The Mortgage Renegotiation Option and Strategic Default.

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Abstract

We argue that lenders and policy makers might help mitigate the credit risk associated with strategic default by US residential homeowners by better understanding characteristics of the strategic negotiation option. We extend the commonly valued default and prepayment mortgage option by proposing and developing a strategic renegotiation option, which is related to the more commonly known strategic default option, where we assume an instantaneous renegotiation between portfolio lender and a US residential mortgage borrower triggered by a declining collateral asset value.

We model those negotiations by considering and sharing future unavoidable foreclosure costs in a Nash bargaining game. We derive closed form solutions for the optimal mortgage loan terms, such as LTV and coupon payment, offered by the portfolio lender to a residential borrower with a strategic default option. We then compare the optimal exercise moment, in terms of the borrower's book LTV, of the strategic negotiation option to the exercise of a conventional default option for borrowers with heterogeneous expectations. We show that the ability of either party to negotiate a larger share of unavoidable foreclosure costs in one's favour has a significant influence on the timing of the optimal *ex post* default decision.

Introduction

We study the strategic negotiation option, where there is an instantaneous renegotiation between the lender and a residential mortgage borrower, triggered by a declining collateral asset value, and value the benefits of such renegotiation to both parties. Our strategic negotiation option is closely related to the strategic default option, where the borrower has the ability to make monthly payments on their mortgage, but chooses not to do so.

The consultants Experian/Oliver Wyman (2010) identify strategic defaulters as those who have not serviced their mortgage for a considerable period (180+ days) but have chosen to continue to service their auto or credit card loans within the agreed period (< 60 days). Timing and definition issues make it problematic for policymakers and lenders to identify potential strategic defaulters at an early enough stage in the mortgage mitigation process to modify their behaviour or future decision. However, we argue that if lenders can screen and identify cases where the strategic renegotiation option might be exercised then the exercise of future strategic or straight default options might be reduced or mitigated.

Negotiation is a necessary first step in a default process for many borrowers. Some borrowers may of course decide not to contact the lender to renegotiate. Whether the borrower actually renegotiates is less relevant to the observation that the lender might be better able to identify that it would be in the borrower's interest to renegotiate. The lender can then take earlier proactive steps to focus on that particular borrower rather than another less fruitful borrower.

Our presentation is a theoretical contribution related to Piskorski, Seru and Vig (2010) (PSV) and Adelino, Gerardi and Willen (2009) (AGW), which empirically investigate the (re)negotiation of (non performing) portfolio and non-portfolio mortgage loans and the so called phenomenon of strategic default and its mitigation. In the recent US housing crisis, borrowers often choose to unilaterally defer a mortgage payment (Guiso, Sapienza and Zingales, (2009) (GSZ), and in the process initiate a default mitigation or negotiation process (generally after being delinquent for at least 60 days). Oliver Wyman (2010) estimates that 19% of all US residential borrowers who were 60 days late in servicing their mortgage could later be considered strategic defaulters. The borrower may threaten the portfolio lender with the exercise of this option for a variety of reasons, triggered by an external event (such as a major purchase, divorce or unemployment) or as we will assume from the desire to maximise their current equity worth by essentially "putting" negative equity to the portfolio lender.

Our treatment is directed at US owner occupied residential mortgages where delinquency does not automatically result in immediate foreclosure. In this case, the credible threat of default or actual delinquency may be of value in negotiating concessions from the lender. We assume that the lender offers no concessions with respect to the loan principal but is prepared to negotiate the mortgage coupon conditional on the irrecoverable costs of foreclosure. Our model is more characteristic of current US "forbearance" programs, where the borrower agrees a reduction in the monthly mortgage payment, rather than "loan modification" programs where the terms of the loan such as principal or maturity are permanently adjusted to the advantage of the borrower. We assume that with declining house prices, the borrower's other options of prepayment, new credit or "short sale" of their illiquid housing asset are of little value.

We recognise that the borrower's decision to defer a mortgage payment may be a temporary or transient reaction to exogenous or endogenous trigger events whereby the borrower believes that the lender will not or cannot immediately foreclose. Hence, this option is seen as an intermediate or "first step" option on a timeline which follows the exercise of the investment option (t=0) but precedes (strategic) default, cure, prepayment or permanent loan term modification. The borrower might eventually decide that (continuing) negative changes in house prices or other adverse trigger events would make exercise of other options optimal. On the other hand, positive changes in property prices or trigger events might make it optimal for the borrower to cure the mortgage to prevent the lender exercising his foreclosure option.

It is, of course, impossible to identify exactly when and whether a borrower might have strategically defaulted until they actually default (perhaps many months later) and their (short term) loan behaviours can be identified (from many months earlier). We therefore argue that if policy makers and lenders can estimate the value and optimal timing of the strategic negotiation option of a borrower it might allow them to better screen, identify and gauge the possible effectiveness of policies designed to reduce future (strategic) default. Our exposition therefore needs to demonstrate in the first instance how the value and optimal timing of the strategic negotiation option behaves in relation to the common default option. Does it lead or lag other options? Does it have any predictive abilities? This paper builds a theoretical basis for future empirical research.

We model both a conventional default option (Model 1) and the strategic negotiation option (Model 2) and compare their optimal exercise moments. In addition, we uniquely strengthen the validity of the comparison by starting the process from mortgage initiation (t=0), ensuring that (financially) optimal LTV and coupon rates have been negotiated for both portfolio lender and borrower. In this manner, by calculating the optimal value of a strategic negotiation option to a borrower, the lender can estimate the maximum cost of the implicit option, even though the actual cost may well be eventually lower.

In the next section, we develop models for mortgage renegotiation and strategic default. The middle section provides numerical illustrations for both options, and discusses some implications for assessing the original credit risk in residential mortgages. The last section summarizes our model and numerical illustrations, and identifies areas for future research, empirically and in additional model development.

I Model Outline and Assumptions

We seek to demonstrate the effect of the negotiation option on underwriting (LTV) requirements compared with the status quo (investment option followed by the default option), and stripping out the effects of different initial LTV at origination. So first we model the optimal *ex-ante* investment and leverage decisions of the homeowner (how much equity should they inject?) and lender (how much debt?) at t=0. The optimal *ex-post* (negotiation or default) decision should be examined conditional on the lender and homeowner having agreed the optimal debt/equity (LTV) ratio at loan origination. [It is easier to model the optimal *ex-post* (negotiation or default) decision for a given (non-optimal) LTV and coupon.]

The homeowner's "spot" asset (property) price (V) follows (Kau and Keenan 1995) a stochastic general Brownian motion process with drift μ.The difference between the drift μ and the risk free rate r is treated as a convenience yield (or market imputed rent) which the homeowner "collects" by living in their property. This market imputed rent will vary proportionally with the "spot" value of the property. In a declining housing market, considerations of maintenance and depreciation are of minor importance. We assume that the original mortgage is a perpetual loan carrying a fixed interest coupon determined at origination.

When the local property market is performing well, homeowners will see a notional increase in their equity value. Homeowners will therefore continue to supply the needed funds to service the debt when and if it is in their interest to do so -- an example being the property still having positive net equity or the imputed market rent being of sufficient value or convenience.

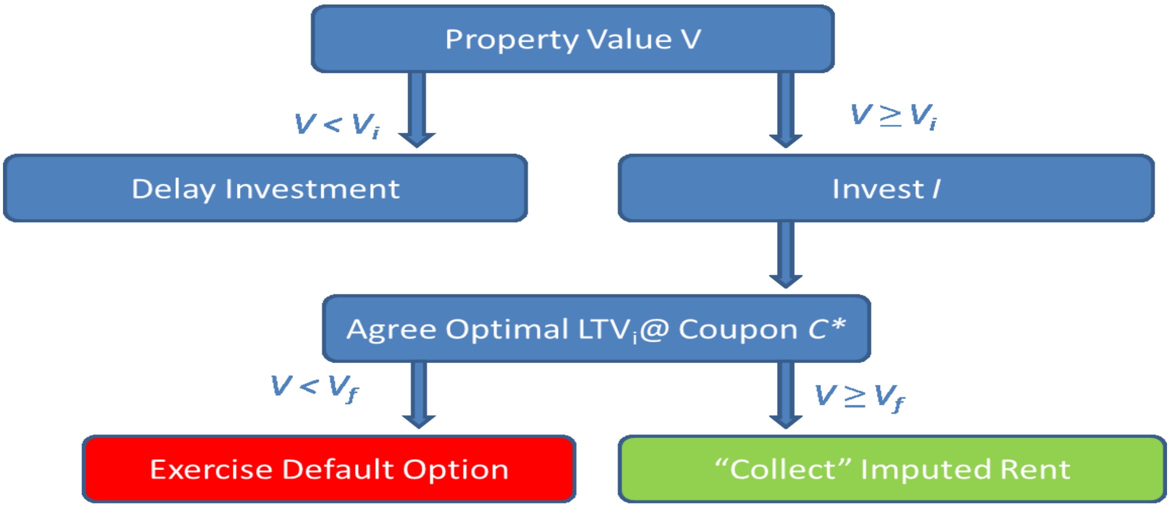
The situation is different if the property market is not performing well, as default is generally costly to the lender and homeowner. The homeowner injects no new equity to prepay and new loan debt (re)mortgaging and equity withdrawal) is not available to the homeowner due to declining house prices or credit restrictions. On default, the homeowner loses all housing equity as the lender repossess the collateral. In addition, the homeowner will also lose the convenience value (market imputed rent) of occupying the home. The lender will only receive the house value less foreclosure costs to cover any outstanding debt. Alternatively, after a successful negotiation (option), the homeowner pays a lower fixed monthly payment and retains ownership of the property and consequent equity and market imputed rent.

Consequently, proactive lenders and homeowners will try to avoid costly foreclosure and attempt to negotiate and agree a forbearance mitigation program. We introduce a parameter ϕ $(0 \leq \phi \leq 1)$ to model the effect and strength of this (re)negotiation regarding the sharing of foreclosure costs which is distributed to the satisfaction of both parties. For ease of exposition, we refer to a homeowner who negotiates a smaller notional share of the unavoidable foreclosure costs as a weak homeowner ($\phi \rightarrow 0$) and one who negotiates a larger share ($\phi \rightarrow 1$) as a strong homeowner. We construct ϕ as a heterogeneous variable indicating the immediate view taken by both the lender and homeowner on how much of the unavoidable foreclosure costs the other would be liable for to condition or influence their *ex-ante* mortgage negotiation.

