quad \text{and} \quad \tilde{c}_0 = \widetilde{C}(\tilde{p}) = \tilde{p}_1^2 + \tilde{p}_2^2.$$

Correspondingly, let $h_0 = H^2_{1,\alpha}(\psi(\tilde{p}))$ and $c_0 = C(\psi(\tilde{p}))$; we have

$$h_0 = \tfrac{1}{\alpha^2}(\tilde{p}_1^2 + \tilde{p}_2^2) - \tfrac{1}{4}\alpha^2 \quad \text{and} \quad c_0 = \tilde{p}_1 + \tfrac{1}{\alpha^2}\tilde{p}_2^2 + \tfrac{1}{2}\tilde{p}_3^2 + \tfrac{1}{\alpha^2}(\tilde{p}_1^2 + \tilde{p}_2^2) + \tfrac{1}{4}\alpha^2.$$

Hence

$$h_0 = \tfrac{1}{\alpha^2}\tilde{c}_0 - \tfrac{1}{4}\alpha^2 \quad \text{and} \quad c_0 = \tilde{h}_0 + \tfrac{1}{4}\alpha^2 + \tfrac{1}{\alpha^2}\tilde{c}_0. \tag{4.2}$$

We can invert these relations to get

$$\tilde{c}_0 = \alpha^2 h_0 + \tfrac{1}{4}\alpha^4 \quad \text{and} \quad \tilde{h}_0 = c_0 - h_0 - \tfrac{1}{2}\alpha^2. \tag{4.3}$$

Therefore, $\tilde{p} \in (\widetilde{H}_\alpha)^{-1}(\tilde{h}_0) \cap \widetilde{C}^{-1}(\tilde{c}_0)$ if and only if $\psi(p) \in (H^2_{1,\alpha})^{-1}(h_0) \cap C^{-1}(c_0)$ whenever (4.2) or (4.3) holds.

We consider the first case for the integral curves of $\tilde{H}_\alpha$ treated in [4]. Let $\tilde{p}(\cdot)$ be an integral curve of $\tilde{H}_\alpha$ and let $\tilde{h}_0 = \tilde{H}_\alpha(\tilde{p}(0))$ and $\tilde{c}_0 = \tilde{C}(\tilde{p}(0))$. If the conditions

$$\tilde{c}_0 - \tfrac{1}{4}\alpha^4 \leq 0, \quad \tilde{h}_0 > \sqrt{\tilde{c}_0}, \quad \tfrac{1}{2}\alpha^2 - \tilde{h}_0 > \sqrt{\tilde{h}_0^2 - \tilde{c}_0} \tag{4.4}$$

hold, then there exist $\sigma \in \{-1,1\}$ and $t_0 \in \mathbb{R}$ such that $\tilde{p}(t) = \bar{\bar{p}}(t+t_0)$, where

$$\begin{cases} \bar{\bar{p}}_1(t) = \sqrt{\tilde{c}_0}\, \dfrac{\sqrt{\tilde{h}_0 - \tilde{\delta}} - \sqrt{\tilde{h}_0 + \tilde{\delta}}\ \mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})}{\sqrt{\tilde{h}_0 + \tilde{\delta}} - \sqrt{\tilde{h}_0 - \tilde{\delta}}\ \mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})} \\ \bar{\bar{p}}_2(t) = -\sigma \sqrt{2\tilde{c}_0 \tilde{\delta}}\ \dfrac{\mathrm{cn}(\tilde{\Omega}\, t,\, \tilde{k})}{\sqrt{\tilde{h}_0 + \tilde{\delta}} - \sqrt{\tilde{h}_0 - \tilde{\delta}}\ \mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})} \\ \bar{\bar{p}}_3(t) = \dfrac{2\sigma\tilde{\delta}}{\tilde{k}'}\ \dfrac{\mathrm{dn}(\tilde{\Omega}\, t,\, \tilde{k})}{\sqrt{\tilde{h}_0 + \tilde{\delta}} - \sqrt{\tilde{h}_0 - \tilde{\delta}}\ \mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})}. \end{cases}$$

Here $\tilde{\delta} = \sqrt{\tilde{h}_0^2 - \tilde{c}_0}$, $\tilde{\Omega} = \sqrt{\frac{2}{\alpha^2}(\tilde{h}_0 + \tilde{\delta})(\frac{1}{2}\alpha^2 - \tilde{h}_0 + \tilde{\delta})}$, $\tilde{k} = \sqrt{\frac{(\tilde{h}_0 - \tilde{\delta})(\frac{1}{2}\alpha^2 - \tilde{h}_0 - \tilde{\delta})}{(\tilde{h}_0 + \tilde{\delta})(\frac{1}{2}\alpha^2 - \tilde{h}_0 + \tilde{\delta})}}$ and $\tilde{k}' = \sqrt{\frac{2\alpha^2\tilde{\delta}}{(\alpha^2 + 2\tilde{\delta} - 2\tilde{h}_0)(\tilde{\delta} + \tilde{h}_0)}}$. We now find the corresponding integral curves of $H^2_{1,\alpha}$. Let $p(\cdot)$ be an integral curve of $H^2_{1,\alpha}$ and let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. We have that $\psi^{-1}(p(\cdot))$ is an integral curve of $\tilde{H}_\alpha$. By (4.3) we have that $\psi^{-1}(p(\cdot))$ satisfies the requisite conditions (4.4) of the above result if and only if the conditions

$$h_0 \leq 0, \quad 2c_0 > 2h_0 + \alpha(\alpha + \sqrt{\alpha^2 + 4h_0}), \quad \alpha^2 + h_0 > c_0 + \sqrt{c_0^2 + h_0^2 - c_0\,(\alpha^2 + 2h_0)}$$

hold. Supposing these conditions hold, there exist $\sigma \in \{-1,1\}$ and $t_0 \in \mathbb{R}$ such that $\psi^{-1}(p(t)) = \bar{\bar{p}}(t+t_0)$, *i.e.*, $p(t) = \psi(\bar{\bar{p}}(t+t_0))$. Finally, we let $\bar{p}(t) = \psi(\bar{\bar{p}}(t))$ and replace $\widetilde{h}_0$ and $\widetilde{c}_0$ with expressions in h_0 and c_0 (using (4.3)) and simplify to obtain the result. □

THEOREM 4.6 (case 1a(ii)). *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case* 1a(ii) *are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1,1\}$ such that $p(t) = \bar{p}(t+t_0)$, where*

$$\bar{p}_1(t) = -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2}\, \frac{\rho_- - \rho_+\ \sin(\Omega t)}{\rho_+ - \rho_-\ \sin(\Omega t)}$$
$$\bar{p}_2(t) = \sigma\eta_0\sqrt{2\delta}\, \frac{\cos(\Omega t)}{\rho_+ - \rho_-\ \sin(\Omega t)}$$
$$\bar{p}_3(t) = -\frac{2\sigma\delta}{\rho_+ - \rho_-\ \sin(\Omega t)}.$$

Here $\Omega = \sqrt{\frac{3\alpha^2 - 4c_0 + \eta_0^2}{2}}$.

Theorem 4.7 (cases 1a(iii) & 2a). *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case* 1a(iii) *or* 2a *are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\bar{p}_1(t) = -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2}\,\frac{\rho_- - \rho_+\ \mathrm{cn}\,(\Omega\, t,\, k)}{\rho_+ - \rho_-\ \mathrm{cn}\,(\Omega\, t,\, k)}$$
$$\bar{p}_2(t) = -\sigma\eta_0\sqrt{2\delta}\,\frac{\mathrm{sn}\,(\Omega\, t,\, k)}{\rho_+ - \rho_-\ \mathrm{cn}\,(\Omega\, t,\, k)}$$
$$\bar{p}_3(t) = -2\sigma\delta\frac{\mathrm{dn}\,(\Omega\, t,\, k)}{\rho_+ - \rho_-\ \mathrm{cn}\,(\Omega\, t,\, k)}.$$

Here $\Omega = \sqrt{2\delta}$ *and* $k = \sqrt{\frac{(3\alpha^2 - 4\delta - 4c_0 + \eta_0^2)(\alpha^2 + 4\delta - 4c_0 + \eta_0^2)}{2\alpha^2\delta}}$.

Theorem 4.8 (case 1b(i)). *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case* 1b(i) *are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\bar{p}_1(t) = -\tfrac{\alpha+\eta_0}{2} - \frac{(\alpha - \eta_0)\,\eta_0}{\eta_0 - \alpha\cosh(\Omega\, t)^2}$$
$$\bar{p}_2(t) = \sigma\eta_0\sqrt{2\alpha\,(\alpha - \eta_0)}\frac{\sinh(\Omega\, t)}{\eta_0 - \alpha\cosh(\Omega\, t)^2}$$
$$\bar{p}_3(t) = \sigma\,(\alpha - \eta_0)\,\sqrt{\alpha\eta_0}\,\frac{\cosh(\Omega\, t)}{\eta_0 - \alpha\cosh(\Omega\, t)^2}.$$

Here $\Omega = \sqrt{\frac{(\alpha - \eta_0)\eta_0}{2}}$.

Theorem 4.9 (case 1b(ii)). *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case* 1b(ii) *are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\bar{p}_1(t) = -\frac{\alpha^3 t^2}{2 + \alpha^2 t^2}, \qquad \bar{p}_2(t) = -\frac{2\sigma\alpha^2 t}{2 + \alpha^2 t^2}, \qquad \bar{p}_3(t) = -\frac{2\sigma\alpha}{2 + \alpha^2 t^2}.$$

Theorem 4.10 (cases 1c & 2c(iii)). *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case* 1c *or* 2c(iii) *are satisfied, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$,* *where*

$$\bar{p}_1(t) = -\tfrac{\alpha}{2} - \varepsilon_1 \frac{\frac{\alpha^3\eta_0 - 2\zeta_2}{\alpha^3\eta_0 + 2\zeta_2}\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\, t,\, k)}{\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\, t,\, k)}$$

$$\bar{p}_2(t) = \varepsilon_2 \frac{\mathrm{sd}\left(\tfrac{1}{2}\Omega\, t,\, k\right)\sqrt{1+k\,\mathrm{cd}\,(\Omega\, t,\, k)}\sqrt{1+\mathrm{nd}\,(\Omega\, t,\, k)}}{\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\, t,\, k)}$$

$$\bar{p}_3(t) = \varepsilon_3 \frac{\mathrm{cn}\left(\tfrac{1}{2}\Omega\, t,\, k\right)\sqrt{1-k\,\mathrm{cd}\,(\Omega\, t,\, k)}\sqrt{1+\mathrm{nd}\,(\Omega\, t,\, k)}}{\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\, t,\, k)}.$$

Here

$$\Omega = \tfrac{1}{\alpha}\sqrt{\zeta_1+\zeta_2-\tfrac{1}{8}\tau^2} \qquad \tau = \alpha\left(\alpha+\eta_0-\sqrt{2}\sqrt{\alpha^2+\eta_0^2-2c_0}\right)$$

$$k = \sqrt{\frac{\zeta_1-\zeta_2-\frac{1}{8}\tau^2}{\zeta_1+\zeta_2-\frac{1}{8}\tau^2}} \qquad \zeta_1 = \tfrac{1}{4}\left(4\alpha^2\left(\alpha+\eta_0\right)\eta_0 - \alpha\left(\alpha+4\eta_0\right)\tau + \tau^2\right)$$

$$k' = \sqrt{\frac{2\zeta_2}{\zeta_1+\zeta_2-\frac{1}{8}\tau^2}} \qquad \zeta_2 = \tfrac{1}{2}\sqrt{\alpha\eta_0\left(\alpha\left(\alpha+\eta_0\right)-\tau\right)\left(2\alpha\left(\alpha+\eta_0\right)-\tau\right)\left(2\alpha\eta_0-\tau\right)}$$

and

$$\varepsilon_1 = \frac{\alpha^3\eta_0 + 2\zeta_2}{2\alpha^2\left(\alpha+2\eta_0\right) - 2\alpha\tau}$$

$$\varepsilon_2 = \tfrac{k'}{2\sqrt{2k\alpha}}\sqrt{\tau\eta_0\left(4\zeta_2-4\zeta_1+\alpha^2\tau\right)}$$

$$\varepsilon_3 = \sqrt{\frac{\zeta_2(\zeta_1-\zeta_2)\left(\alpha^2\left(\zeta_2+\alpha^2\eta_0(\alpha+\eta_0)\right)-\left(\zeta_2+\alpha^3\eta_0\right)\tau\right)}{k\alpha^3\eta_0\left(\alpha^2+2\alpha\eta_0-\tau\right)^2}}.$$

THEOREM 4.11 (case 2b). *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case* 2b *are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{p}(t+t_0)$, where*

$$\bar{p}_1(t) = -\alpha - \frac{\Omega^2}{\alpha - \sigma_1\eta_0\cosh(\Omega\, t)}$$

$$\bar{p}_2(t) = -\sigma_1\sigma_2\eta_0\Omega\,\frac{\sinh(\Omega\, t)}{\alpha-\sigma_1\eta_0\cosh(\Omega\, t)}$$

$$\bar{p}_3(t) = -\frac{\sigma_2\Omega^2}{\alpha-\sigma_1\eta_0\cosh(\Omega\, t)}.$$

Here $\Omega = \sqrt{\frac{\eta_0^2-\alpha^2}{2}}$.

THEOREM 4.12 (case 2c(i)). *Let* $p(\cdot)$ *be an integral curve of the system* $H^2_{1,\alpha}$ *through* $p(0)$. *Let* $h_0 = H^2_{1,\alpha}(p(0))$ *and* $c_0 = C(p(0))$. *If the conditions of case* 2c(i) *are satisfied, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$, *where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2} \frac{k' \rho_+ - \sigma\rho_- \operatorname{dn}(\Omega\, t,\, k)}{k'\rho_- - \sigma\rho_+ \operatorname{dn}(\Omega\, t,\, k)} \\
\bar{p}_2(t) &= \tfrac{2\delta\eta_0}{\Omega}\ \frac{\operatorname{cn}(\Omega\, t,\, k)}{k'\rho_- - \sigma\rho_+ \operatorname{dn}(\Omega\, t,\, k)} \\
\bar{p}_3(t) &= -2\sigma\delta k'\ \frac{\operatorname{sn}(\Omega\, t,\, k)}{k'\rho_- - \sigma\rho_+ \operatorname{dn}(\Omega\, t,\, k)}.
\end{aligned}$$

Here

$$\Omega = \frac{1}{2}\sqrt{\alpha^2 - 4c_0 + 3\eta_0^2 + 4\delta}$$

$$k = 2\sqrt{\frac{2\delta}{\alpha^2 - 4c_0 + 3\eta_0^2 + 4\delta}} \quad \textit{and} \quad k' = \sqrt{\frac{\alpha^2 - 4c_0 + 3\eta_0^2 - 4\delta}{\alpha^2 - 4c_0 + 3\eta_0^2 + 4\delta}}.$$

THEOREM 4.13 (case 2c(ii)). *Let* $p(\cdot)$ *be an integral curve of the system* $H^2_{1,\alpha}$ *through* $p(0)$. *Let* $h_0 = H^2_{1,\alpha}(p(0))$ *and* $c_0 = C(p(0))$. *If the conditions of case* 2c(ii) *are satisfied, then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$, *where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{\alpha+\eta_0}{2} \ - \frac{(\alpha - \eta_0)\,\eta_0}{\eta_0 - \alpha\cos(\Omega\, t)^2} \\
\bar{p}_2(t) &= -\eta_0\sqrt{2\alpha\,(\eta_0 - \alpha)}\ \frac{\sin(\Omega\, t)}{\eta_0 - \alpha\cos(\Omega\, t)^2} \\
\bar{p}_3(t) &= (\alpha - \eta_0)\,\sqrt{\alpha\eta_0}\ \frac{\cos(\Omega\, t)}{\eta_0 - \alpha\cos(\Omega\, t)^2}.
\end{aligned}$$

Here $\Omega = \sqrt{\frac{\eta_0(\eta_0-\alpha)}{2}}$.

4.4 – System $H^2_{2,\alpha}$

The system $H^2_{2,\alpha}(p) = \alpha p_2 + p_1^2 + \frac{1}{2}p_2^2$, $\alpha > 0$ has equations of motion

$$\dot{p}_1 = -(\alpha + p_2)p_3, \qquad \dot{p}_2 = 2p_1p_3, \qquad \dot{p}_3 = p_1(\alpha - p_2).$$

The equilibria are $\mathsf{e}_1^\nu = (\nu, \alpha, 0)$, $\mathsf{e}_2^\mu = (0, \mu, 0)$, and $\mathsf{e}_3^\nu = (0, -\alpha, \nu)$.

Theorem 4.14. *The equilibrium states have the following behaviour:*

(i) *The states* e_1^ν *are stable.*
(ii) *The states* e_2^μ, $\mu \in (-\infty, -\alpha) \cup (\alpha, \infty)$ *are (spectrally) unstable.*
(iii) *The states* e_2^μ, $-\alpha \leq \mu \leq \alpha$ *are stable.*
(iv) *The states* e_3^ν *are stable.*

There are five cases for the intersection of a parabolic cylinder $(H_{2,\alpha}^2)^{-1}(h_0)$ and a sphere $C^{-1}(c_0)$. (We note that if the intersection is nonempty, then $h_0 \leq c_0 + \frac{1}{2}\alpha^2$.) We further subdivide one of these cases into two subcases to facilitate integration. An index of the conditions defining these cases appears in Table 2. In Figure 9 we graph the critical energy states (h_0, c_0); in Figure 10 we graph the corresponding typical configurations. (The value $\alpha = 1$ was used in both these figures.)

Conditions		Index
$\frac{c_0}{2} + \alpha\sqrt{c_0} < h_0$		a
$\frac{c_0}{2} + \alpha\sqrt{c_0} = h_0$		b
$\frac{c_0}{2} - \alpha\sqrt{c_0} < h_0 < \frac{c_0}{2} + \alpha\sqrt{c_0}$	$c_0 = 2h_0$	c(i)
	$c_0 \neq 2h_0$	c(ii)
$\frac{c_0}{2} - \alpha\sqrt{c_0} = h_0$		d
$\frac{c_0}{2} - \alpha\sqrt{c_0} > h_0$		e

Table 2: Index of cases for integral curves of $H_{2,\alpha}^2$.

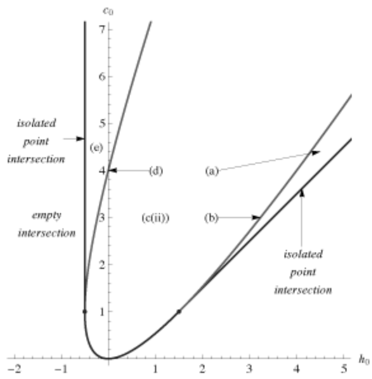
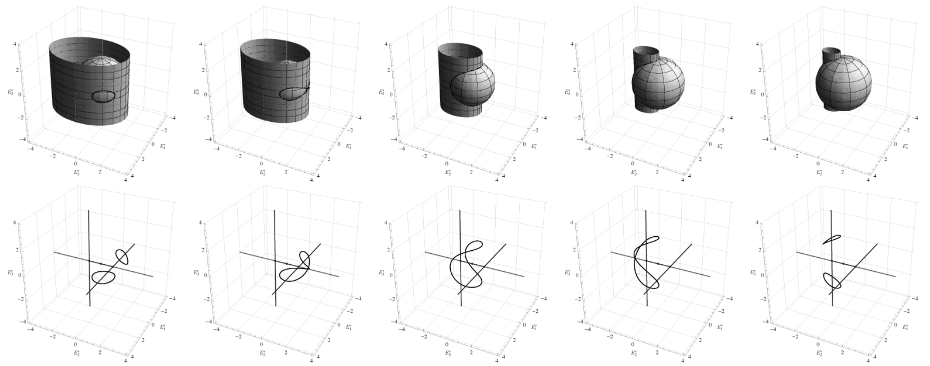


Figure 9: Critical energy states for $H_{2,\alpha}^2$.

We will find the following lemma useful when verifying that a given curve is an integral curve.



(a) Case a (b) Case b (c) Case c(ii) (d) Case d (e) Case e

Figure 10: Typical configurations for $H^2_{2,\alpha}$.

LEMMA 4.15. *If $p(\cdot)$ is a curve such that $H^2_{2,\alpha}(p(t)) = h_0$, $C(p(t)) = c_0$, and $\dot{p}_2(t) = 2p_1(t)p_3(t)$ for $t \in \mathbb{R}$, then $p(\cdot)$ is an integral curve of the system $H^2_{2,\alpha}$.*

PROOF. As $\frac{d}{dt}C(p(t)) = 2p_1\dot{p}_1 + 2p_2\dot{p}_2 + 2p_3\dot{p}_3 = 0$, $\frac{d}{dt}H^2_{2,\alpha}(p(t)) = \alpha\dot{p}_2 + 2p_1\dot{p}_1 + p_2\dot{p}_2 = 0$, and $\dot{p}_2 = 2p_1p_3$, we have $p_1\dot{p}_1 = -(\alpha + p_2)p_1p_3$ and $p_3\dot{p}_3 = (\alpha - p_2)p_1p_3$. It follows that $p(\cdot)$ is an integral curve of the system $H^2_{2,\alpha}$. $\square$

We now present the expressions for the integral curves in the first case.

THEOREM 4.16 (case a). *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} + \alpha\sqrt{c_0} < h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{\delta}\sqrt{c_0 + \delta - \alpha^2}\,\frac{\mathrm{dn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)} \\
\bar{p}_2(t) &= -\tfrac{1}{2\alpha}\,\frac{(\delta + c_0 - 2h_0)\sqrt{\rho+\delta} + (\delta - c_0 + 2h_0)\sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)} \\
\bar{p}_3(t) &= -\sigma\sqrt{\delta}\sqrt{\alpha^2 + 2c_0 - 2h_0}\,\frac{\mathrm{cn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)}.
\end{aligned}$$

Here $\delta = \sqrt{c_0^2 + 4h_0^2 - 4c_0\,(\alpha^2 + h_0)}$, $\Omega = \sqrt{c_0 + \delta - \alpha^2}$, $k = \sqrt{\frac{\alpha^2+\delta-c_0}{\alpha^2-\delta-c_0}}$, and $\rho = 2h_0 - c_0 - 2\alpha^2$.

REMARK 4.17. If we take the limit of the expression for $\bar{p}(t)$, as α tends to 0, then we obtain integral curves for the first case of the system H^2.

One might consider limiting h_0 to $\frac{c_0}{2} + \alpha\sqrt{c_0}$ in case (a) in order to produce integral curves for case (b). However, this limit degenerates and so a more direct approach is required.

THEOREM 4.18 (case b). *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} + \alpha\sqrt{c_0} = h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \tfrac{2\sigma\left(c_0-\alpha^2\right)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}-\alpha}} \, \frac{\cosh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)} \\
\bar{p}_2(t) &= \sqrt{c_0} - \frac{2(c_0 - \alpha^2)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)} \\
\bar{p}_3(t) &= \tfrac{2\sigma\left(c_0-\alpha^2\right)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}+\alpha}} \, \frac{\sinh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)}.
\end{aligned}$$

Here $\Omega = 2\sqrt{c_0 - \alpha^2}$.