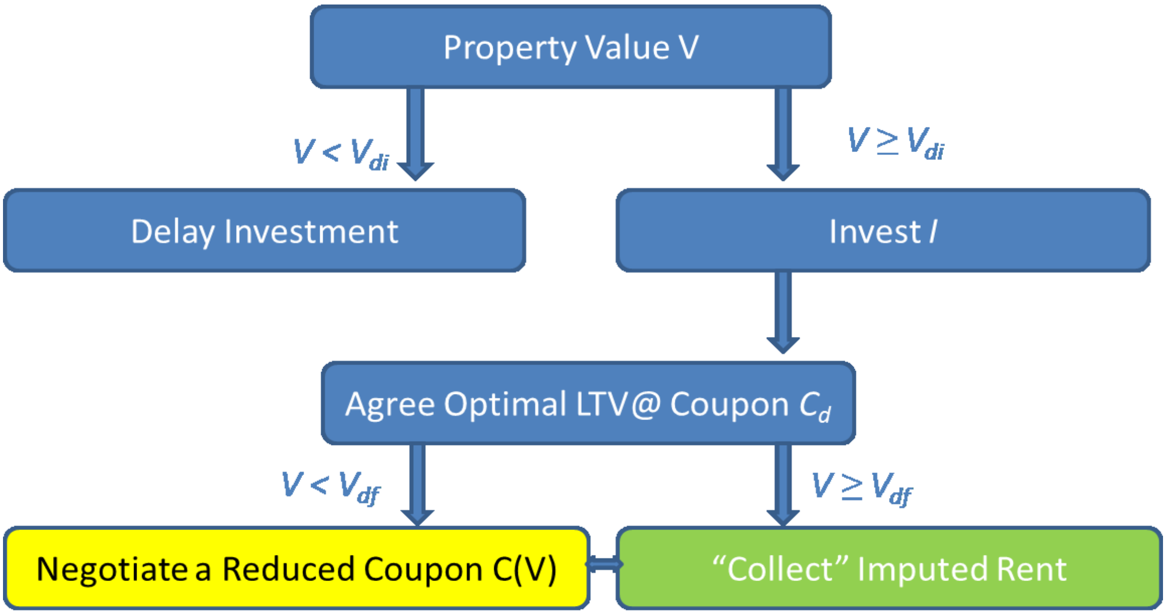
We believe that elements of real world individual characteristics such as a homeowner's FICO credit rating or US state residency could be used to estimate the negotiation strength ϕ. It is reasonable to assume that those homeowners with strong credit scores may be able to negotiate and extract different and better concessions and terms from lenders than homeowners with weaker scores. PSV (2010) and AGW (2009) demonstrate that default and cure rates are different for securitised and non-securitised loans from homeowners with high and low FICO scores after mortgage (re)negotiation. Recent empirical papers (PSV 2010, AGW 2009) make comparisons between homeowners and lenders with heterogeneous characteristics. We therefore uniquely attempt to introduce a heterogeneous element, such as for example the FICO score, to the negotiated mortgage contract outcome both *ex* and *post* any negotiation or default event. We also find support for this approach in the Experian –Oliver Wyman Reports (2010) which observe that strategic defaulters are more likely to be credit worthy borrowers with high FICO scores.

In contrast to the traditional option theoretic approach, as described by Kau and Keenan (1995), we arrive at our solution by adapting from Fan and Sundaresan (2000) to cover the borrower's irreversible negotiation and investment options. Their methodology is an expansion on the endogenous default approach to corporate debt found in corporate finance literature e.g. Leland (1994) whereby the management chooses the timing of default to maximise equity value. In general, traditional option theoretic models proceed, using a backwards numerical solutions approach, to calculate the value of the default and prepayment options using two stochastic factors (property prices and interest rates) and a finite mortgage term, Azevedo-Pereria, Newton and Paxson (2002). To ensure tractability and obtain closed form solutions we employ just one stochastic factor with a perpetual mortgage term. We believe this approach is justified as the stochastic interest rate factor is mainly of influence on the prepayment option (which we assume is valueless for distressed borrowers in the recent economic climate) and where new credit is readily available (which we again assume is unlikely with declining house prices, conditional of course on federal monetary policy).

Figure 1 Owner Occupied Negotiation Option Model Flow Diagrams



Model 1 – The Invest and Default Option



Model 2 – The Invest and Negotiate Option

<u>Legend</u>

V : "Spot" Property Price.

V_i : Optimal property price at where the homeowner would invest with a default option only.

V_f Optimal property price at where the borrower would default with a default option only.

V_{di}: Optimal property price at where the borrower would invest with a negotiation option.

V_{df} Optimal property price at where the borrower would negotiate.

C^* . Perpetual mortgage payment to the lender for the default only option (tax deductible).

C_d : Perpetual mortgage payment to the lender for the negotiation option (tax deductible).

C(V): Renegotiated mortgage payment (after negotiation) which depends on property price.

I : Initial property investment made at the critical investment thresholds.

Figure 1 summarises the options available to the homeowner. The blue boxes in both Models 1 and 2 are the initial purchase option followed by the box in yellow (Model 2) which represent the negotiation option and the default option in the red box (Model 1).

As a consequence of the assumption that the US residential mortgage contract is incomplete but not asymmetrical, given the widely available amount of data on residential house prices, the lender and homeowner play a generalised Nash cooperative game (Yellow box, Model 2, Figure 1) to avoid foreclosure costs. They have incomplete but no asymmetric knowledge of each other's options, circumstances and costs, and having *ex-ante* negotiated the initial mortgage contract (LTV and mortgage payment in the blue boxes) conditional on anticipated default, may *ex-post* renegotiate the contract should a credible threat of default arise due to a unfavourable shock to spot property prices. We then allow the mortgage to be modified with a new lower mortgage coupon (C(V)) on a successful negotiation. What happens should property prices later recover above the negotiation trigger point is immaterial in this exposition as we are primarily interested in identifying the value and negotiation option to identify and screen borrowers and mitigate a future strategic or actual default.

Should the homeowner or lender not negotiate, then Model 1 applies, and with continuing negative property price shocks or trigger events, the homeowner will optimally exercise the default option (Red box, Model 1, Figure 1). We assume, in this regard, that the homeowner has limited liability and can default on the mortgage contract at any time with no long-term consequences to a subsequent credit rating – a common outcome in both US recourse and non-recourse states.

The question arises as to whether lender and homeowner can *ex- ante* discover each other's relative negotiation strength ϕ. Whether they can or not, it is however possible for both to agree that ϕ will be in a range from 0 to 1 just as it is also possible for both to *ex-ante* know all possible future property prices and foreclosure costs from 0% to 100%. Both parties can therefore *ex-ante* calculate relevant trigger points and decide on what conditions they will optimally instantaneously agree a (re)negotiated mortgage coupon, based on an anticipated share of the unavoidable foreclosure costs. Under these assumptions, both lender and homeowner will *ex-ante* anticipate the same range of negotiated outcomes.

Although our model is driven by stochastic property prices, we present and discuss the optimal negotiation trigger points by transforming the stochastic property price to a book LTV (defined as the nominal debt/appraised house value). It is common for policymakers and lenders to measure the likelihood that a homeowner will default in terms of their book LTV whereby a book LTV greater than 100% indicates negative equity i.e. the property value is less than the outstanding loan. This is reasonable given that no principal payments are made in the interim.

We show, for typical US mortgage loan values, that optimal negotiation option exercise should normally occur earlier than a comparative default option exercise for all homeowners but strong negotiators should exercise their negotiation option earlier than weak negotiators. We show that the lenders *ex-ante* mortgage yield spread should increase to pay for the homeowner's *ex-post* strategic negotiation option. We show that the optimal equity down payment or deposit is conditional on the homeowner's negotiation ability (ϕ) in the case of a negotiation but not a default option and that a lender could offer a larger initial LTV mortgage to a weaker homeowner. This is because the weaker (as so the stronger) homeowner always has the ability to pay but his negotiating ability allows the strong lender to profit from the lower foreclosure costs and hence offer a larger loan. This is not the case with the stronger homeowner where the weak lender will offer a smaller loan. Finally, we show that while increasing property price volatility should motivate homeowners to ***delay*** exercising the default option it will on the other hand ***accelerate*** exercise of their negotiation option.

In the next section we outline the derivation of the compound investment and negotiation options as well as of the comparative compound investment and default options with a glossary of notation in Appendix A and detailed derivations included in Appendix B

In the subsequent section using stylised US mortgage data, we examine and highlight the fundamental differences between the *ex-post* behaviour of the negotiation option versus the more traditional default option and highlight the effect of heterogeneous (ϕ) negotiation on the endogenous exercise threshold expressed in terms of negative equity, mortgage yield spread and LTV ratios. We finally critically assess whether this approach might help screen strategic or other types of defaulters at earlier stages.

Default and Negotiation Option Models

The property price process is exogenous and the homeowner and lender have homogenous and rational expectations while transactions are sufficiently small to have no effect on local property prices. The homeowner will make the constant and perpetual mortgage payment c^* to the lender in the default option (Model 1) and c_d in the comparative case where the homeowner has a negotiation option (Model 2). Assuming that the mortgage payment is only interest, c^* or c_d is tax deductible. The homeowner thus chooses a mixture of equity and (risky) debt to finance the property investment *I* at an endogenously chosen time *T*.

We assume that the homeowner has only one property with a property price process given by a geometric Brownian motion.

$$dV = \mu V dt + \sigma V dW \qquad [1]$$

where W is a standard Brownian motion, μ the net property price drift and σ is the volatility.

Let $r > 0$ denote the risk free interest rate. Assume $r > \mu$ for convergence. We view the difference $(r - \mu)$ as a convenience yield or the flow of benefits that ownership of the property provides in addition to the expected capital gain μ per unit change of V. This is then treated as a form of imputed or implied housing rent which is proportional to the current value of the property V and equal to $(r - \mu)V$.

Let the tax rate be $0 \leq \tau < 1$. The funded property asset value is given by $F(V) = E(V) + D(V)$ where $E(V)$ is the value of equity and $D(V)$ the value of debt. The homeowner decides when to exercise the purchase option by purchasing the property for a fixed cost I and then benefits from the net stochastic property price increase/decrease of V $(V \geq 0)$ as well as collecting the convenience yield or market imputed rent by occupying the property.

For Model 1, after purchasing the property and taking on the mortgage liability, if the funded property value *F(V)* is sufficiently lower than the value of the nominal debt $D(V)$, that is E(V) is negative, the homeowner may consider defaulting on the mortgage payments, forcing the lender to consider repossession or foreclosure. In this case, if the lender does foreclose, the liquidation value (to the lender) is given by $(1-\alpha)\,F(V)$ where $0\ \leq \alpha\ \leq 1$ is the estimated foreclosure or deadweight costs as a proportion of the property at the moment of foreclosure sale, while the homeowner's equity value *E(V)* is zero if the mortgage is non-recourse. Alternatively, for Model 2 the homeowner may "ruthlessly" exercise the negotiation option with the lender. In this case, the lender may agree to renegotiate the mortgage contract resulting in a new lower and more affordable mortgage payment for the homeowner. This negotiation cannot happen under duress and the lender is still free to repossess their collateral should the homeowner consequently be delinquent and miss a payment.

The new mortgage payment is conditional on the "surplus" equity generated by avoiding foreclosure being "notionally" divided between the homeowner and lender based on their relative negotiating strength, (ϕ and $1-\phi$ whereby $\phi = 1$ implies that the homeowner has the greater share). The preservation of this "surplus" equity is the only potential "asset" of value, over which both a lender and homeowner may want to negotiate. We model the process as a cooperative Nash bargaining game (Fan and Sundaresan, 2000).