We provide a detailed proof for case c(i) to show how the integral curves may be obtained. For this case, in the reduction to standard form, the roots of the two quadratics need to be deinterlaced. Consequently, the expressions for the corresponding integral curves are more involved.

THEOREM 4.19 (case c(i)). *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $c_0 = 2h_0$, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \alpha k \sqrt{\alpha}\, \sqrt[4]{c_0}\, \frac{\mathrm{cn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\ \mathrm{dn}(\Omega\, t, k)} \sqrt{\frac{1 + \mathrm{dn}(\Omega\, t, k)}{k' + \mathrm{dn}(\Omega\, t, k)}} \\
\bar{p}_2(t) &= \sqrt{c_0}\, \frac{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} - \alpha\ \mathrm{dn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\ \mathrm{dn}(\Omega\, t, k)} \\
\bar{p}_3(t) &= \alpha k \sqrt{\alpha}\, \sqrt[4]{c_0}\, \frac{\mathrm{sn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\ \mathrm{dn}(\Omega\, t, k)} \sqrt{\frac{k' + \mathrm{dn}(\Omega\, t, k)}{1 + \mathrm{dn}(\Omega\, t, k)}}.
\end{aligned}$$

Here $\Omega = \frac{\alpha^2}{\sqrt{\alpha^2+c_0}-\sqrt{c_0}}$, $k = \frac{2\sqrt[4]{c_0}\,\sqrt[4]{\alpha^2+c_0}}{\sqrt{\alpha^2+c_0}+\sqrt{c_0}}$, and $k' = \frac{\sqrt{\alpha^2+c_0}-\sqrt{c_0}}{\sqrt{\alpha^2+c_0}+\sqrt{c_0}}$.

PROOF. We start by explaining how the expression for $\bar{p}(\cdot)$ was found. Suppose $\bar{p}(\cdot)$ is an integral curve of $H^2_{2,\alpha}$ such that $c_0 = 2h_0$, where $h_0 = H^2_{2,\alpha}(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. Note that $\frac{c_0}{2} - \alpha\sqrt{c_0} < h_0 < \frac{c_0}{2} + \alpha\sqrt{c_0}$ is trivially satisfied when $c_0 > 0$. As $\bar{p}(\cdot)$ satisfies $(\frac{d\bar{p}_2}{dt})^2 = 4\bar{p}_1^2\bar{p}_3^2$, $H^2_{2,\alpha}(\bar{p}(\cdot)) = \frac{c_0}{2}$, and $C(\bar{p}(\cdot)) = c_0$, we have

$$\frac{d\bar{p}_2}{dt} = \sqrt{(c_0 - 2\alpha p_2 - p_2^2)(c_0 + 2\alpha p_2 - p_2^2)}.$$

After deinterlacing the roots of the two quadratics we get

$$\frac{d\bar{p}_2}{dt} = \sqrt{(c_0 + 2\sqrt{\alpha^2 + c_0}\, p_2 + p_2^2)(c_0 - 2\sqrt{\alpha^2 + c_0}\, p_2 + p_2^2)}.$$

We transform this equation into standard form. Making the change of variables $s = -\frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$ yields

$$t = \frac{1}{(r_1 - r_2)\sqrt{-A_1 A_2}} \int_{\frac{r_1 - \bar{p}_2}{\bar{p}_2 - r_2}}^{\sqrt{-\frac{B_1}{A_1}}} \frac{ds}{\sqrt{\left(-\frac{B_1}{A_1} - s^2\right)\left(s^2 - \left(-\frac{B_2}{A_2}\right)\right)}}.$$

Here

$$\begin{aligned} A_1 &= \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} < 0 & \qquad A_2 &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} > 0 \\ B_1 &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} > 0 & \qquad B_2 &= \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} < 0 \\ r_1 &= \sqrt{c_0} & \qquad r_2 &= -\sqrt{c_0}. \end{aligned}$$

By applying the elliptic integral formula (see [6, 28])

$$\int_x^a \frac{dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{1}{a}\,x,\ \frac{\sqrt{a^2 - b^2}}{a}\right), \quad b \le x \le a$$

we obtain

$$\bar{p}_2(t) = \sqrt{c_0}\,\frac{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} - \alpha\,\mathrm{dn}\,(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\,\mathrm{dn}\,(\Omega\, t, k)}$$

where $\Omega = \frac{\alpha^2}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0}}$ and $k = \frac{2\sqrt[4]{c_0}\,\sqrt[4]{\alpha^2 + c_0}}{\sqrt{\alpha^2 + c_0} + \sqrt{c_0}}$. As $\bar{p}_1(t)^2 = \frac{c_0}{2} - \alpha\bar{p}_2(t) - \frac{1}{2}\bar{p}_2(t)^2$, we have

$$\begin{aligned} \bar{p}_1(t)^2 &= \frac{\alpha\sqrt{c_0}\,(1 + \mathrm{dn}\,(\Omega\, t, k))\left(2\sqrt{c_0\,(\alpha^2 + c_0)} - 2c_0 - \alpha^2 + \alpha^2\,\mathrm{dn}\,(\Omega\, t, k)\right)}{\left(\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\,\mathrm{dn}\,(\Omega\, t, k)\right)^2} \\ &= \frac{\alpha^3\sqrt{c_0}\,(1 + \mathrm{dn}\,(\Omega\, t, k))\,(\mathrm{dn}\,(\Omega\, t, k) - k')}{\left(\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\,\mathrm{dn}\,(\Omega\, t, k)\right)^2} \end{aligned}$$

where $k' = \frac{\sqrt{\alpha^2 + c_0} - \sqrt{c_0}}{\sqrt{\alpha^2 + c_0} + \sqrt{c_0}}$. We now multiply this equation by

$$\frac{\mathrm{cn}\,(\Omega\, t, k)^2}{\mathrm{cn}\,(\Omega\, t, k)^2} = \frac{k^2\,\mathrm{cn}\,(\Omega\, t, k)^2}{\mathrm{dn}\,(\Omega\, t, k)^2 - (k')^2} = \frac{k^2\,\mathrm{cn}\,(\Omega\, t, k)^2}{(\mathrm{dn}\,(\Omega\, t, k) - k')(\mathrm{dn}\,(\Omega\, t, k) + k')}$$

and take the square root to obtain

$$\bar{p}_1(t) = \sigma_1 \frac{\alpha k \sqrt{\alpha\sqrt{c_0}}\ \mathrm{cn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\ \mathrm{dn}(\Omega\, t, k)} \sqrt{\frac{1 + \mathrm{dn}(\Omega\, t, k)}{k' + \mathrm{dn}(\Omega\, t, k)}}$$

for some $\sigma_1 \in \{-1, 1\}$. Similarly, using $c_0 = \bar{p}_1(t)^2 + \bar{p}_2(t)^2 + \bar{p}_3(t)^2$ and multiplying by

$$\frac{\mathrm{sn}\,(\Omega\, t, k)^2}{\mathrm{sn}\,(\Omega\, t, k)^2} = \frac{k^2\ \mathrm{sn}\,(\Omega\, t, k)^2}{(1 - \mathrm{dn}\,(\Omega\, t, k))(1 + \mathrm{dn}\,(\Omega\, t, k))}$$

yields

$$\bar{p}_3(t) = \sigma_2 \frac{\alpha k \sqrt{\alpha\sqrt{c_0}}\ \mathrm{sn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\ \mathrm{dn}(\Omega\, t, k)} \sqrt{\frac{k' + \mathrm{dn}(\Omega\, t, k)}{1 + \mathrm{dn}(\Omega\, t, k)}}$$

for some $\sigma_2 \in \{-1, 1\}$.

We show that $\bar{p}(\cdot)$ is an integral curve for certain values of σ_1 and σ_2. We have

$$\frac{d}{dt}\bar{p}_2(t) - 2\bar{p}_1(t)\bar{p}_3(t) = \frac{2k^2\alpha^3\sqrt{c_0}(1 - \sigma_1\sigma_2)\ \mathrm{cn}(\Omega\, t, k)\ \mathrm{sn}(\Omega\, t, k)}{\left(\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\ \mathrm{dn}(\Omega\, t, k)\right)^2}.$$

Therefore $\frac{d}{dt}\bar{p}_2(t) = 2\bar{p}_1(t)\bar{p}_3(t)$ whenever $\sigma_1 = \sigma_2 = 1$. We have by construction that, $H^2_{2,\alpha}(\bar{p}(t)) = h_0$ and $C(\bar{p}(t)) = c_0$. Consequently, by Lemma 4.15, it follows that $\bar{p}(\cdot)$ (as stated in the theorem) is an integral curve; it is not difficult to show that $0 < k < 1$ and that $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2_{2,\alpha}(p(0))$, $c_0 = C(p(0))$, and suppose that $c_0 = 2h_0$. We claim that $p(t) = \bar{p}(t + t_0)$ for some $t_0 \in \mathbb{R}$. We have $\alpha\, p_2(0) + p_1(0)^2 + \frac{1}{2}p_2(0)^2 = \frac{c_0}{2}$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$. Therefore $\alpha p_2(0) + \frac{1}{2}p_2(0)^2 \leq \frac{c_0}{2}$ and so $-\alpha - \sqrt{\alpha^2 + c_0} \leq p_2(0) \leq -\alpha + \sqrt{\alpha^2 + c_0}$. We also have $p_1(0)^2 + p_2(0)^2 \leq c_0$, which implies that $\alpha - \sqrt{\alpha^2 + c_0} \leq p_2(0) \leq \alpha + \sqrt{\alpha^2 + c_0}$. Thus

$$\alpha - \sqrt{\alpha^2 + c_0} \leq p_2(0) \leq -\alpha + \sqrt{\alpha^2 + c_0}.$$

Now $\bar{p}_2(0) = \alpha - \sqrt{\alpha^2 + c_0}$ and $\bar{p}_2(\frac{K}{\Omega}) = -\alpha + \sqrt{\alpha^2 + c_0}$. Thus there exists $t_2 \in [0, \frac{K}{\Omega}]$ such that $p_2(0) = \bar{p}_2(t_2)$. As

$$p_1(0)^2 = \tfrac{c_0}{2} - 2\alpha p_2(0) - \tfrac{1}{2}p_2(0)^2 = \tfrac{c_0}{2} - 2\alpha\bar{p}_2(t_2) - \tfrac{1}{2}\bar{p}_2(t_2)^2 = \bar{p}_1(t_2)^2$$

it follows that $p_1(0) = \pm\bar{p}_1(t_2)$. Furthermore $\bar{p}_1(t + \frac{2K}{\Omega}) = -\bar{p}_1(t)$ and $\bar{p}_2(t + \frac{2K}{\Omega}) = \bar{p}_2(t)$. Thus there exists $t_1 \in \mathbb{R}$ ($t_1 = t_2$ or $t_1 = t_2 + \frac{2K}{\Omega}$) such that $p_1(0) = \bar{p}_1(t_1)$ and $p_2(0) = \bar{p}_2(t_1)$. On the other hand

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_1)^2$$

and so $p_3(0) = \pm\bar{p}_3(t_1)$. Furthermore $\bar{p}_1(-t) = \bar{p}_1(t)$, $\bar{p}_2(-t) = \bar{p}_2(t)$, and $\bar{p}_3(-t) = -\bar{p}_3(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1$) such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Case c(ii) is very similar to case c(i), although the computations are more involved. The identity $\operatorname{cn}\left(\frac{1}{2}\Omega\, t + \frac{1}{2}K, k\right)^2 = \frac{k'(1-\operatorname{sn}(\Omega\, t,k))}{k'+\operatorname{dn}(\Omega\, t,k)}$ proved to be useful in deriving the below expression for $\bar{p}_1(t)$.

Theorem 4.20 (case c(ii)). *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} - \alpha\sqrt{c_0} < h_0 < \frac{c_0}{2} + \alpha\sqrt{c_0}$ and $c_0 \neq 2h_0$, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t+t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \varsigma\, \varepsilon_1\, \frac{\operatorname{cn}(\frac{1}{2}\Omega\, t + \frac{1}{2}K, k)\sqrt{1 + k\, \operatorname{sn}(\Omega\, t, k)}\sqrt{k' + \operatorname{dn}(\Omega\, t, k)}}{\sqrt{\omega+\rho} - \varsigma\sqrt{\omega-\rho}\, \operatorname{sn}(\Omega\, t, k)}\\
\bar{p}_2(t) &= \varepsilon_2\, \frac{\frac{\rho-2\alpha(\delta+\eta_0)}{\rho+2\alpha(\delta+\eta_0)}\sqrt{\omega+\rho} + \varsigma\sqrt{\omega-\rho}\, \operatorname{sn}(\Omega\, t, k)}{\sqrt{\omega+\rho} - \varsigma\sqrt{\omega-\rho}\, \operatorname{sn}(\Omega\, t, k)}\\
\bar{p}_3(t) &= \varsigma\, \varepsilon_3\, \frac{\operatorname{cn}(\frac{1}{2}\Omega\, t - \frac{1}{2}K, k)\sqrt{1 - k\, \operatorname{sn}(\Omega\, t, k)}\sqrt{k' + \operatorname{dn}(\Omega\, t, k)}}{\sqrt{\omega+\rho} - \varsigma\sqrt{\omega-\rho}\, \operatorname{sn}(\Omega\, t, k)}.
\end{aligned}$$

Here

$$\begin{aligned}
\Omega &= \tfrac{1}{2}\sqrt{2\rho - \tau} & \eta_0 &= \sqrt{\alpha^2 + 2h_0}\\
k &= \sqrt{\frac{\tau + 2\rho}{\tau - 2\rho}} & \tau &= \delta^2 - 4\alpha^2 - 6\delta\eta_0 + \eta_0^2\\
k' &= \sqrt{\frac{4\rho}{2\rho - \tau}} & \rho &= 2\sqrt{\delta\eta_0\,(2\alpha + \delta - \eta_0)\,(2\alpha - \delta + \eta_0)}\\
\delta &= \sqrt{2\,(\alpha^2 + c_0) - \eta_0^2} & \omega &= 2\alpha(\delta + \eta_0) - (\delta - \eta_0)^2\\
\varsigma &= \operatorname{sgn}(\delta - \eta_0) & &
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_1 &= \tfrac{1}{(\delta-\eta_0)\sqrt{2k'}}\sqrt{(\omega+\rho)\left(\eta_0^2 + (2\alpha - \delta)\eta_0 - \tfrac{1}{2}\rho\right)\left(\eta_0^2 - (2\alpha + \delta)\eta_0 + \tfrac{1}{2}\rho\right)}\\
\varepsilon_2 &= \frac{\rho + 2\alpha(\delta + \eta_0)}{2(\delta - \eta_0)}\\
\varepsilon_3 &= \tfrac{1}{(\delta-\eta_0)\sqrt{2k'}}\sqrt{(\omega+\rho)\left(\delta^2 + (2\alpha - \eta_0)\,\delta - \tfrac{1}{2}\rho\right)\left(\delta^2 - (2\alpha + \eta_0)\,\delta + \tfrac{1}{2}\rho\right)}.
\end{aligned}$$

One might consider limiting h_0 to $\frac{c_0}{2} - \alpha\sqrt{c_0}$ in case (e) in order to produce integral curves for case (d). However, like for case (b), this limit degenerates and again a more direct approach is required.

THEOREM 4.21 (case d). *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} - \alpha\sqrt{c_0} = h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{2\sigma(c_0-\alpha^2)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}+\alpha}}\,\frac{\sinh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\,\cosh(\Omega\, t)}\\
\bar{p}_2(t) &= -\sqrt{c_0} + \frac{2(c_0 - \alpha^2)}{\sqrt{c_0} + \alpha\,\cosh(\Omega\, t)}\\
\bar{p}_3(t) &= \tfrac{2\sigma(c_0-\alpha^2)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}-\alpha}}\,\frac{\cosh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\,\cosh(\Omega\, t)}.
\end{aligned}$$

Here $\Omega = 2\sqrt{c_0 - \alpha^2}$.

We present the expressions for the integral curves of case (e).

THEOREM 4.22 (case e). *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} - \alpha\sqrt{c_0} > h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{\delta}\sqrt{\alpha^2 + 2h_0}\,\frac{\mathrm{cn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)}\\
\bar{p}_2(t) &= \tfrac{1}{2\alpha}\,\frac{(\delta - c_0 + 2h_0)\sqrt{\rho+\delta} + (\delta + c_0 - 2h_0)\sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)}\\
\bar{p}_3(t) &= \sigma\sqrt{\delta}\sqrt{\delta + c_0 - \alpha^2}\,\frac{\mathrm{dn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\ \mathrm{sn}(\Omega\, t, k)}.
\end{aligned}$$

Here $\delta = \sqrt{c_0^2 + 4h_0^2 - 4c_0\,(\alpha^2 + h_0)}$, $\Omega = \sqrt{\delta + c_0 - \alpha^2}$, $k = \sqrt{\frac{\alpha^2+\delta-c_0}{\alpha^2-\delta-c_0}}$, and $\rho = c_0 - 2\alpha^2 - 2h_0$.

REMARK 4.23. If we take the limit of the expression for $\bar{p}(t)$, as α tends to 0, then we obtain integral curves for the third case of the system H^2.

5 – Inhomogeneous systems of type II

Among the inhomogeneous systems on $\mathfrak{so}(3)^*_-$, there are four kinds of systems whose equilibria cannot be expressed as unions of lines and planes (type II). In fact, there is one one-parameter family of systems, two two-parameter families of systems, and one three-parameter family of systems (see Theorem 2.4). The stability nature of all equilibria is determined for the system $H^1_{2,\alpha}$. On the other hand, for each of the remaining systems (*i.e.*, those with homogeneous part H^2) we determine the stability nature of all but one or two equilibrium points. Again, we omit proofs for stability results, except where instability does not follow from spectral instability. However, a full proof is provided for the system $H^2_{5,\boldsymbol{\alpha}}$, as the argument and computations are more involved. We found it unfeasible to compute expressions for the integral curves, due to computational complexity. Some indication of this complexity can be inferred from the graphs of the critical energy states.

5.1 – System $H^1_{2,\alpha}$

The system $H^1_{2,\alpha}(p) = p_1 + \alpha p_2 + \frac{1}{2}p_1^2,\ \alpha > 0$ has equations of motion

$$\dot{p}_1 = -\alpha p_3, \qquad \dot{p}_2 = (1 + p_1)p_3, \qquad \dot{p}_3 = \alpha p_1 - (1 + p_1)p_2.$$

The equilibria are $\mathsf{e}_1^\mu = (\mathrm{e}^\mu - 1,\ \alpha(1 - \mathrm{e}^{-\mu}), 0)$ and $\mathsf{e}_2^\mu = (-\,\mathrm{e}^\mu - 1,\ \alpha(1 + \mathrm{e}^\mu),\ 0)$.

In Figure 11 we graph the critical energy states (h_0, c_0); in Figure 12 we graph the corresponding typical configurations. (The value $\alpha = \frac{1}{2}$ was used for both these figures.)

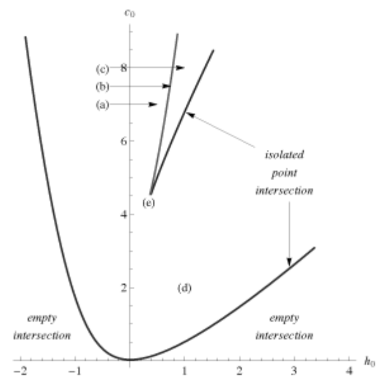
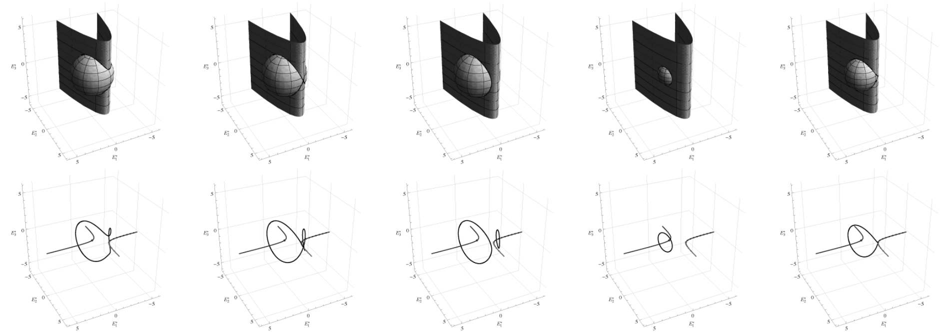


Figure 11: Critical energy states for $H^1_{2,\alpha}$.

Theorem 5.1. *The equilibrium states have the following behaviour:*

(i) *The states* e_1^μ *are stable.*
(ii) *The states* $\mathsf{e}_2^\mu,\ \mu < \frac{2}{3}\ln\alpha$ *are (spectrally) unstable.*
(iii) *The state* $\mathsf{e}_2^\mu,\ \mu = \frac{2}{3}\ln\alpha$ *is unstable.*
(iv) *The states* $\mathsf{e}_2^\mu,\ \mu > \frac{2}{3}\ln\alpha$ *are stable.*



(a) (b) (c) (d) (e)

Figure 12: Typical configurations for $H^1_{2,\alpha}$.

Proof. (iii) Let $\mu = \frac{2}{3}\ln\alpha$; we consider the equilibrium state $\mathsf{e}_2^\mu = (-1 - \alpha^{\frac{2}{3}}, \alpha^{\frac{1}{3}} + \alpha, 0)$. We have that

$$p(t) = \left(\frac{4\alpha^{\frac{2}{3}}}{1+\alpha^{\frac{4}{3}}t^2} - 1 - \alpha^{\frac{2}{3}},\ \alpha^{\frac{1}{3}} + \alpha - \frac{12\alpha^{\frac{1}{3}} + 4\alpha^{\frac{5}{3}}t^2}{(1+\alpha^{\frac{4}{3}}t^2)^2}, \frac{8\alpha t}{(1+\alpha^{\frac{4}{3}}t^2)^2}\right)$$

is an integral curve of the system $H^1_{2,\alpha}$ such that $\lim_{t\to-\infty} p(t) = \mathsf{e}_2^\mu$. Let $\mathcal{B}_\varepsilon$ be the open ball of radius $\varepsilon = \alpha^{\frac{1}{3}}$ centred at the point e_2^μ. For any neighbourhood $V \subset \mathcal{B}_\varepsilon$ of e_2^μ there exists $t_0 < 0$ such that $p(t_0) \in V$. Furthermore $\|p(0) - \mathsf{e}_2^\mu\| = 4\alpha^{\frac{1}{3}}\sqrt{1+\alpha^{\frac{2}{3}}} > \varepsilon$, *i.e.*, $p(0) \notin \mathcal{B}_\varepsilon$. Thus the state e_2^μ, $\mu = \frac{2}{3}\ln\alpha$ is unstable. □

5.2 – System $H^2_{3,\boldsymbol{\alpha}}$

The system $H^2_{3,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$, $\alpha_1, \alpha_2 > 0$ has equations of motion

$$\dot{p}_1 = -(\alpha_2 + p_2)p_3, \qquad \dot{p}_2 = (\alpha_1 + 2p_1)p_3 \qquad \dot{p}_3 = \alpha_2 p_1 - (\alpha_1 + p_1)p_2.$$

The equilibria of this system are given by

$$\mathsf{e}_1^\mu = \left(\mathrm{e}^\mu - \alpha_1,\ \alpha_2(1 - \alpha_1\,\mathrm{e}^{-\mu}),\ 0\right), \qquad \mathsf{e}_2^\mu = \left(-\,\mathrm{e}^\mu - \alpha_1,\ \alpha_2(1 + \alpha_1\,\mathrm{e}^{-\mu}),\ 0\right)$$
$$\mathsf{e}_3^\nu = \left(-\tfrac{\alpha_1}{2},\ -\alpha_2,\ \nu\right).$$

In Figure 13 we graph the critical energy states (h_0, c_0); in Figure 14 we graph the corresponding typical configurations. (The values $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$ were used in both the figures.)