The methodology is similar to solving a perpetual American (scale) option entry/exit problem [four equations with 4 unknowns, see Dixit (1989)] and a solution is found for the different ODEs in terms of the critical entry/exit thresholds for the default or negotiation options, respectively Model 1, V_i/V_{di} and Model 2, $V_f/\,V_{df}$. Solutions are of the form $F(V) = A_0 + A_1V^{\gamma} + A_2V^{\beta}$ with the appropriate boundary conditions leading to different specific solutions.

Conventionally modelled default results in the lender repossessing the property. However, in the case of the negotiation option, homeowner and lender (originally) negotiate a mortgage contract, conditional on sharing the avoidable foreclosure costs, at the negotiation trigger point V_{df} with both willing to change the original (incomplete) contract terms. The lender now agrees a new mortgage coupon *C(V)* based on the current property price, lower than the initial mortgage coupon c_d (agreed at the investment threshold V_{di}) and the homeowner continues to own the property and collect the market imputed rent.

For Model 2, $F(V, C)$ is the property asset value before investment. At origination, $F(V, C)$ is simply the purchase price *I*. After purchase, $F(V, C)$ will be a complex function of the (market) property value V, foreclosure costs and tax benefits. The homeowner chooses the optimal investment threshold V_{di} and the optimal mortgage repayment c_d to maximise his equity position $E(V, C)$. As the property price V approaches infinity, the mortgage becomes riskless and hence the property value must satisfy an upper boundary condition whereby

$$\lim_{V \to \infty} F(V, C) = V + \frac{\tau C}{r} \qquad [2]$$

Lower boundary conditions for the negotiation option differ from the default option as lender/homeowner are prepared to vary the contract terms at the lower threshold, where the total value of the property funding arrangement $F(V_{df}, c_d)$ includes the value of future tax benefits.

The homeowner and lender thus bargain over a larger amount (when $V \leq V_{df}$) resulting in a property asset value *F(V)* of

$$F(V) = V + \frac{\tau c_d}{r}\left[1 - (\frac{\beta}{\beta - \gamma})\left(\frac{V}{V_{df}}\right)^{\gamma}\right] when\ V \geq\ V_{df} \quad [3]$$

$$F(V) =\ V + \frac{\tau c_d}{r}\left(\frac{-\gamma}{\beta - \gamma}\right)\left(\frac{V}{V_{df}}\right)^{\beta}\ when\ V < V_{df} \quad [4]$$

where $\beta > 1, \gamma < 0\ are\ the\ roots\ of\ \frac{\sigma^2 V^2}{2} + (\mu - \sigma^2/2)V - r = 0$

The equity equation *E(V)* ($V < V_{df}$) is also adjusted to account for the new mortgage payment $C(V)$ which is now a function of the current property value and $(r - \mu)V$ the market imputed rent.

$$\frac{1}{2}\sigma^2 V^2 \mathrm{E}_{VV}(V) + \mu V \mathrm{E}_V(V) - rE(V) + (r - \mu)V - C(V) = 0\ \ when\ V < V_{df} \quad [5]$$

With upper boundary conditions the same for both the negotiation and default options (Models 1 and 2), we obtain revised lower boundary conditions from the "extra" value of $F(V)$ using equation [4] and the Nash negotiation sharing rule to get

$$\lim_{V \downarrow V_{df}} E(V) = \phi\left(\alpha V_{df} - \frac{\tau C}{r}\frac{\gamma}{\beta - \gamma}\right) \quad [6]$$

Differentiating [6] gives

$$\lim_{V \downarrow V_{df}} E_V(V) = \phi(\alpha - \frac{\tau C}{V_{df} r}\frac{\gamma\beta}{\beta - \gamma}) \quad [7]$$

Further development (see Appendix B) leads to closed form expressions for the key outcomes for the negotiation option model and the comparable outcomes for the default option model.

a) The homeowner's investment threshold for the negotiation option V_{di} is given by

$$V_{di} = \frac{\beta}{\beta - 1}\left[1 + \frac{\tau}{gL}\right]^{-1} I \qquad \text{[8] or [A45]}$$

$$where\ g = \left[\frac{\beta}{\beta-\gamma}(1-\gamma)\right]^{-\frac{1}{\gamma}} = \frac{V_{di}}{V_{df}} \text{ and } L = \frac{1-\tau(1-\phi)}{1-\phi\alpha}$$

The investment threshold for the default option V_i is given by

$$V_i = \frac{\beta}{\beta - 1}\left[1 + \frac{\tau}{h}\right]^{-1} I \qquad \text{[9] or [A8]}$$

$$where\ h = \left[1 - \frac{\gamma(\tau + \alpha)}{\tau}\right]^{-1/\gamma} = \frac{V_i}{V_f}$$

b) The mortgage coupon for the negotiation option c_d (for $V \geq V_{df}$) is given by

$$c_d = r\frac{\gamma - 1}{\gamma}\frac{\beta}{\beta - 1}(gL + \tau)^{-1} I \qquad \text{[10] or [A46]}$$

The mortgage coupon for the default option c^* (for $V \geq V_f$) is given by

$$c^* = r\frac{\gamma - 1}{\gamma}\frac{\beta}{\beta - 1}[h + \tau]^{-1} I \qquad \text{[11] or [A7]}$$

We show in the next section that the consequence of these different outcomes for the default and negotiation option is that lenders *ex-ante* mortgage yield spread should increase significantly to pay for the homeowner's *ex-post* negotiation option.

c) Homeowners attempt to renegotiate with lenders when $V(t) < V_{df}$, where V_{df} is the endogenously determined negotiation threshold given by

$$V_{df} = \frac{\beta}{\beta - 1}\left[g + \frac{\tau}{L}\right]^{-1} I \qquad [12] \text{ or } [A14]$$

Homeowners default when $V(t) < V_f$, where V_f is the endogenously determined default threshold given by

$$V_f = \frac{\beta}{\beta - 1}[h + \tau]^{-1} I \qquad [13] \text{ or } [A6]$$

We show that the implications of these equations are that negotiation option exercise will occur earlier than the default option exercise for all homeowners but strong negotiators will exercise their negotiation option earlier than weak negotiators will.

d) The homeowner renegotiates a new coupon

$$C(V) = (1 - \alpha\phi)[(r - \mu)V] \qquad [14] \text{ or } [A41]$$

In other words, the renegotiated mortgage coupon is the current notional market imputed rent $[(r - \mu)V]$ times the factor $(1 - \alpha\phi)$ which is either equal to or less than 1 depending on the homeowner/lender heterogeneous bargaining power and the probable foreclosure costs. If $(r - \mu)$ is small, the new coupon is very small if α is large.

e) We define the optimal LTV_{di} at mortgage origination V_{di} for the negotiation option as the book value of debt divided by the property value at mortgage origination

$$LTV_{di} = \frac{D(V_{di}, c_d)}{F(V_{di}, c_d)}$$

This can be shown to be equivalent to

$$LTV_{di} = \frac{\gamma - [(1 - g^{\gamma})(1 + \tau(\phi - 1))]}{\gamma(gL + \tau)} \qquad [15] \text{ or } [A48]$$

f) The ex-post yield spread at origination is defined as

$$YS_{di} = \frac{c_d}{D(V_{di}, c_d)} - r \qquad [16]$$

and

$$YS_i = \frac{c_*}{D(V_i, c^*)} - r \qquad [17]$$

for both options respectively where D(.) is the value of debt at the investment threshold V_i or V_{di}.

We can show but more simply demonstrate in the next section that both the renegotiation threshold V_{df} given by [12] and the optimal investment threshold V_{di} given by [8] increase in the homeowner's bargaining power : $\frac{dV_{di}}{d\phi} > 0, \frac{dV_{df}}{d\phi} > 0$. The optimal coupon payment c_d given in [10] decreases in the homeowner's bargaining power : $\frac{dc_d}{d\phi} < 0$. When the homeowner's bargaining power is stronger, they can extract more out of the anticipated foreclosure costs and lenders therefore anticipate lower future coupons. As a result, the renegotiation threshold V_{df} increases with ϕ which lowers tax benefits. This implies that LTV, that is, the amount lenders are prepared to offer and the optimal coupon level decreases with homeowners increasing bargaining power. Hence, the incentive to invest decreases in ϕ and the stronger homeowner waits longer before purchasing a property. Another implication of both models is that the inefficiency of foreclosure costs α enters directly into the determination of the optimal investment thresholds and coupon payments. Given that g, τ, γ are constant while L depends on ϕ and α, we show the interesting result in [15] that the optimal loan to value ratio LTV_{di} at origination is dependent only on the bargaining power ϕ and the anticipated foreclosure costs α with the lender assuming that the homeowner can always pay the coupon.

When compared to the traditional default only scenario (Model 1), intuitively, a higher bargaining power gives homeowners more incentives to initiate or accelerate ex-post negotiation. However, this incentive, especially for those homeowners with weak bargaining power, must be balanced with the benefit of avoiding costly foreclosure. The result however is that the (Model 2) renegotiation threshold V_{df} dominates or is higher than the (Model 1) default threshold V_f i.e. $g + \frac{\tau}{L} > h + \tau$. While $g\ (> 1)$ is independent of the bargaining power ϕ, *L(≥0)* on the other hand is dependent on ϕ.

It is also important to note that the optimal investment threshold V_{di}, the optimal renegotiation threshold V_{df} and finally the optimal coupon payment c_d are all proportional to the initial purchase price *I*. As discussed earlier, what a homeowner occupier is prepared to pay for a property *I* may well be different from what a homeowner investor is prepared to pay for the same property due to the untaxed benefit of the market-imputed rent discussed earlier.

II Negotiation and Default Option Numerical Illustrations

The negotiation option represents the relationship between the purchase and financing decisions, where the initial *ex-ante* purchase decision is dependent on a (potential) renegotiation between lender and homeowner. On the other hand, the default (non-bargaining) option represents the relationship where the homeowner makes the purchase decision knowing that non-payment of the mortgage will result in the forfeiture of all equity.

This section demonstrates the effects of a negotiation option in a graphical manner and compares the fundamentally different quantitative results that arise from the two options using stylised US mortgage data and the equations derived in the preceding section. Where appropriate we transform the stochastic property price V($) to a LTV (%) where a LTV greater than 100% represents so-called negative equity. The parameter ϕ represents heterogeneous characteristics of the homeowner in relation to the lender impacting on their ability to negotiate. Recognising the impreciseness and difficulty of measuring this parameter, we observe how ***the negotiation region***, delineated by the extreme corner values of ϕ = (0, 1), in the various graphs, compares to the single ***default point***. *Ex-ante* mortgage origination, a homeowner and lender will only know that their relative bargaining positions *ex-post* a negotiation event must lie between these two extremes.

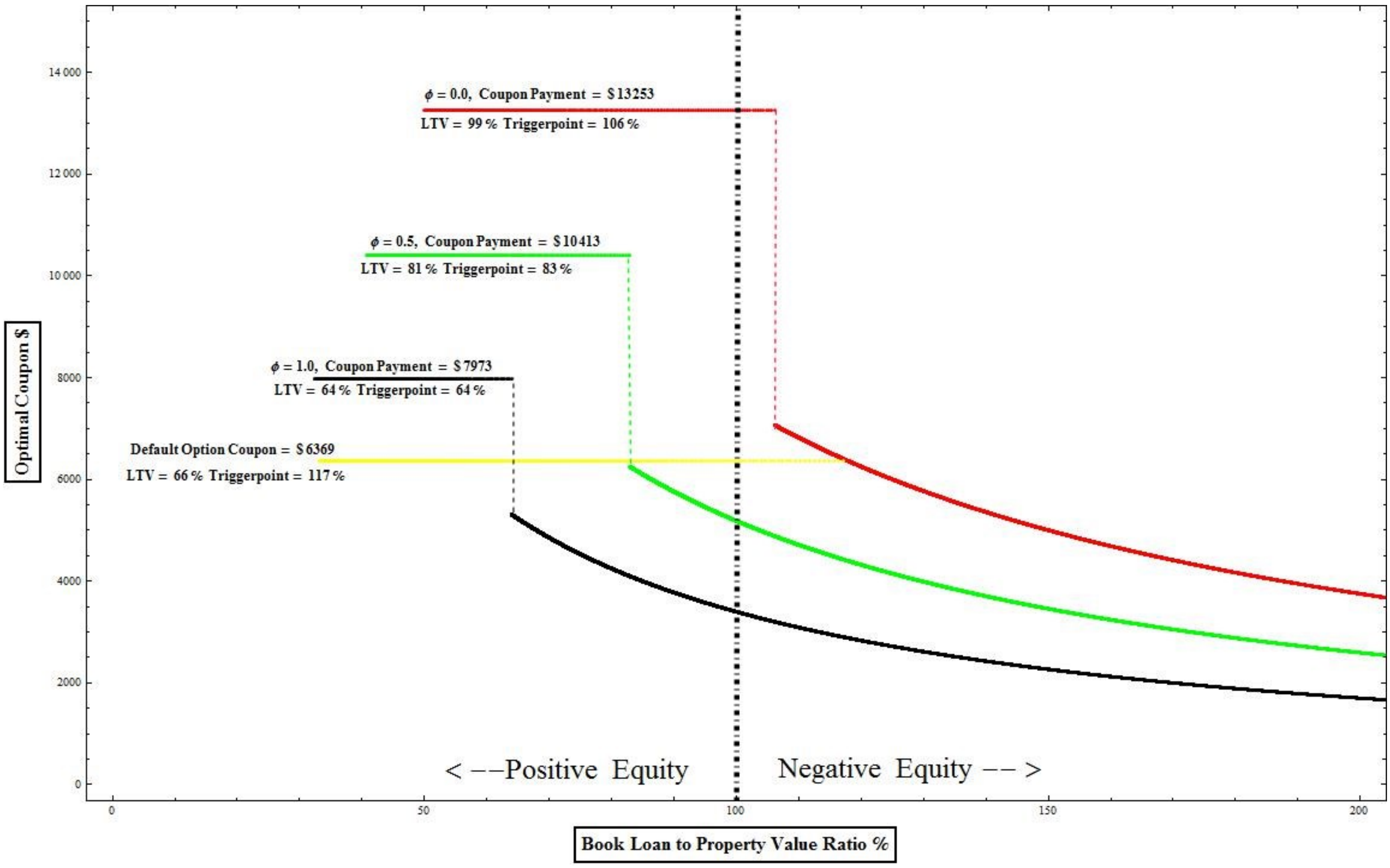
The homeowner decides to buy a property financed partly with debt paying the optimal coupon to a willing lender. The analysis proceeds as follows:

a) Calculate the mortgage (book) loan and payment at the optimal investment point.
b) Establish the critical negotiation region and default point as a function of BLTV.
c) Calculate the lender's risk spread (over the riskless rate).
d) Illustrate (some) model sensitivities to foreclosure costs and volatility.

We define BLTV as the nominal loan/spot value of the property. The nominal loan is constant and does not include missed payments as we assume that negotiation is triggered by the homeowner based on the current "spot" property value. Figure 2 plots mortgage coupon against the LTV for the range of negotiation strengths $0 \leq \phi \leq 1$.The mortgage coupon is constant until the LTV decreases to the negotiation exercise point at which moment the homeowner and lender negotiate a reduced coupon conditional on the spot value of the property. The new lower coupon *C(V)* is the product of the current market imputed rent $(r - \mu)V$ and a combination of the unavoidable foreclosure costs and their own negotiation ability$(1 - \alpha\phi)$.

Figure 2 Optimal Coupon Payment $ as a function of LTV and Bargaining Power φ.

The three discontinuous curves labelled φ=0.0, 0.5 and 1.0 are the coupon payment curves using equations [10] and [11]
For φ=0.0 a coupon c_d of $13253 is paid up to the negotiation exercise point and thereafter a decreasing coupon depending on the Book LTV
Coupon payments decrease as φ increases reflecting the lower mortgage (debt capacity) offered by the lender. The LTV quoted is at origination
The lower (heavy yellow) straight line is the constant coupon for the default option and terminates at the default exercise point of 117%
A coupon is always paid after exercising the negotiation option which becomes more affordable with increasing BLTV or negative equity
Parameter values : I = $ 250000, r = 0.03, μ = 0.00, τ = 0.20, α = 0.3 and σ = 0.10

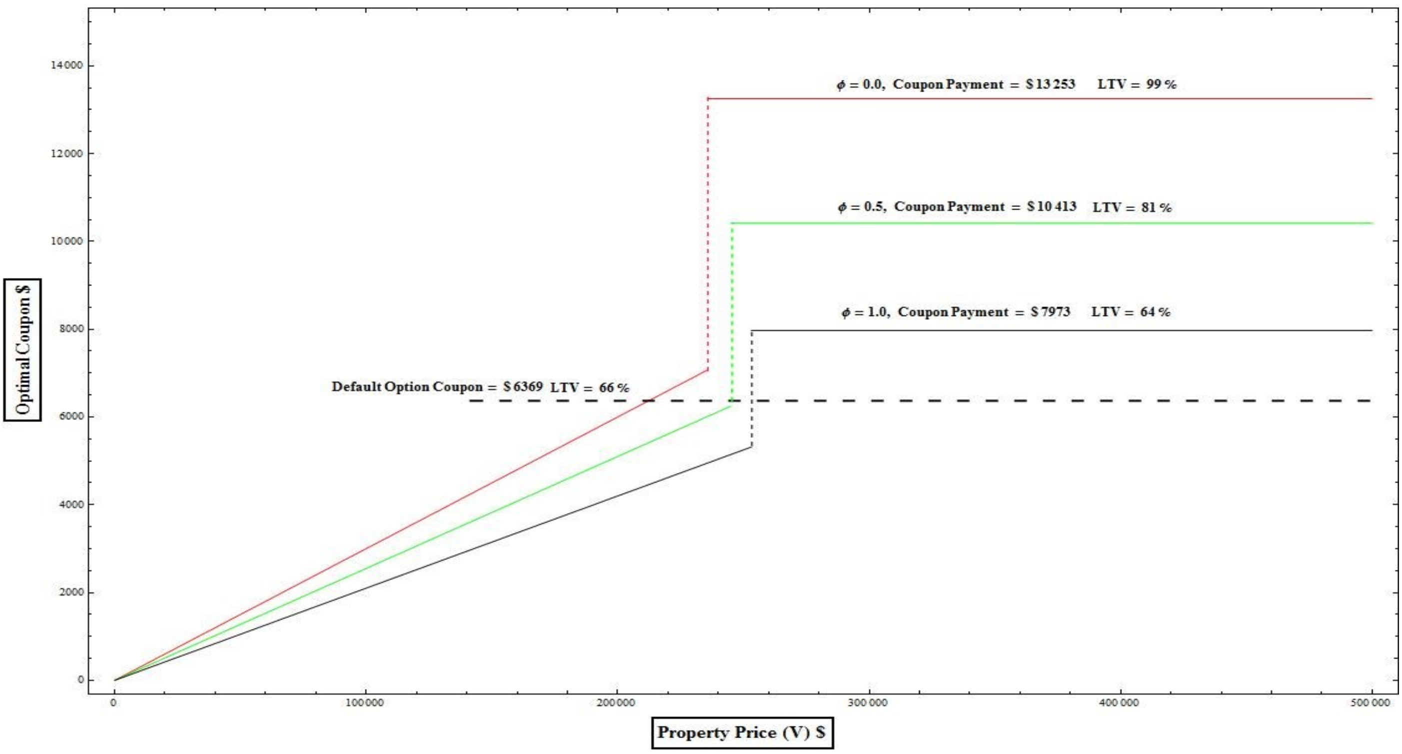


We note that a weak borrower would according to these model parameters negotiate once their LTV reached 106% or a change of +7% from their LTV at origination. A strong borrower would negotiate almost immediately should their LTV increase above their LTV at origination. A borrower with a default only option would in contrast default as soon as their LTV reached 117% compared to their LTV at origination of 64%.

We plot in Figure 3 the coupon payment as a function of property value F(V) instead of LTV %. We note that although a strong borrower with a negotiation option has a comparable LTV at origination to a borrower with a default only option (64% vs. 66%), the strong borrower pays a much higher monthly coupon for this right ($7973 vs. $6369).

Figure 3 Optimal Coupon Payment $ as a Function of Property Price V and Bargaining Power φ.

The three discontinuous curves labelled φ=0.0, 0.5 and 1.0 are the coupon payment curves using equations [10] and [11]
For φ=0.0 a coupon c_d of $13253 is paid up to the negotiation exercise point and thereafter a decreasing coupon depending on the Book LTV
Coupon payments decrease as φ increases reflecting the lower mortgage (debt capacity) offered by the lender. The LTV quoted is at origination
The lower (heavy yellow) straight line is the constant coupon for the default option and terminates at the default exercise point of 117%
A coupon is always paid after exercising the negotiation option which becomes more affordable with increasing BLTV or negative equity
Parameter values : I = $ 250000, r = 0.03, μ = 0.00, τ = 0.20, α = 0.3 and σ = 0.10



A strong negotiator should negotiate earlier than the weaker negotiator which might result in paying a reduced coupon, retaining ownership, collecting the market-imputed rent and retaining the "hope" that property values may bounce back recovering some lost equity.

The lender offers the highest mortgage (LTV_{di} = 99%) to the weak homeowner ($\phi = 0$) who pays the highest coupon. In the other case where the homeowner is strong ($\phi = 1$) the lender offers the lowest mortgage loan (LTV_{di} = 64%). This contrasts with the default option only where the mortgage ($LTV_d = 66\%$) and constant coupon is not dependent on negotiation ability or heterogeneous characteristics of the homeowner.