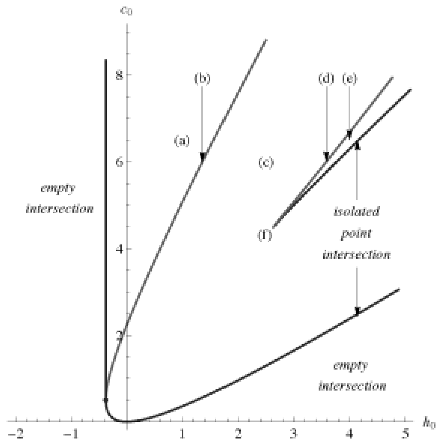


Figure 13: Critical energy states for $H^2_{3,\boldsymbol{\alpha}}$.

Theorem 5.2. *The equilibrium states have the following behaviour:*

(i) *The states* e_1^μ, $\mu < \ln \frac{\alpha_1}{2}$ *are (spectrally) unstable.*
(ii) *The states* e_1^μ, $\ln \frac{\alpha_1}{2} \leq \mu$ *are stable.*
(iii) *The states* e_2^μ, $\mu < \frac{1}{3} \ln \alpha_1 \alpha_2^2$ *are (spectrally) unstable.*
(iv) *The states* e_2^μ, $\mu > \frac{1}{3} \ln \alpha_1 \alpha_2^2$ *are stable.*
(v) *The states* e_3^ν *are stable.*

Remark 5.3. The equilibrium state e_2^μ, $\mu = \frac{1}{3} \ln \alpha_1 \alpha_2^2$ is spectrally stable. However, we were unable to determine its Lyapunov stability nature. We suspect that this state is unstable (see Figure 14f).

5.3 – System $H^2_{4,\boldsymbol{\alpha}}$

The system $H^2_{4,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2} p_2^2$, $\alpha_1 \geq \alpha_3 > 0$ has equations of motion

$$\dot{p}_1 = (\alpha_3 - p_3) p_2, \qquad \dot{p}_2 = -\alpha_3 p_1 + (\alpha_1 + 2p_1) p_3, \qquad \dot{p}_3 = -(\alpha_1 + p_1) p_2.$$

The equilibria of this system are given by

$$\mathsf{e}_1^\mu = \left(\tfrac{1}{2}(\mathrm{e}^\mu - \alpha_1),\, 0,\, \tfrac{\alpha_3}{2}(1 - \alpha_1 \mathrm{e}^{-\mu})\right), \qquad \mathsf{e}_2^\mu = \left(-\tfrac{1}{2}(\mathrm{e}^\mu + \alpha_1),\, 0,\, \tfrac{\alpha_3}{2}(1 + \alpha_1 \mathrm{e}^{-\mu})\right)$$
$$\mathsf{e}_3^\nu = (-\alpha_1,\, \nu,\, \alpha_3)\,.$$

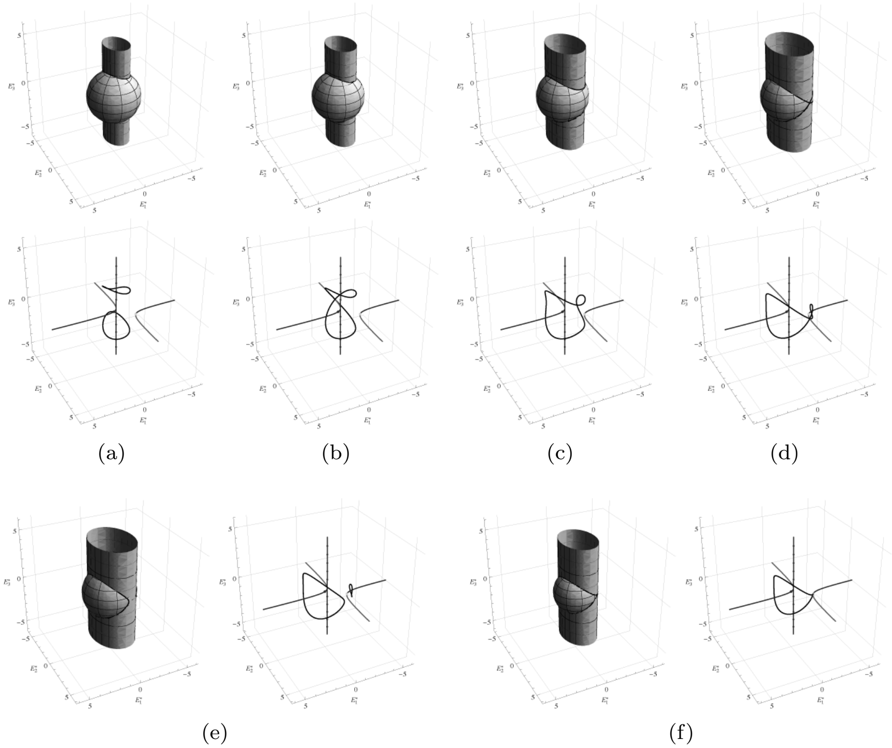
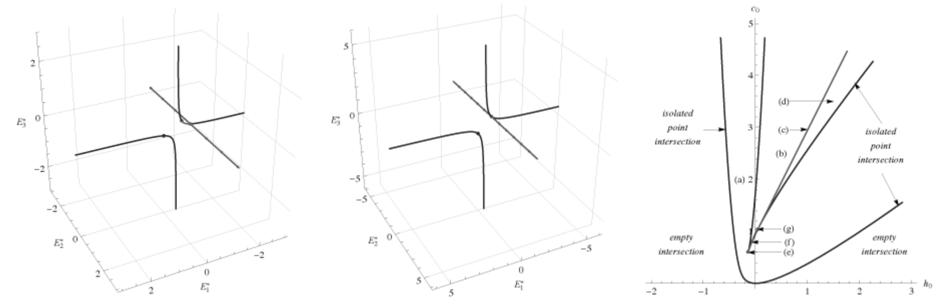


Figure 14: Typical configurations for $H^2_{3,\boldsymbol{\alpha}}$.

When $\alpha_1 = \alpha_3$ the set of unstable equilibria degenerates (see Figure 15); we treat this case separately. In Figures 15iii and 16 we graph the critical energy states (h_0, c_0) and the corresponding typical configurations. (We used the values $\alpha_1 = 1$, $\alpha_3 = \frac{1}{5}$ for Figures. 15i, 15iii, and 16 and the values $\alpha_1 = \alpha_3 = 1$ for Figure 15ii.)

Theorem 5.4. *If $\alpha_1 > \alpha_3 > 0$, then the equilibrium states have the following behaviour:*

(i) *The states e_1^μ are stable.*
(ii) *The states e_2^μ, $\frac{1}{3}\ln\alpha_1\alpha_3^2 < \mu < \ln\alpha_1$ are (spectrally) unstable.*
(iii) *The state e_2^μ, $\mu = \ln\alpha_1$ is unstable.*
(iv) *The states e_2^μ, $\mu \in \left(-\infty, \frac{1}{3}\ln\alpha_1\alpha_3^2\right) \cup (\ln\alpha_1, \infty)$ are stable.*
(v) *The states e_3^ν are (spectrally) unstable.*



(i) $\alpha_1 > \alpha_3$ (ii) $\alpha_1 = \alpha_3$ (iii)

Figure 15: Equilibria and critical energy states for $H^2_{4,\boldsymbol{\alpha}}$.

If $\alpha_1 = \alpha_3 > 0$, *then the equilibrium states have the following behaviour:*

(vi) *The states* e_1^μ *are stable.*

(vii) *The state* e_2^μ, $\mu = \ln \alpha_1$ *is unstable.*

(viii) *The states* e_2^μ, $\mu \neq \ln \alpha_1$ *are stable.*

(ix) *The states* e_3^ν *are* (*spectrally*) *unstable.*

Remark 5.5. The equilibrium state e_2^μ, $\mu = \frac{1}{3} \ln \alpha_1 \alpha_3^2$, $\alpha_1 \neq \alpha_3$ is spectrally stable. However, we were unable to determine its Lyapunov stability nature. We suspect it is unstable (see Figure 16e).

Proof. (iii) Consider the equilibrium state $\mathsf{e}_2^{\ln \alpha_1} = (-\alpha_1, 0, \alpha_3)$. We have that

$$p(t) = \left(\frac{4(\alpha_1 + \alpha_3)}{4 + 2(\alpha_1 + \alpha_3)^2 t^2} - \alpha_1, \, \frac{-2(\alpha_1 + \alpha_3)^2 t}{2 + (\alpha_1 + \alpha_3)^2 t^2}, \, \alpha_3 - \frac{4(\alpha_1 + \alpha_3)}{4 + 2(\alpha_1 + \alpha_3)^2 t^2} \right)$$

is an integral curve of the system $H^2_{4,\boldsymbol{\alpha}}$ such that $\lim_{t \to -\infty} p(t) = \mathsf{e}_2^{\ln \alpha_1}$. Let $\mathcal{B}_\varepsilon$ be the open ball of radius $\varepsilon = \alpha_1 + \alpha_3$ centred at the point $\mathsf{e}_2^{\ln \alpha_1}$. For any neighbourhood $V \subset \mathcal{B}_\varepsilon$ of $\mathsf{e}_2^{\ln \alpha_1}$ there exists $t_0 < 0$ such that $p(t_0) \in V$. Furthermore $\|p(0) - \mathsf{e}_2^{\ln \alpha_1}\| = \sqrt{2}(\alpha_1 + \alpha_3) > \varepsilon$, *i.e.*, $p(0) \notin \mathcal{B}_\varepsilon$. Hence the state $\mathsf{e}^{\ln \alpha_1}$ is unstable. $\square$

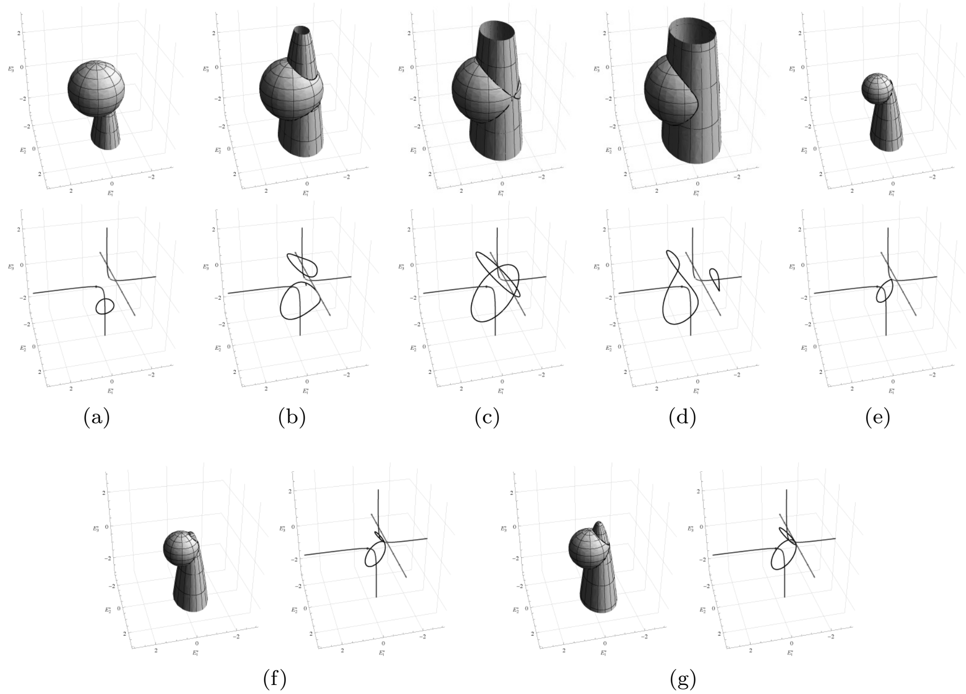


Figure 16: Typical configurations for $H^2_{4,\boldsymbol{\alpha}}$.