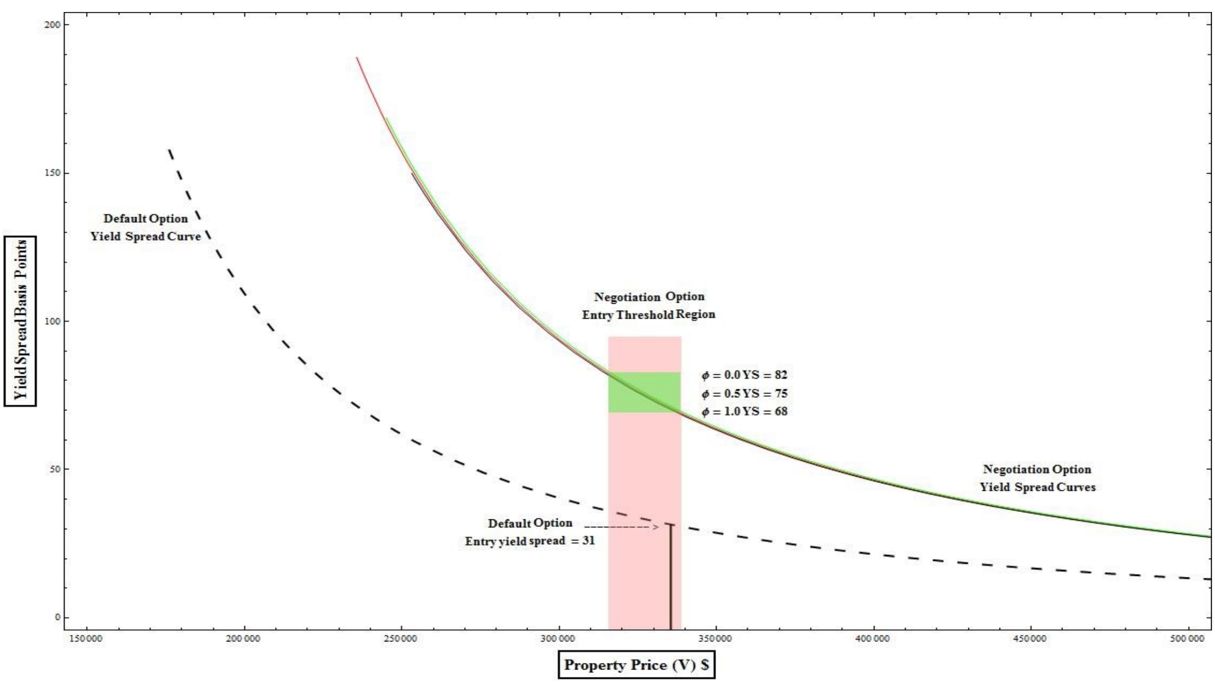
Finally, the weaker the homeowner is as a negotiator the closer the negotiation trigger point is to the optimal default point and the more likely, given unfavourable property shocks, that a homeowner may very quickly move from exercising their negotiation option to exercising a default option. The stronger the homeowner is as a negotiator, the earlier that the homeowner, who may not yet be in negative equity, exercises his negotiation option and the less likely that default will eventually result. It is clear that, with decreasing property values (i.e. increasing LTV), the economic consequences of the negotiation option are that the homeowner should endogenously choose to enter negotiation earlier and start paying a more affordable mortgage. With a default option, the homeowner will default later as soon as the critical threshold (LTV=117%) is reached. The more bargaining power the homeowner has, the earlier that negotiation will occur as the more financial concessions that may be extracted.

The overall direction of these results are consistent with PSV(2010) who claim that significant differences exist between the delinquency and default behaviour of securitised and non-securitised loans with larger effects for borrowers with a high FICO credit rating. Our model also suggests that significant differences should exist between the default behaviour of borrowers were they able to be characterised on the basis of their negotiation ability.

We compare the lender's yield spread over the risk free rate for a negotiation and default option in Figure 4. A borrower's negotiation option increases the lender's required risk spread compared to a default option. Variation between yield curves for the three values of parameter ϕ in the negotiation case is relatively small compared to that of the default option. Differences in yield spreads are more dependent on the optimal investment entry points with the weaker negotiator paying a higher yield because they make the investment earlier than a strong negotiator does. The existence of *any* measure of bargaining or sharing introduces a fundamental change to the contract whereby the lender requires a higher yield spread.

Figure 4 Yield Spread (Basis Points) as a Function of Property Price V and Bargaining Power φ.

The three convex curves labelled φ=0.0, 0.5 and 1.0 are the yield spread curves at motgage origination using equations [16] and [17]
For φ=0.0 the yield spread at the entry threshold is 82 basis points and decreases as φ increases reflecting greater bargaining power.
The lower dashed line is the yield spread curve for the default option with a value of 31 basis points at the investment entry threshold
Yield curves for the negotiation option coincide closely (in this example) but all differ significantly from that of the default option
Parameter values : I = \$ 250000, r = 0.03, μ = 0.00, τ = 0.20, α = 0.3 and σ = 0.10



We summarise key graphical data from Figures 2 and 4 in Table 1 overleaf, showing the effect of increasing foreclosure costs α and changes in property volatility σ. Increases in property price volatility σ with no changes in other parameters behaves as expected that is delaying investment, increasing yield spreads and reducing debt capacity (LTV) at origination. The higher the foreclosure costs, the lower the LTV the lender should agree with the strong negotiator while continuing to offer the same LTV to the weak negotiator. A large decrease in (LTV) lending capacity from 89% to 72% can be observed for the average value of ϕ=0.5 as foreclosure costs increase from 10% to 50%. Whether a lender in a strong negotiation position might lend to a homeowner in a weak negotiation position with probable large foreclosure costs is perhaps self-evident after reflection on lending practises and the effects of securitisation in the recent 2007/2008 US subprime crises.

Increasing volatility has a surprising effect resulting in an earlier exercise of the negotiation option but later exercise of the default option. This might indicate that a homeowner facing certain foreclosure "sits tighter" longer during periods of high volatility while a homeowner with a negotiation option will negotiate earlier for a more affordable mortgage coupon.

This is in itself not a surprising result and appears intuitively correct. From an option viewpoint, the interesting point is that volatility cause one option to accelerate and the other to decelerate. Unfortunately, this phenomenon, if correct, would make it very difficult to distinguish those homeowners who intend to default from those homeowners who only wish to negotiate. In this model, one might suppose that the first wave of homeowners contacting the lender are actually those who wish to negotiate rather than default. Homeowners who are poor negotiators would only contact the lender at the very last moment. This type of behaviour again would seem intuitively correct.

Table 1 Negotiation Option Sensitivity to Asset Volatility and Foreclosure Costs.

Summary of the outcomes for different values of α and σ illustrating the sensitivity of the output to different input parameters.
The second sub-table from the top down are the results for the base parameter case used for Figures 1 and 2
The first three sub-tables demonstrate that a lender offers a smaller loan (LTV@origination) as foreclosure costs α increase.
Increasing volatility σ increases the required yield spread, lowers the negotiation trigger point but increases the default trigger point
Parameter values : I = $ 250000, r = 0.03, μ = 0.00, τ = 0.20.

φ	Mortgage Coupon $	Book LTV % @ Default or Negotiation	Yield Spread @ Origination Basis Points	LTV @ Origination %
Foreclosure % α = 10% ,σ=0.10,μ=0.00				
0	13253	106	82	99
0.5	11472	93	75	89
1	9981	82	68	81
Default	7670	113	36	77
Foreclosure % α = 30% ,σ=0.10,μ=0.00				
0	13253	106	82	99
0.5	10413	83	75	81
1	7973	64	68	64
Default	6369	117	31	66
Foreclosure % α =50% , σ=0.10,μ=0.00				
0	13253	106	82	99
0.5	9323	73	75	72
1	5853	45	68	47
Default	5572	119	27	59
Foreclosure % α = 30% ,σ=0.20,μ=0.00				
0	29321	61	255	85
0.5	23051	46	243	68
1	17658	35	233	52
Default	9002	130	91	71

III Conclusions

We have combined two different aspects of real options that of irreversible investment and debt pricing/capital structure, to develop closed form solutions by which the borrower can choose the optimal *ex-ante* mortgage terms (LTV and mortgage coupon) and *ex-post* timing to exercise their negotiation option. We achieve this by applying methodological aspects of strategic endogenous default (Leland 1994) developed for corporate bond valuation to the *ex-ante* valuation of negotiated mortgages.

The default model can be regarded as the worst case from a lender's viewpoint as the model has been developed by maximising the homeowner's equity. In the real (option) world, exercise of the negotiation option may be initiated not only by a desire to optimise the homeowner's equity but also by a sub optimal trigger event. In both cases, a prudent lender (or policymaker) who has *ex-ante* priced their mortgages based on an optimal negotiation exercise may be in a stronger negotiation position by assuming that a sub optimal exercise will cost the homeowner more than the lender. From a lender's viewpoint, the optimal exercise behaviour may be less deserving than the second sub optimal exercise. The model suggests that in a declining house market those stronger negotiators, whether deserving or not, will initiate a negotiation earlier. Consequently, lenders may need to screen these first wave of homeowner applicants more closely with consequent higher screening and monitoring costs.

Policymakers and lenders should also be aware that with increasing property price volatility the model suggests some homeowners may try to accelerate the moment of negotiation while paradoxically other homeowners may try to delay the moment of default depending on which real option they intend to exercise.

This makes perfect sense from an ethical viewpoint but is typical of the moral hazard faced by the lender. If default results in certain foreclosure then homeowners will not be anxious to default. However, if negotiation results in a more affordable mortgage coupon, then homeowners in contrast will accelerate the exercise of that option. The question therefore arises as to whether this effect is observable in the current housing market as suggested by the Experian- Oliver Wyman (2010) report.

We emphasise that the option to renegotiate the mortgage payment is not a “free ride” or a costless option for the homeowner. The lender charges *ex-ante* higher yield spreads for this right compared to the default option. We have shown that the lender is no worse off in whatever bargaining position he finds himself and in most cases will be better off. Ultimately, if the lender cannot agree a new mortgage payment with a homeowner, then he can always foreclose with inevitable costs. We emphasise that it is a temporary mortgage modification which can revert to the original coupon when property prices recover. The homeowner remains responsible for paying off the full mortgage principal.

Implicit in our modelling is that the lender and homeowner should always agree new (sliding) mortgage payments conditional on the current property value. This is surely an abstraction from reality where in practise, due to the same aforementioned monitoring and screening costs, only one new lower affordable mortgage payment may be agreed, whereupon non-performance might lead to irrevocable foreclosure. The current HAMP program is such that, after renegotiation, non-performance after 3 months leads to inevitable foreclosure. However, the purpose of the model is to investigate the theoretical conditions for the initial negotiation rather than subsequent negotiations.

We have introduced an additional bargaining parameter ϕ (related to future unavoidable foreclosure costs) and compared to the traditional option theoretic mortgage default treatment. This parameter ϕ is a convenient construct to easily divide the benefits of avoiding foreclosure costs between lender and homeowner. The parameter is heterogeneous in that two homeowners with the same lender (or servicing agent) may have different values resulting in different outcomes of the (re)negotiated mortgage payment. In any case, we are less interested in the exact value of ϕ and more interested in delineating the maximum and minimum boundaries of the critical LTV region where renegotiation of the mortgage coupon may occur as a result of both parties wishing to avoid foreclosure costs. Better understanding of this region, compared to the traditional default region, may help lenders better screen (weak) homeowners who contact them later from those (strong) homeowners who contact them earlier and try to take advantage of lender weakness.

The strategic negotiation option has been demonstrated to have *ex-post* distinct economic and financial consequences. It remains to empirically investigate whether this idea of homeowners strategically delaying payments and negotiating actually occurs within an option theoretic equity optimising framework or rather within some other “affordability optimising” framework. We believe that given the large number of US homeowners, believed to strategically default, that an empirical investigation as to whether strategic negotiation might serve as a leading indicator could from the above exposition have merit and value.

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Appendix A - Glossary of Notation

V: "Spot" Property Price.

V_i: Optimal property price at where the borrower would invest with a default option only.

V_f: Optimal property price at where the borrower would default with a default option only.

V_{di}: Optimal property price at where the borrower would invest with a negotiation option.

V_{df}: Optimal property price at where the borrower would negotiate.

σ: Net property price volatility.

μ: Net property price drift.

β, γ: Roots of $\frac{\sigma^2 V^2}{2} + (\mu - \sigma^2/2)V - r = 0$

α : Lender's foreclosure costs as a percentage of the property price V

τ: Borrowers tax rate

r: Risk free rate of return

$c*$: Perpetual mortgage payment to the lender for the default only option (tax deductible).

c_d: Perpetual mortgage payment to the lender for the negotiation option (tax deductible).

$C(V)$: Renegotiated mortgage payment (after delinquency) which depends on property price.

ϕ: Heterogeneous bargaining or sharing parameter which lies between 0 and 1.

I : Initial property investment made at the critical investment thresholds V_i or V_{di}.

YS_i: Risk adjusted Yield Spread for the default only option at origination V_i.

YS_{di}: Risk adjusted Yield Spread for the negotiation option at origination V_{di}

$F(V, C)$: Asset value as a function of property price and mortgage payment.

$E(V, C)$: Equity value as a function of property price and mortgage payment.

$D(V, C)$: Debt value as a function of property price and mortgage payment.

$F_{ae}(V_f, 0)$: Property asset value at default with all equity (*ae*) financing and no taxes.

$F_{ae}(V_{df}, 0)$: Property asset value at delinquency with all equity (*ae*) financing and no taxes.

L: defined as $\frac{1-\tau(1-\phi)}{1-\phi\alpha}$

g: defined as $\left[\frac{\beta}{\beta-\gamma}(1-\gamma)\right]^{-\frac{1}{\gamma}} = \frac{V_{di}}{V_{df}}$ used for the negotiation option.

h: defined as $\left[1 - \frac{\gamma(\tau+\alpha)}{\tau}\right]^{-1/\gamma} = \frac{V_i}{V_f}$ used for the default only option.

Appendix B - Detailed Model Derivation

Default (No Bargaining) Option – Model 1

We proceed by first deriving a "default no bargaining" case based on Leland (1994), where the lender and homeowner do not bargain or share the value of the assets at the optimally chosen critical default threshold V_f. The lender forecloses and the property automatically becomes all equity financed and is presumed sold with no future tax benefits. At which point the value of the equity claim is $E(V_f) = 0 \; and \; D(V_f) = (1-\alpha)F_{ae}(V_f, 0))$ where α is the loss severity percentage, $E(V_f)$ is value of equity and $D(V_f)$ the value of debt at the default threshold V_f while $F_{ae}(V_f, 0)$ is the property value of an all equity financed investment with 0 coupon. Thus $\alpha F_{ae}(V_f, 0)$ is taken away by outsiders, $(1-\alpha)F_{ae}(V_f, 0)$ by the lender, and the homeowner gets nothing in strategic default.

If V follows a general Brownian motion given by [A1] then the solution for the general differential equation $F(V)$ where $aV^2F_{VV} + bVF_V + cF = Vd + e$

is given by $F(V) = A_1V^\gamma + A_2V^\beta + \frac{Vd}{b+c} + \frac{e}{c}$

where $\gamma < 0 \; and \; \beta > 0$ are roots with $\gamma, \beta = \frac{(a-b)\pm\sqrt{(b-a)^2-4ac}}{2a}$

and A_1, A_2 are determined from the appropriate boundary conditions.

The "spot" property price of the property, denoted by V, follows the general Brownian motion process given by

$$dV = \mu V dt + \sigma V dW \qquad [A1]$$

Where μ is the instantaneous expected rate of return of the property gross of any payout, σ is the instantaneous variance of the property price and dW is a standard Brownian motion.

Given [A1] the asset value $F(V)$ of a claim paying $Vd + e$ satisfies the equilibrium condition

$$rF(V_t) = Vd + e + \frac{1}{dt}\mathbb{E}_t^Q[F(V_{t+dt})]$$

The first two terms are the expected cash flow while the third term is the expected capital gain of $F(V_t)$ from t to t+dt.

Using [A1] and applying Ito's lemma to the third term we get a general ODE

$$rF(V) = Vd + e + \frac{1}{2}\sigma^2 V^2 F_{VV}(x) + \mu V F_V(V)$$

$$with\ a = \frac{1}{2}\sigma^2, b = \ \mu\ , c = -r \text{ and } \gamma, \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\left[\frac{1}{2} - \frac{\mu}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}}$$

where r > μ is the risk neutral instantaneous rate of return.

This has a general solution of the form

$$F(V) = A_1 V^\gamma + A_2 V^\beta + \frac{Vd}{(r-\mu)} + \frac{e}{r} \qquad \text{[A2]}$$

Solutions are sought for three value functions namely:

a) D(V) →the lenders debt value ,

b) E(V) → the borrower's equity value,

c) F(V) →the total property financing arrangement value.

The borrower has control over capital structure and decides on the mix of debt $D(V,C)$and equity $E(V,C)$ to finance the property whereby $F(V,C) = D(V,C) + E(V,C)$. The all equity financed property value is given by $F_{ae}(V,0)$. Hence at origination (t=0) $D(V_i, c^*) +$

$E(V_i, c^*) = I$ where V_i is the (initial threshold) property value that induces the investor to invest and c^* is the optimally chosen perpetual coupon agreed with the lender and I the initial investment. Debt value $D(V, C)$ is derived by letting d=0 and e=c in [A2] and solving with the following smooth pasting and value matching conditions

$$\lim_{V\to\infty} D(V, C) = \frac{C}{r}$$

$$D(V_f, C) = (1 - \alpha)\, F_{ae}(V_f, 0)$$

Resulting in

$$D(V, C) = \frac{C}{r}\left[1 - \left(\frac{V}{V_f}\right)^{\gamma}\right] + (1 - \alpha)F_{ae}(V_f, 0)\left(\frac{V}{V_f}\right)^{\gamma} \; for V > V_f \qquad \text{[A3]}$$

a) Equity value $E(V, C)$ is derived by letting d = $r - \mu$ and e = (1- τ)C in [A2] and solving with the following smooth pasting and value matching conditions

$$\lim_{V\to\infty} E(V, C) = V - \frac{(1 - \tau)C}{r}$$

$$E(V_f, C) = 0$$

Resulting in

$$E(V, C) = (1 - \tau)\left[\left(V - \frac{C}{r}\right) - \left(V_f - \frac{C}{r}\right)\left(\frac{V}{V_f}\right)^{\gamma}\right] for\ V > V_f \qquad \text{[A4]}$$

b) Property value $F(V, C)$ is derived by letting d = $r - \mu$ and e = τC in [A2] and solving with the following smooth pasting and value matching conditions

$$\lim_{V\to\infty} F(V, C) = V + \frac{\tau C}{r}$$

$$F(V_f, C) = 0$$

$$F(V, C) = F_{ae}(V, 0) + \frac{\tau C}{r}\left[1 - \left(\frac{V}{V_f}\right)^{\gamma}\right] - \alpha F_{ae}(V_f, 0)\left(\frac{V}{V_f}\right)^{\gamma} \; for\ V > V_f \qquad \text{[A5]}$$

Property value can be thus decomposed into the sum of three components

1) The all equity property value

2) Plus the tax shield value of benefits and

3) Less the foreclosure costs of V which the lender suffers in strategic default (V<D)

The tax benefit value of debt $\frac{\tau C}{r}\left[1-\left(\frac{V}{V_f}\right)^{\gamma}\right]$ is an increasing and concave function of V whereby if $\tau > 0$ there is some range which indicates that tax savings are achieved by higher levels of C. However $\frac{\tau C}{r}\left[1-\left(\frac{V}{V_f}\right)^{\gamma}\right]$ starts to decline with *C* at a certain point with the potential loss of tax benefits on foreclosure. Similarly $\alpha F_{ae}(V_f, 0)\left(\frac{V}{V_f}\right)^{\gamma}$ is increasing and convex in *C* but decreasing and convex in V. In a similar vein the endogenous default policy that maximises equity value for a given debt level solves $E_V(V) = \frac{\partial E(V,C)}{\partial V} = 0 \; at \; V = V_f$ and can be shown to be given by

$$V_f = \frac{\gamma}{\gamma - 1}\frac{C}{r}$$

Or using [A8] $$V_f = \frac{\beta}{\beta-1}[h+\tau]^{-1}I$$ [A6]or[13]

$$\text{Where } h = \left[1 - \frac{\gamma(\tau+\alpha)}{\tau}\right]^{-1/\gamma} = \frac{V_i}{V_f}$$

Rewriting [A5] we obtain

$$F(V,C) = F_{ae}(V,0) + \frac{\tau C}{r} - \left(\frac{V}{V_f}\right)^{\gamma}\left[\frac{\tau C}{r} + \alpha F_{ae}(V_f, 0)\right]$$

And substituting V_f from [A6] gives

$$= F_{ae}(V,0) + \frac{\tau C}{r} - \left(\frac{V}{\frac{\gamma}{\gamma-1}\frac{C}{r}}\right)^{\gamma}\left[\frac{\tau C}{r} + \alpha\frac{\gamma}{\gamma-1}\frac{C}{r}\right]$$

Grouping C in the 3rd term together and some cancellations gives

$$F_{ae}(V,0)+\frac{\tau C}{r}-\left(\frac{V}{\frac{\gamma}{\gamma-1}\frac{1}{r}}\right)^{\gamma}\left[\frac{\tau C^{1-\gamma}}{r}+\alpha C^{1-\gamma}\frac{\gamma}{\gamma-1}\frac{1}{r}\right]$$