5.4 – System $H^2_{5,\boldsymbol{\alpha}}$

The system $H^2_{5,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$, with $\alpha_2 > 0,\ \alpha_1 > |\alpha_3| > 0$ or $\alpha_2 > 0,\ \alpha_1 = \alpha_3 > 0$, has equations of motion

$$\dot{p}_1 = \alpha_3\, p_2 - (\alpha_2 + p_2)\, p_3, \quad \dot{p}_2 = -\alpha_3\, p_1 + (\alpha_1 + 2p_1)\, p_3, \quad \dot{p}_3 = \alpha_2\, p_1 - (\alpha_1 + p_1)\, p_2.$$

The equilibria are $\left(x, \frac{\alpha_2 x}{\alpha_1 + x}, \frac{\alpha_3 x}{\alpha_1 + 2x}\right)$, $x \neq -\alpha_1$, $x \neq -\frac{1}{2}\alpha_1$. These points are the union of three curves which have respective parametrizations

$$\begin{aligned}
\mathsf{e}_1^{\mu} &= \left(-\operatorname{e}^{\mu} - \alpha_1,\ \alpha_2 + \alpha_1\alpha_2 \operatorname{e}^{-\mu},\ \frac{\alpha_3(\alpha_1 + \operatorname{e}^{\mu})}{\alpha_1 + 2\operatorname{e}^{\mu}}\right)\\
\mathsf{e}_2^{\mu} &= \left(\tfrac{1}{4}\alpha_1 \tanh(\mu) - \tfrac{3}{4}\alpha_1,\ \alpha_2 - \frac{4\alpha_2}{1 + \tanh(\mu)},\ \tfrac{1}{2}\alpha_3(2 + \operatorname{e}^{2\mu})\right)\\
\mathsf{e}_3^{\mu} &= \left(\operatorname{e}^{\mu} - \tfrac{1}{2}\alpha_1,\ \alpha_2 - \frac{2\alpha_1\alpha_2}{\alpha_1 + 2\operatorname{e}^{\mu}},\ \tfrac{1}{4}\alpha_3(2 - \alpha_1 \operatorname{e}^{-\mu})\right).
\end{aligned}$$

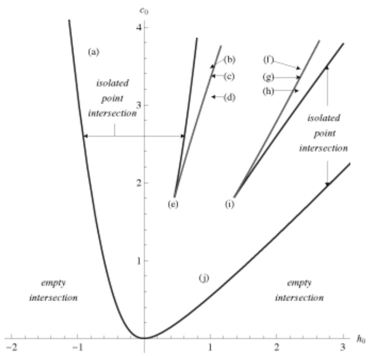


Figure 17: Critical energy states for $H^2_{5,\boldsymbol{\alpha}}$.

The first case corresponds to $x < -\alpha_1$, the second to $-\alpha_1 < x < -\frac{1}{2}\alpha_1$, and the third to $x > -\frac{1}{2}\alpha_1$.

The paraboloid $(H^2_{5,\boldsymbol{\alpha}})^{-1}(h_0)$ and sphere $C^{-1}(c_0)$ are tangent at $p \in \mathfrak{so}\,(3)^*$ if and only if $h_0 = H^2_{5,\boldsymbol{\alpha}}(p)$, $c_0 = C(p)$, and $\begin{bmatrix}\alpha_1 + 2p_1 & \alpha_2 + p_2 & \alpha_3\end{bmatrix} = \kappa \begin{bmatrix}2p_1 & 2p_2 & 2p_3\end{bmatrix}$ for some $\kappa \in \mathbb{R}$.

For $p \neq 0$, this yields $p = \left(\frac{\alpha_1}{2(\kappa-1)}, \frac{\alpha_2}{2(\kappa-\frac{1}{2})}, \frac{\alpha_3}{2\kappa}\right)$, $\kappa \neq 0, \frac{1}{2}, 1$. In other words, apart from at the origin, the level surfaces of $H^2_{5,\boldsymbol{\alpha}}$ and C are tangent at the points

$$\mathsf{e}^{\kappa} = \left(\frac{\alpha_1}{2(\kappa-1)}, \frac{\alpha_2}{2(\kappa-\frac{1}{2})}, \frac{\alpha_3}{2\kappa}\right), \qquad \kappa \neq 0, \tfrac{1}{2}, 1.$$

We shall find it more convenient to use this parametrization of the equilibria (covering all equilibrium points except the origin) in determining the stability nature of the equilibria. We note that

$$\begin{aligned}
&\mathsf{e}^{\kappa} = \mathsf{e}^{\mu}_1 \quad \text{for} \quad \tfrac{1}{2} < \kappa = \frac{2\,\mathrm{e}^{\mu} + \alpha_1}{2\left(\mathrm{e}^{\mu} + \alpha_1\right)} < 1,\ \mu \in \mathbb{R} \\
&\mathsf{e}^{\kappa} = \mathsf{e}^{\mu}_2 \quad \text{for} \quad 0 < \kappa = \frac{1}{2 + \mathrm{e}^{2\mu}} < \tfrac{1}{2},\ \mu \in \mathbb{R} \\
&\mathsf{e}^{\kappa} = \mathsf{e}^{\mu}_3 \quad \text{for} \quad \kappa = \frac{2\,\mathrm{e}^{\mu}}{2\,\mathrm{e}^{\mu} - \alpha_1} < 0,\ \mu < \ln\frac{\alpha_1}{2} \quad \text{or} \quad \kappa = \frac{2\,\mathrm{e}^{\mu}}{2\,\mathrm{e}^{\mu} - \alpha_1} > 1,\ \mu > \ln\frac{\alpha_1}{2}.
\end{aligned}$$

In Figure 17 we graph the critical energy states (h_0, c_0); in Figure 18 we graph the corresponding typical configurations. (We used the values $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{2}{5}$, $\alpha_3 = \frac{1}{2}$ for these figures.)

The polynomial

$$P_{\boldsymbol{\alpha}}(\kappa) = -\alpha_3^2\left(1 - 3\kappa + 2\kappa^2\right)^3 + \kappa^3\left(-8\alpha_2^2(\kappa-1)^3 - \alpha_1^2(2\kappa-1)^3\right)$$

will be central to our discussion of the stability nature of the equilibria.

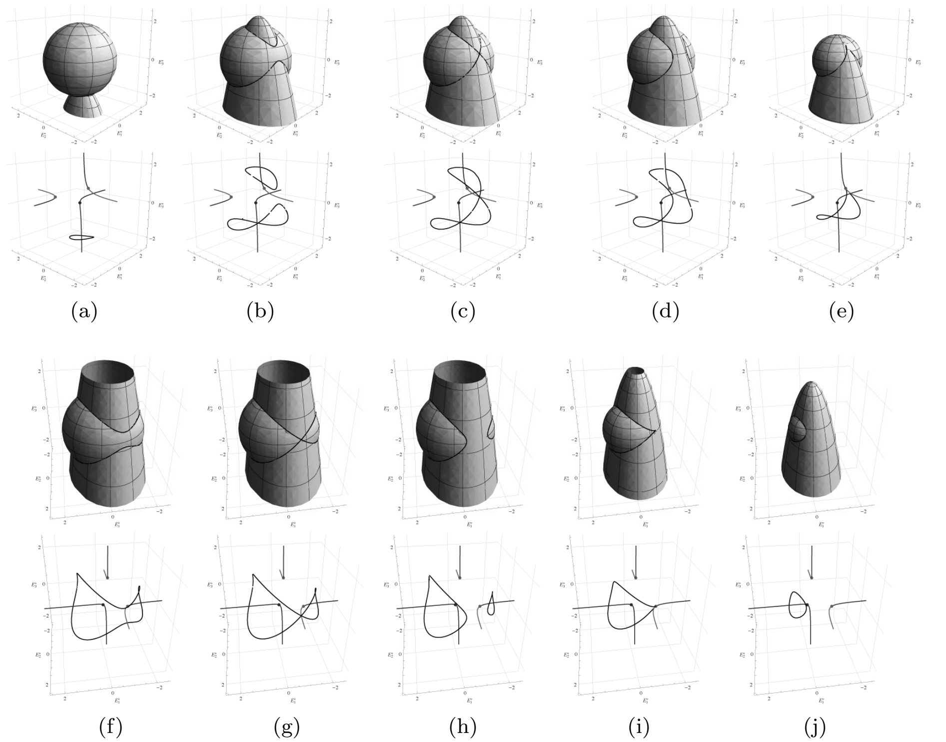


Figure 18: Typical configurations for $H^2_{5,\boldsymbol{\alpha}}$.

LEMMA 5.6. *The polynomial* $P_{\boldsymbol{\alpha}}(\kappa)$ *has exactly two real roots* $\kappa_1 \in (0, \frac{1}{2})$ *and* $\kappa_2 \in (\frac{1}{2}, 1)$.

PROOF. We have that

$$\begin{aligned} P_{\boldsymbol{\alpha}}(-\kappa) = & -\alpha_3^2 - 9\alpha_3^2\kappa - 33\alpha_3^2\kappa^2 - \left(\alpha_1^2 + 8\alpha_2^2 + 63\alpha_3^2\right)\kappa^3 \\ & - \left(6\alpha_1^2 + 24\alpha_2^2 + 66\alpha_3^2\right)\kappa^4 - \left(12\alpha_1^2 + 24\alpha_2^2 + 36\alpha_3^2\right)\kappa^5 \\ & - \left(8\alpha_1^2 + 8\alpha_2^2 + 8\alpha_3^2\right)\kappa^6. \end{aligned}$$

Thus $P_{\boldsymbol{\alpha}}(-\kappa) < 0$ for $\kappa \geq 0$ and so $P_{\boldsymbol{\alpha}}$ has no nonpositive real roots. Furthermore, $P_{\boldsymbol{\alpha}}(0) = -\alpha_3^2 < 0$, $P_{\boldsymbol{\alpha}}(\frac{1}{2}) = \frac{\alpha_2^2}{8} > 0$, and $P_{\boldsymbol{\alpha}}(1) = -\alpha_1^2 < 0$. Therefore $P_{\boldsymbol{\alpha}}$ has at least one root in $(0, \frac{1}{2})$ and at least one root in $(\frac{1}{2}, 1)$. As $\left(1 - 3\kappa + 2\kappa^2\right)^3 > 0$, $(\kappa - 1)^3 > 0$, and $(2\kappa - 1)^3 > 0$ for $\kappa > 1$, it follows that $P_{\boldsymbol{\alpha}}(\kappa) < 0$ for $\kappa > 1$. Thus $P_{\boldsymbol{\alpha}}$ has no real roots in $(1, \infty)$.