Then taking $F_c(c)$

$$\frac{\tau}{r}-\left(\frac{V}{\frac{\gamma}{\gamma-1}\frac{1}{r}}\right)^{\gamma}\left[(1-\gamma)\frac{\tau c^{-\gamma}}{r}+\alpha(1-\gamma)c^{-\gamma}\frac{\gamma}{\gamma-1}\frac{1}{r}\right]$$

Substituting c^* , extracting $c^{-\gamma}$ back and setting the expression equal to 0 gives

$$\frac{\tau}{r}=\left(\frac{V}{c^*\frac{\gamma}{\gamma-1}\frac{1}{r}}\right)^{\gamma}\left[(1-\gamma)\frac{\tau}{r}-\frac{\alpha\gamma}{r}\right]$$

$$\frac{\tau}{r}=\left(\frac{V}{c^*\frac{\gamma}{\gamma-1}\frac{1}{r}}\right)^{\gamma}\left[\frac{\tau-\gamma\tau-\alpha\gamma}{r}\right]$$

Giving

$$c^*=\frac{Vr}{h}\frac{\gamma-1}{\gamma}$$

where $h=\left[1-\frac{\gamma(\tau+\alpha)}{\tau}\right]^{-1/\gamma}$

We now assume that the borrower only agrees the optimal coupon c^* with the lender and obtains the required funds when the property value $V=V_i$ the entry level threshold thus

$$c^*=\frac{V_i r}{h}\frac{\gamma-1}{\gamma}$$

Or using [A8] $$c^*=r\frac{\gamma-1}{\gamma}\frac{\beta}{\beta-1}[h+\tau]^{-1}I \qquad \text{[A7]or[11]}$$

We next examine the relationship of V_i with the initial investment I but show the derivation in more detail in the next section.

By investing the borrower collects $E(V,c) - \big(I - D(V,c)\big) = F(V,c) - I$

It can be shown that this implies that the optimal investment threshold V_i is given by

$$V_i = \frac{\beta}{\beta - 1}\left[\left(I - \frac{\tau c^*}{r}\right) + \frac{\beta - \gamma}{\beta}\left(\alpha V_f + \frac{\tau c^*}{r}\right)\left(\frac{V}{V_f}\right)^{\gamma}\right]$$

Using V_f in [A6] and letting $\frac{V_i}{V_f} = h$

$$V_i(\beta - 1) = \ \beta I - \beta\frac{\tau c^*}{r} + \frac{(\beta - \gamma)}{(\gamma - 1)}\frac{c^*}{r}[-\tau + (\alpha + \tau)\gamma]h^{\gamma}$$

$$\beta I - \beta\frac{\tau c^*}{r} + \frac{(\beta - \gamma)}{(\gamma - 1)}\frac{c^*}{r}\left[\frac{-\tau}{-\tau}(-\tau + (\alpha + \tau)\gamma)\right]h^{\gamma}$$

$$\beta I - \beta\frac{\tau c^*}{r} + \frac{(\beta - \gamma)}{(\gamma - 1)}\frac{c^*}{r}\left[-\tau(1 - \frac{\gamma(\tau + \alpha)}{\tau})\right]h^{\gamma}$$

$$= \beta I - \beta\frac{\tau c^*}{r} + \frac{(\beta - \gamma)}{(\gamma - 1)}\frac{c^*}{r}[-\tau h^{-\gamma}]h^{\gamma}$$

$$= \beta I + \frac{\tau c^*}{r}\left[-\beta + \frac{\beta - \gamma}{1-\gamma}\right]$$

$$= \beta I + \frac{\tau c^*}{r}\left[\gamma \frac{\beta - 1}{1-\gamma}\right]$$

And then substituting [A7]

$$= \beta I - \left[\tau(\beta - 1)\frac{V_i}{h}\right]$$

Bringing terms in I and x_i to opposite sides and rearranging gives

$$V_i = \frac{\beta}{\beta - 1}\left[1 + \frac{\tau}{h}\right]^{-1} I \qquad \text{[A8]or[9]}$$

Negotiation (Bargaining) Option- Model 2

With a negotiation option, in contrast to the default option, the lender and homeowner agree a new coupon conditional on sharing the avoided foreclosure costs at the optimally chosen critical negotiation threshold, denoted by V_{df}. The bargaining process will thus result in the reinstatement of the mortgage at the new lower payment. It is presumed that in this (actual or threatened) negotiation the lender mitigates his foreclosure costs as much as possible but is finally indifferent to whether the homeowner or an outsider "profits" from the value of the potential foreclosure costs such as administration, legal fees, social costs, loss on the property sale, etc. The sharing rule is now given by $E(V_{df}) = \alpha\phi F_{ae}(V_{df}, 0)$and $D(V_{df}) = (1 - \alpha\phi)\, F_{ae}(V_{df}, 0)$ where α is the loss severity and ϕ the bargaining strength $0 \leq \phi \leq 1$.

If $\phi = 0$ then the weak homeowner gets nothing while the lender gets $F_{ae}(V_{df}, 0)$. If $\phi = 1$ then the strong homeowner gets $\alpha F_{ae}(V_{df}, 0)$ (which outsiders would otherwise have received) while the weak lender receives$(1-\alpha)F_{ae}(v_{df}, 0)$. If $\phi = 1/2$ then the homeowner gets

$\alpha F_{ae}(V_{df},0)/2$(which outsiders would otherwise have received) while the lender receives$(1 - \alpha/2)F_{ae}(V_{df},0)$. Thus in all cases the lender is either better off or indifferent compared to the default case while the homeowner may be better off, but rather resume mortgage payments at a new more affordable coupon C(V).

The first section 4.9.2.1 of the derivation examines the case when borrower and lender bargain over the equity value at foreclosure, while the second section 4.9.2.2 extends the case to where they bargain over the added option value of including the homeowner's tax benefits. The choice of either specification will be dependent on the tax regime applicable to residential homeowners which differs from country to country

Without Sharing Tax Benefits – Model 2

We proceed by solving for $\mathbf{E(V)}$

Where $E(V) = \frac{1}{2}\sigma^2 V^2 E_{VV} + \mu V E_V - rE + (r-\mu)V - C(1-\tau) = 0$

As the value of the assets approaches infinity debt becomes riskless and

$$\lim_{V\to\infty} E(x) = V - \frac{(1-\tau)C}{r}$$

The new lower boundary conditions follow from the bargaining game whereby

$$\lim_{V\to V_{df}} E(V) = \phi\alpha V \qquad \text{[A9a]}$$

$$\lim_{V\to V_{df}} E_V(V) = \phi\alpha \qquad \text{[A9b]}$$

Using the general solution $E(V) = A_0 + A_1 V^\gamma + A_2 V^\beta$

[A10]

as $V \to \infty$, $V^\beta \to \infty \quad \Longrightarrow A_2 = 0$

while as $V \to \infty$, $V^\gamma \to 0 \Longrightarrow A_0 = V - \frac{C(1-\tau)}{r}$

Thus [A10] becomes

$$E(V) = V - \frac{C(1-\tau)}{r} \ + A_1 V^{\gamma} \qquad \text{[A11]}$$

Differentiating [A11] w.r.t. V and substituting [A9b] implies

$$E_V(V) = 1 - 0 + \gamma A_1 V^{\gamma-1} = \phi\alpha \qquad \text{[A12]}$$

While substituting [A9a] in [A11] implies

$$E(V_{df}) = V_{df} - \frac{C(1-\tau)}{r} \ + A_1 V_{df}^{\gamma} = \phi\alpha V_{df} \qquad \text{[A13]}$$

And then eliminating A_1 from [A12] and [A13]

$$\phi\alpha V_{df} \qquad = V_{df} - \tfrac{C(1-\tau)}{r} \ + \tfrac{\phi\alpha-1}{\gamma} V_{df}^{\gamma-\gamma+1}$$

$$= V_{df} - \frac{C(1-\tau)}{r} + \frac{\phi\alpha-1}{\gamma} V_{df}$$

$$\Rightarrow \tfrac{C(1-\tau)}{r} = V_{df}\left[(1-\phi\alpha) + \tfrac{(\phi\alpha-1)}{\gamma}\right]$$

$$= V_{df}(1-\phi\alpha)\left[1 - \frac{1}{\gamma}\right]$$

$$= V_{df}(1-\phi\alpha)\frac{\gamma-1}{\gamma}$$

$$V_{df} = \frac{C(1-\tau)}{r(1-\phi\alpha)} \frac{\gamma}{\gamma-1} \qquad \text{[A14] or [12]}$$

$$= \frac{\beta}{\beta-1}\left[g + \frac{\tau}{L}\right]^{-1} I$$

Taking A_1 from [A13] $\quad = \quad \left(\phi\alpha V_{df} - V_{df} + \frac{C(1-\tau)}{r}\right) V_{df}^{-\gamma}$ and substituting in [A11]

$$E(V) = V - \frac{C(1-\tau)}{r} + \left(\phi\alpha V_{df} - V_{df} + \frac{C(1-\tau)}{r}\right) V_{df}^{-\gamma} V^{\gamma}$$

$$E(V) = V - \frac{C(1-\tau)}{r}\left[1 - \left(\frac{V}{V_{df}}\right)^{\gamma}\right] - V_{df}\left(\frac{V}{V_{df}}\right)^{\gamma}(1-\phi\alpha) \qquad [A15]$$

We repeat the derivation for $D(V)$ but using $D(V_{df}) = (1-\alpha\phi)V_{df}$ to obtain

$$D(V) = \frac{C}{r}\left[1 - \left(\frac{V}{V_{df}}\right)^{\gamma}\right] + V_{df}\left(\frac{V}{V_{df}}\right)^{\gamma}(1-\phi\alpha) \qquad [A16]$$

Thus the total property value $F(x) = E(x) + D(x)$ is given by

$$V - \frac{C(1-\tau)}{r}\left[1 - \left(\frac{V}{V_{df}}\right)^{\gamma}\right] - V_{df}\left(\frac{V}{V_{df}}\right)^{\gamma}(1-\phi\alpha) + \frac{C}{r}\left[1 - \left(\frac{V}{V_{df}}\right)^{\gamma}\right] + V_{df}\left(\frac{V}{V_{df}}\right)^{\gamma}(1-\phi\alpha)$$

or $V + \frac{C}{r}\left[1 - \left(\frac{V}{V_{df}}\right)^{\gamma}\right](-1 + \tau + 1)$

Reducing to

$$F(V) = V + \frac{C\tau}{r}\left[1 - \left(\frac{V}{V_{df}}\right)^{\gamma}\right] \qquad [A17]$$

Sharing Tax Benefits – Model 2

This next section further extends the negotiation option whereby homeowner and lender negotiate over the extra option value arising from the tax benefit. Note that with an owner occupied mortgage in contrast to a non-owner occupied mortgage the notional income arising from property value increases is not immediately taxable.

We first solve for the total property value *F(x)*.