It remains to be shown that $P_{\boldsymbol{\alpha}}$ has at most one real root in $(0, \frac{1}{2})$ and at most one real root in $(\frac{1}{2}, 1)$. Suppose $\kappa \in (\frac{1}{2}, 1)$. Then we have that $1 - 6\kappa + 8\kappa^2 \geq 0$ and so

$$\begin{aligned}
&\frac{\frac{d}{d\kappa} P_{\boldsymbol{\alpha}}(\kappa)}{\kappa - \frac{1}{2}} \\
&= -48\alpha_2^2(\kappa-1)^2\kappa^2 - 6\alpha_1^2\kappa^2\left(1 - 6\kappa + 8\kappa^2\right) - 6\alpha_3^2(\kappa-1)^2(3 + 2\kappa(4\kappa - 5)) \\
&< -6\alpha_3^2\kappa^2\left(1 - 6\kappa + 8\kappa^2\right) - 6\alpha_3^2(-1+\kappa)^2(3 + 2\kappa(-5 + 4\kappa)) \\
&= -6(\alpha_3 - 2\alpha_3\kappa)^2(3 + 4(\kappa - 1)\kappa) < 0.
\end{aligned}$$

Hence $P_{\boldsymbol{\alpha}}$ is strictly decreasing on $(\frac{1}{2}, 1)$. Therefore $P_{\boldsymbol{\alpha}}$ has at most one real root in $(\frac{1}{2}, 1)$. Similar computations, although somewhat more involved, show that $P_{\boldsymbol{\alpha}}$ is strictly increasing on $(0, \frac{1}{2})$; hence $P_{\boldsymbol{\alpha}}$ has at most one real root in $(0, \frac{1}{2})$. □

THEOREM 5.7. *The equilibrium states have the following behaviour:*

(i) *The states* e^{κ}, $\kappa \in (\frac{1}{2}, \kappa_2)$, *or correspondingly* e_1^{μ}, $\mu < \ln \frac{\alpha_1(2\kappa_2 - 1)}{2(1-\kappa_2)}$, *are (spectrally) unstable.*
(ii) *The states* e^{κ}, $\kappa \in (\kappa_2, 1)$, *or correspondingly* e_1^{μ}, $\mu > \ln \frac{\alpha_1(2\kappa_2 - 1)}{2(1-\kappa_2)}$, *are stable.*
(iii) *The states* e^{κ}, $\kappa \in (0, \kappa_1)$, *or correspondingly* e_2^{μ}, $\mu > \frac{1}{2}\ln \frac{1-2\kappa_1}{\kappa_1}$, *are stable.*
(iv) *The states* e^{κ}, $\kappa \in (\kappa_1, \frac{1}{2})$, *or correspondingly* e_2^{μ}, $\mu < \frac{1}{2}\ln \frac{1-2\kappa_1}{\kappa_1}$, *are (spectrally) unstable.*
(v) *The states* e^{κ}, $\kappa \in (-\infty, 0) \cup (1, \infty)$, *or correspondingly* e_3^{μ}, $\mu \neq \ln \frac{\alpha_1}{2}$, *are stable.*
(vi) *The origin* e_3^{μ}, $\mu = \ln \frac{\alpha_1}{2}$ *is stable.*

REMARK 5.8. The states e^{κ_1} and e^{κ_2} are spectrally stable. However, we were unable to determine their Lyapunov stability nature. We suspect that they are unstable (see Figures. 18e and 18i).

PROOF. The linearization of the system at e^{κ} has eigenvalues $\frac{\pm\sqrt{P_{\boldsymbol{\alpha}}(\kappa)}}{\sqrt{2}\sqrt{\kappa^2(1-3\kappa+2\kappa^2)^2}}$ and 0. Hence, as $\kappa^2\left(1 - 3\kappa + 2\kappa^2\right)^2 > 0$ for $\kappa \neq 0$, $\kappa \neq \frac{1}{2}$, $\kappa \neq 1$, we have a positive real eigenvalue if and only if $P_\alpha(\kappa) > 0$. We have that $P_{\boldsymbol{\alpha}}(0) = -\alpha_3^2 < 0$, $P_{\boldsymbol{\alpha}}(\frac{1}{2}) = \frac{\alpha_2^2}{8} > 0$, and $P_{\boldsymbol{\alpha}}(1) = -\alpha_1^2 < 0$. Furthermore, by the foregoing lemma, $P_{\boldsymbol{\alpha}}$ has exactly two real roots $\kappa_1 \in (0, \frac{1}{2})$ and $\kappa_2 \in (\frac{1}{2}, 1)$. Therefore $P_{\boldsymbol{\alpha}}(\kappa) > 0$ for $\kappa \in (\kappa_1, \kappa_2)$ and $P_{\boldsymbol{\alpha}}(\kappa) \leq 0$ for $\kappa \in (-\infty, \kappa_1] \cup [\kappa_2, \infty)$. Consequently, the equilibrium states e^{κ}, $\kappa \in (\kappa_1, \frac{1}{2})$ and e^{κ}, $\kappa \in (\frac{1}{2}, \kappa_2)$ are spectrally unstable; all other states are spectrally stable.

Consider the energy function $H_\lambda = \lambda H_{5,\boldsymbol{\alpha}}^2 - \lambda\kappa C$. We have $\mathbf{d}\, H_\lambda(\mathsf{e}^{\kappa}) = 0$ and $\mathbf{d}^2\, H_\lambda(\mathsf{e}^{\kappa}) = \operatorname{diag}(2(1-\kappa)\lambda,\, \lambda(1-2\kappa),\, -2\kappa\lambda)$. Suppose $\kappa \in (-\infty, 0) \cup (1, \infty)$ and let $\lambda = -\kappa$. Then $\mathbf{d}^2 H_\lambda = \operatorname{diag}(2(\kappa-1)\kappa,\, \kappa(2\kappa-1),\, 2\kappa^2\lambda)$ is positive definite. Therefore the states e^κ, $\kappa \in (-\infty,0)\cup(1,\infty)$ are stable.

On the other hand, assume that $\kappa \in (0,1)$. It is a simple matter to show that $p \in \ker \mathbf{d}\, C(\mathsf{e}^\kappa)$ if and only if $p_1 = \frac{(1-\kappa)(2\alpha_2\kappa p_2 + \alpha_3(2\kappa-1)p_3)}{\alpha_1\kappa(2\kappa-1)}$, *i.e.*, $\ker \mathbf{d}\, C(\mathsf{e}^\kappa)$ has basis

$$\left(\left(\frac{2\alpha_2(1-\kappa)}{\alpha_1(2\kappa-1)}, 1, 0\right),\ \left(\frac{\alpha_3(1-\kappa)}{\alpha_1\kappa}, 0, 1\right)\right).$$

The restriction of $\mathbf{d}^2 H_\lambda(\mathsf{e}^\kappa)$ to $\ker \mathbf{d}\, C(\mathsf{e}^\kappa)$ is

$$Q = \begin{bmatrix} -\frac{\left(8\alpha_2^2(\kappa-1)^3 + \alpha_1^2(2\kappa-1)^3\right)\lambda}{\alpha_1^2(1-2\kappa)^2} & -\frac{4\alpha_2\alpha_3(\kappa-1)^3\lambda}{\alpha_1^2\kappa(2\kappa-1)} \\ -\frac{4\alpha_2\alpha_3(\kappa-1)^3\lambda}{\alpha_1^2\kappa(2\kappa-1)} & -\frac{2\left(\alpha_3^2(\kappa-1)^3 + \alpha_1^2\kappa^3\right)\lambda}{\alpha_1^2\kappa^2} \end{bmatrix}.$$

Suppose $\kappa \in (0,\kappa_1)$ and let $\lambda = 1$. Then the first minor $-\frac{8\alpha_2^2(\kappa-1)^3+\alpha_1^2(2\kappa-1)^3}{\alpha_1^2(1-2\kappa)^2} > 0$ and $\det Q = -\frac{2P_\alpha(\kappa)}{\alpha_1^2(1-2\kappa)^2\kappa^2}$. Hence, as $P_{\boldsymbol{\alpha}}$ is negative on $(0,\kappa_1)$, we have $\det Q > 0$ and so the states e^κ, $\kappa \in (0,\kappa_1)$ are stable. Suppose $\kappa \in (\kappa_2, 1)$ and let $\lambda = -1$. We have that $\frac{2\left(\alpha_3^2(\kappa-1)^3+\alpha_1^2\kappa^3\right)}{\alpha_1^2\kappa^2} > 0$ and $\det Q = -\frac{2P_{\boldsymbol{\alpha}}(\kappa)}{\alpha_1^2(1-2\kappa)^2\kappa^2}$. Hence, as $P_{\boldsymbol{\alpha}}$ is negative on $(\kappa_2, 1)$, we have $\det Q > 0$ and so the states e^κ, $\kappa \in (\kappa_2, 1)$ are stable. $\square$

6 – Comments and concluding remarks

Quadratic Hamilton-Poisson systems play a notable role in the context of invariant optimal control. To each invariant optimal control affine problem on a Lie group one can associate, via the Pontryagin Maximum Principle, a quadratic Hamilton-Poisson system on the dual space of the corresponding Lie algebra. The extremal controls are (linearly) related to the integral curves of this Hamiltonian system. (For more details see, *e.g.*, [5, 13, 26, 23].)

Accordingly, in order to obtain the extremal trajectories for an optimal control problem on $\mathsf{SO}(3)$, one needs to find the integral curves of the associated quadratic Hamilton-Poisson system on $\mathfrak{so}(3)^*_-$. Any such quadratic Hamilton-Poisson system is equivalent to one of the normal forms enumerated in Theorem 2.4. Two illustrative examples follow.

Example 6.1. Consider the optimal control problem on $\mathsf{SO}(3)$ specified by

$$\begin{cases} \dot g = g(E_1 + u_1E_2 + u_2E_3), \quad g \in \mathsf{SO}(3), \quad u = (u_1,u_2) \in \mathbb{R}^2 \\ g(0) = g_0 \quad \text{and} \quad g(T) = g_T \\ \mathcal{J} = \displaystyle\int_0^T \left(c_1u_1(t)^2 + c_2u_2(t)^2\right) dt \to \min, \quad c_1 \geq c_2 > 0. \end{cases}$$

The normal extremal trajectories are solutions to $\dot{g} = g(E_1 + u_1 E_2 + u_2 E_3)$, where $u_1(t) = \frac{1}{c_1} p_2(t)$, $u_2(t) = \frac{1}{c_2} p_3(t)$ and $p(t)$ is an integral curve of the Hamiltonian system on $\mathfrak{so}(3)^*_-$ specified by

$$H(p) = p_1 + \tfrac{1}{2c_1} p_2^2 + \tfrac{1}{2c_2} p_3^2.$$

If $c_1 > c_2$, then H is equivalent to $H^2_{1,\alpha}$, $\alpha = \frac{\sqrt{2}(c_1 - c_2)}{\sqrt{c_1 c_2}}$. Indeed,

$$\psi \,:\, p \mapsto p \begin{bmatrix} -\sqrt{2c_1c_2} & 0 & 0 \\ 0 & -\frac{c_1\sqrt{c_2}}{\sqrt{c_1 - c_2}} & 0 \\ 0 & 0 & -\frac{c_2\sqrt{2c_1}}{\sqrt{c_1 - c_2}} \end{bmatrix} + \begin{bmatrix} 2c_2 - c_1 & 0 & 0 \end{bmatrix}$$

is an affine isomorphism such that $T\psi \cdot \vec{H}^2_{1,\alpha} = \vec{H} \circ \psi$. Accordingly, the extremal controls are $\bar{u}_1(t) = -\sqrt{\frac{c_2}{c_1 - c_2}} p_2(t)$ and $\bar{u}_2(t) = -\sqrt{\frac{2c_1}{c_1 - c_2}} p_3(t)$, where $p(\cdot)$ is an integral curve of the system $H^2_{1,\alpha}$ (see Theorems 4.5-4.13). On the other hand, if $c_1 = c_2 = c$, then H is equivalent to H^1_0. Indeed, $\psi : p \mapsto p\Psi + q$, $\Psi = \operatorname{diag}(-c, 1, 1)$, $q = \begin{bmatrix} c & 0 & 0 \end{bmatrix}$ is an affine isomorphism such that $\psi \cdot \vec{H}^1_0 = \vec{H} \circ \psi$. Accordingly, the extremal controls are $\bar{u}_1(t) = \frac{1}{c} p_2(t)$ and $\bar{u}_2(t) = \frac{1}{c} p_3(t)$, where $p(\cdot)$ is an integral curve of the system H^1_0 (see Section 3.1).

Example 6.2. The attitude control of a spacecraft has been modelled as a left-invariant control system on the Lie group $\mathsf{SO}(3)$, see [29]. Assuming that the spacecraft can only be controlled about two axes, the differential equation describing the attitude of the spacecraft is given by

$$\dot{g} = g(u_1 E_1 + u_2 E_2).$$

In the spacecraft attitude control problem one wishes to minimize some energy-type cost function which is quadratic in the controls.

Typically, such a problem takes the form

$$\begin{cases} \dot{g} = g(u_1 E_1 + u_2 E_2), \quad g \in \mathsf{SO}(3), \quad u = (u_1, u_2) \in \mathbb{R}^2 \\ g(0) = g_0 \quad \text{and} \quad g(T) = g_T \\ \mathcal{J} = \displaystyle\int_0^T \left(c_1 u_1(t)^2 + c_2 u_2(t)^2 \right) dt \to \min, \quad c_1, c_2 > 0. \end{cases}$$

The associated Hamilton-Poisson system is again easily calculated and is equivalent to one of the homogeneous systems listed in Theorem 2.4. The extremal controls can be obtained as in Example 6.1; the optimal trajectories $g(\cdot)$ on $\mathsf{SO}(3)$ can then be calculated (see, *e.g.*, [10, 24]).

Acknowledgements

Rory Biggs hereby acknowledges the financial assistance of the National Research Foundation (DAAD-NRF) and Rhodes University, as well as the Claude Leon Foundation.

REFERENCES

[1] R. Abraham – J. E. Marsden: *Foundations of mechanics, 2nd ed.*, Benjamin/Cummings, Reading, Mass., 1978.
[2] R. M. Adams – R. Biggs – C. Remsing: *Quadratic Hamiltonian-Poisson systems on* $\mathfrak{so}^*_-(3)$*: classification and integration*, In: "Geometry, integrability and quantization XV", Avangard Prima, Sofia, 2014, 55–66.
[3] R. M. Adams – R. Biggs – C. C. Remsing: *On some quadratic Hamilton-Poisson systems*, Appl. Sci., **15** (2013), 1–12.
[4] R. M. Adams – R. Biggs – C. C. Remsing, *Two-input control systems on the Euclidean group* $\mathsf{SE}(2)$, ESAIM Control Optim. Calc. Var., **19** (2013), 947–975, http://dx.doi.org/10.1051/cocv/2012040.
[5] A. A. Agrachev – Y. L. Sachkov: *Control theory from the geometric viewpoint*, Springer, Berlin, 2004, http://dx.doi.org/10.1007/978-3-662-06404-7.
[6] J. V. Armitage – W. F. Eberlein: *Elliptic functions*, Cambridge University Press, Cambridge, 2006, http://dx.doi.org/10.1017/CBO9780511617867.
[7] D. I. Barrett – R. Biggs – C. C. Remsing: *Quadratic Hamilton-Poisson systems on* $\mathfrak{se}(1,1)^*_-$*: the homogeneous case*, Int. J. Geom. Methods Mod. Phys., **12** (2015), 1550011, 17, http://dx.doi.org/10.1142/S0219887815500115.
[8] I. Basak: *Explicit solution of the Zhukovski-Volterra gyrostat*, Regul. Chaotic Dyn., **14** (2009), 223–236, http://dx.doi.org/10.1134/S1560354709020038.
[9] I. Basak: *Bifurcation analysis of the Zhukovskii-Volterra system via bi-Hamiltonian approach*, Regul. Chaotic Dyn., **15** (2010), 677–684, http://dx.doi.org/10.1134/S1560354710060043.
[10] J. Biggs – W. Holderbaum: *Integrable Hamiltonian systems defined on the Lie groups* $\mathsf{SO}(3)$ *and* $\mathsf{SU}(2)$*: an application to the attitude control of a spacecraft*, in 5th Wismar Symp. Aut. Control, 2008, 1–7.
[11] J. Biggs – W. Holderbaum: *Integrable quadratic Hamiltonians on the Euclidean group of motions*, J. Dyn. Control Syst., **16** (2010), 301–317, http://dx.doi.org/10.1007/s10883-010-9094-8.
[12] R. Biggs – C. C. Remsing: *A classification of quadratic Hamilton-Poisson systems in three dimensions*, In: "Geometry, integrability and quantization XV", Avangard Prima, Sofia, 2014, 67–78, http://dx.doi.org/10.7546/giq-15-2014-67-78.
[13] R. Biggs – C. C. Remsing: *Cost-extended control systems on Lie groups*, Mediterr. J. Math., **11** (2014), 193–215, http://dx.doi.org/10.1007/s00009-013-0355-0.
[14] A. V. Bolsinov – A. T. Fomenko: *Integriruemye gamiltonovy sistemy*, 1, 2 (in Russian), Izdatelskiĭ Dom Udmurt·ski"ĭ Universitet", Izhevsk, 1999.

[15] A. Borisov – I. Mamaev: *Rigid Body Dynamics: Hamiltonian Methods, Integrability*, Chaos (in Russian), 2nd ed., Moscow-Izhevsk: RCD, Inst. Comp. Sci., 2005.
[16] F. Bullo – A. D. Lewis: *Geometric control of mechanical systems*, Springer, New York, 2005,
http://dx.doi.org/10.1007/978-1-4899-7276-7.
[17] sc C. Dăniasă – A. Gîrban – R. M. Tudoran: *New aspects on the geometry and dynamics of quadratic Hamiltonian systems on* ($\mathfrak{so}(3)$), Int. J. Geom. Methods Mod. Phys., **8** (2011), 1695–1721,
http://dx.doi.org/10.1142/S0219887811005889.
[18] A. Elipe – M. Arribas – A. Riaguas: *Complete analysis of bifurcations in the axial gyrostat problem*, J. Phys. A, **30** (1997), 587–601,
http://dx.doi.org/10.1088/0305-4470/30/2/021.
[19] A. Elipe – V. Lanchares: *Exact solution of a triaxial gyrostat with one rotor*, Celestial Mech. Dynam. Astronom., **101** (2008), 49–68,
http://dx.doi.org/10.1007/s10569-008-9129-6.
[20] J. Frauendiener: *Quadratic Hamiltonians on the unit sphere*, Mech. Res. Comm., **22** (1995), 313–317,
http://dx.doi.org/10.1016/0093-6413(95)00030-U.
[21] V. Ivancevic: *Symplectic rotational geometry in human biomechanics*, SIAM Rev., **46** (2004), 455–474,
http://dx.doi.org/10.1137/S003614450341313X.
[22] V. Jurdjevic, *Non-Euclidean elastica*, Amer. J. Math., **117** (1995), 93–124,
http://dx.doi.org/10.2307/2375037.
[23] V. Jurdjevic: —em Geometric control theory, Cambridge University Press, Cambridge, 1997.
[24] E. W. Justh – P. S. Krishnaprasad: *Optimal natural frames*, Commun. Inf. Syst., **11** (2011), 17–34,
http://projecteuclid.org/euclid.cis/1290608671.
[25] M. P. Kharlamov: *Topological analysis of classical integrable systems in the dynamics of a rigid body*, Dokl. Akad. Nauk SSSR, **273** (1983), 1322–1325.
[26] P. S. Krishnaprasad: *Optimal control and Poisson reduction*, Tech. Rep. T.R. 93-87, Inst. Systems Research, University of Maryland, 1993.
[27] C. Laurent-Gengoux – A. Pichereau – P. Vanhaecke, *Poisson structures*, Springer, Heidelberg, 2013,
http://dx.doi.org/10.1007/978-3-642-31090-4.
[28] D. F. Lawden: *Elliptic functions and applications*, Springer, New York, 1989,
http://dx.doi.org/10.1007/978-1-4757-3980-0.
[29] N. E. Leonard – P. S. Krishnaprasad: *Motion control of drift-free, left-invariant systems on Lie groups*, IEEE Trans. Automat. Control, **40** (1995), 1539–1554,
http://dx.doi.org/10.1109/9.412625.
[30] J. E. Marsden – T. S. Ratiu: *Introduction to mechanics and symmetry*, 2nd ed., Springer, New York, 1999,
http://dx.doi.org/10.1007/978-0-387-21792-5.
[31] J. P. Ortega – V. Planas-Bielsa – T. S. Ratiu: *Asymptotic and Lyapunov stability of constrained and Poisson equilibria*, J. Differential Equations, **214** (2005), 92–127,
http://dx.doi.org/10.1016/j.jde.2004.09.016.
[32] G. E. Shilov: *Linear algebra*, english ed., Dover Publications, New York, 1977.

[33] R. M. Tudoran: *The free rigid body dynamics: generalized versus classic*, J. Math. Phys., **54** (2013), 072704, 10,
http://dx.doi.org/10.1063/1.4816550.
[34] R. M. Tudoran – R. A. Tudoran: *On a large class of three-dimensional Hamiltonian systems*, J. Math. Phys., **50** (2009), 012703, 9,
http://dx.doi.org/10.1063/1.3068405.
[35] V. Volterra: *Sur la théorie des variations des latitudes*, Acta Math., **22** (1899), 201–357,
http://dx.doi.org/10.1007/BF02417877.
[36] J. T. Y. Wen – K. Kreutz-Delgado: *The attitude control problem*, IEEE Trans. Automat. Control, **36** (1991), 1148–1162,
http://dx.doi.org/10.1109/9.90228.

Lavoro pervenuto alla redazione il 15 settembre 2015
ed accettato per la pubblicazione il 27 ottobre 2015

Indirizzi degli Autori:
R. M. Adams – Department of Mathematics (Pure and Applied) – Rhodes University – PO Box 94 – 6140 Grahamstown – South Africa
Email address: rossadamsm@gmail.com
R. Biggs – Department of Mathematics (Pure and Applied) – Rhodes University – PO Box 94 – 6140 Grahamstown– South Africa
Email address: rorybiggs@gmail.com
W. Holderbaum – School of Systems Engineering – University of Reading – Whiteknights Reading – Birkshire – RG6 6AY, UK
Email address: w.holderbaum@reading.ac.uk
C. C. Remsing – Department of Mathematics (Pure and Applied) – Rhodes University – PO Box 94 – 6140 Grahamstown – South Africa
Email address: c.c.remsing@ru.ac.za