The total value of the property $F(V)$ satisfies the following differential equations

$$F(V) = \frac{1}{2}\sigma^2 V^2 F_{VV} + \mu V F_V - rF + (r-\mu)V + \tau C = 0 \ \ when\ V \qquad [A18]$$

and

$$F(V) = \frac{1}{2}\sigma^2 V^2 F_{VV} + \mu V F_V - rF + (r-\mu)V = 0 \;\; when\, V \le \;\; V_{df} \tag{A19}$$

with an upper boundary condition given by

$$\lim_{V\to\infty} F(V) = V + \frac{\tau C}{r} \tag{A20}$$

The lower boundary conditions follow from the bargaining game whereby

$$\lim_{V\uparrow V_{df}} F(V) = \lim_{V\downarrow V_{df}} F(V) \tag{A21}$$

$$\lim_{V\uparrow V_{df}} F_V(V) = \lim_{V\downarrow V_{df}} F_V(V) \tag{A22}$$

$$\lim_{V\downarrow 0} F(V) = 0 \tag{A23}$$

We find a solution of the form $F(V) = A_0 + A_1 V^{\gamma} + A_2 V^{\beta}$

Firstly for $V > V_{df}$

As V→ ∞ , $V^{\beta} \to \infty$ $\Longrightarrow A_2 = 0$

while as V→ ∞ , $V^{\gamma} \to 0$ $\Longrightarrow A_0 = V + \frac{\tau C}{r}$

Secondly for $V \le V_{df}$

as $V \to 0$, $V^{\gamma} \to 0$ $\Longrightarrow A_1 = 0$ while $\frac{\tau C}{r} = 0$

Giving

$$F(V) = V + \frac{\tau C}{r} + A_1 \left(\frac{V}{V_{df}}\right)^{\gamma} when\, V > V_{df} \tag{A24}$$

and

$$F(V) = V + A_2 \left(\frac{V}{V_{df}}\right)^{\beta} when\, V \le V_{df} \tag{A25}$$

However using boundary conditions [A21] when $V = V_{df} \Longrightarrow \frac{\tau C}{r} + A_1 = A_2$

Differentiating [A24] and [A25] and using [A22] when $V = V_{df} \Longrightarrow \gamma A_1 = \beta A_2$

Solving for $A_1 and A_2$ gives $A_1 = \frac{\tau C}{r}(\frac{\beta}{\gamma-\beta})$ and $A_2 = \frac{\tau C}{r}(\frac{\gamma}{\gamma-\beta})$

$$F(V) = V + \frac{\tau C}{r}\left[1 - (\frac{\beta}{\beta - \gamma})\left(\frac{V}{V_{df}}\right)^{\gamma}\right] when\ V > V_{df} \qquad [A26]$$

$$F(V) = V + \frac{\tau C}{r}\left(\frac{-\gamma}{\beta - \gamma}\right)\left(\frac{V}{V_{df}}\right)^{\beta} when\ V \leq V_{df} \qquad [A27]$$

The total value of the property financing arrangement F(V) includes the value of tax benefits and is thus higher than just the property value V. The borrower and lender thus bargain over a larger amount (when $V \leq V_{df}$) and from a bargaining viewpoint the share is such that $\widetilde{E}(V) = \tilde{\alpha}F(V)\ and\ \widetilde{D}(V) = (1 - \tilde{\alpha})F(V)$.The Nash solution $\tilde{\theta}^{*}$can be rewritten as

$$\tilde{\theta}^{*} = \arg\max\{\tilde{\alpha}F(V) - 0\}^{\phi}\{(1 - \tilde{\alpha})F(V) - max[(1 - \phi)V, 0]\}^{1-\phi}$$

$$= min\left[\phi - \phi\frac{(1 - \alpha)}{F(V)}V, \phi\right]$$

$$= min\left[\phi(1 - \frac{(1 - \alpha)}{F(V)}V), \phi\right]$$

$$= min[\phi(F(V) - (1 - \alpha)V), \phi]$$

We next solve for the equity value E(V) by setting up the ODE for the equity relationship

The equity value of the property E(V) satisfies the following differential equations

$$E(V) = \frac{1}{2}\sigma^2 V^2 E_{VV} + \mu V E_V - rE + (r - \mu)V - C(1 - \tau) = 0\ \ when\ V \qquad [A28]$$

and

$$E(V) = \frac{1}{2}\sigma^2 V^2 E_{VV} + \mu V E_V - rE + (r - \mu)V - C(V) = 0\ \ when\ V \leq V_{df} \qquad [A29]$$

where $C(V)$ is the coupon paid to the lender after renegotiation with boundary conditions

$$\lim_{V \to \infty} E(V) = V - \frac{C(1 - \tau)}{r} \qquad [A30]$$

Lower boundary conditions follow from the extra value of *F(V)* using [A27] with $V = V_{df}$

$$\lim_{V \downarrow V_{df}} E(V) = \phi(\alpha V_{df} - \frac{\tau C}{r}\frac{\gamma}{\beta - \gamma}) \qquad [A31]$$

Differentiating [A27] and again substituting $V = V_{df}$ gives

$$\lim_{V \downarrow V_{df}} E_V(V) = \phi(\alpha - \frac{\tau C}{rV_{df}}\frac{\gamma\beta}{\beta - \gamma}) \qquad [A32]$$

We find a solution of the form $E(V) = A_0 + A_1V^\gamma + A_2V^\beta$

Firstly for $V > V_{df}$

as $V \to \infty$, $V^\beta \to \infty$ $\qquad \Longrightarrow A_2 = 0$

while as V→ ∞ , $V^\gamma \to 0$ $\qquad \Longrightarrow A_0 = V - \frac{C(1-\tau)}{r}$

Giving

$$E(V) = V - \frac{C(1-\tau)}{r} + A_1V^\gamma \; when \; V > V_{df} \qquad [A33]$$

Differentiating [A33] w.r.t. V implies

$$E_V(V) = 1 - 0 + \gamma A_1 V^{\gamma - 1} \qquad [A34]$$

Setting [A34] equal to [A33] then multiplying across by $\frac{V}{\gamma}$ and substituting $V = V_{df}$ gives,

$$\frac{V_{df}}{\gamma} + A_1 V_{df}^\gamma = \frac{\phi\alpha V_{df}}{\gamma} - \phi\frac{\tau C}{r}\frac{\beta}{\beta - \gamma} \qquad [A35]$$

But then using [A31] and [A33] with $V = V_{df}$ we also get

$$V_{df} - \frac{C(1-\tau)}{r} + A_1 V_{df}^{\gamma} = \phi\alpha V_{df} - \phi\frac{\tau C}{r}\frac{\gamma}{\beta - \gamma} \qquad \text{[A36]}$$

and by eliminating $A_1 V_{df}^{\gamma}$ from [A35] and [A36]

$$V_{df} - \frac{C(1-\tau)}{r} + \frac{\phi\alpha V_{df}}{\gamma} - \phi\frac{\tau C}{r}\frac{\beta}{\beta-\gamma} = \frac{V_{df}}{\gamma} + \phi\alpha V_{df} - \phi\frac{\tau C}{r}\frac{\gamma}{\beta-\gamma}$$

Gathering terms

$$\frac{C(1-\tau)}{r} + \phi\frac{\tau C}{r}\left(\frac{\beta}{\beta-\gamma} - \frac{\gamma}{\beta-\gamma}\right) = V_{df}\left[(1-\phi\alpha) - \frac{1}{\gamma}(1-\phi\alpha)\right]$$

$$\frac{C(1-\tau)}{r} + \phi\frac{\tau C}{r} = V_{df}(1-\phi\alpha)\left[1 - \frac{1}{\gamma}\right]$$

$$V_{df} = \frac{(1-\tau(1-\phi))}{(1-\phi\alpha)}\frac{C}{r}\frac{\gamma}{\gamma-1}$$

$$V_{df} = L\frac{C}{r}\frac{\gamma}{\gamma-1} \qquad \text{[A37] or [12]}$$

Where $$L = \frac{(1-\tau(1-\phi))}{(1-\phi\alpha)} \qquad \text{[A38]}$$

We next solve for A_1 in [A35]

$$\frac{V_{df}}{\gamma} + A_1 V_{df}^{\gamma} = \frac{\phi\alpha V_{df}}{\gamma} - \phi\frac{\tau C}{r}\frac{\beta}{\beta-\gamma}$$

Giving

$$V_{df}^{\gamma} A_1 = \frac{\phi\alpha V_{df}}{\gamma} - \phi\frac{\tau C}{r}\frac{\beta}{\beta-\gamma} - \frac{V_{df}}{\gamma}$$

$$= \frac{V_{df}}{\gamma}(\phi\alpha - 1) - \frac{\phi\tau C}{r}\frac{\beta}{\beta-\gamma}$$

And then substituting for V_{df} on the RHS from [A37] gives

$$= \frac{C}{r}\frac{(\tau(1-\phi)-1)}{\gamma-1} - \frac{\phi\tau C}{r}\frac{\beta}{\beta-\gamma}$$

Resulting in

$$E(V) = V - \frac{C(1-\tau)}{r} + \left[\frac{C}{r}\frac{(\tau(1-\phi)-1)}{\gamma-1} - \frac{\phi\tau C}{r}\frac{\beta}{\beta-\gamma}\right]\left(\frac{V}{V_{df}}\right)^{\gamma} when\ V \qquad \text{[A39]}$$

Using the Nash bargaining game equity value when $V \leq V_{df}$ after negotiation is given by

$$E(V) = \phi[F(V) - (1-\alpha)V] \; when \; V \leq V_{df} \qquad [A40]$$

By differentiating [A40] once and then twice and substituting the results in [A29] we get

$$E_V(V) = \phi[F_V - (1-\alpha)]$$

$$E_{VV}(V) = \phi[F_{VV}]$$

$$C(V) = \frac{1}{2}\sigma^2 V^2 \phi[F_{VV}] + \mu V \phi[F_V - (1-\alpha)] - r\phi[F(V) - (1-\alpha)V] + (r-\mu)V$$

And gathering terms

$$= \phi\left[\frac{1}{2}\sigma^2 V^2 F_{VV} + \mu V F_V - rF(V)\right] + \phi(1-\alpha)[rV - \mu V] + (r-\mu)V$$

And then substituting in the first term using [A19]

$$= \phi[-(r-\mu)V] + \phi(1-\alpha)V[r-\mu] + (r-\mu)V$$

$$= (r-\mu)V[-\phi + \phi - \alpha\phi + 1]$$

$$C(V) = (1-\alpha\phi)(r-\mu)V \qquad [14]or[A41]$$

As in the Model 1 example we now proceed to calculate the optimal coupon $c_d$using V_{df} from [A37] and $F(V)$ from [A26] and maximising $F_c(V) = 0$

$$F(V) = V + \frac{\tau c}{r}\left[1 - \left(\frac{\beta}{\beta-\gamma}\right)\left(\frac{V}{V_{df}}\right)^{\gamma}\right]$$

Substituting for V_{df} from [A37]

$$= V + \frac{\tau}{r}\left[c - (\frac{\beta}{\beta-\gamma})V^{\gamma}c^{1-\gamma}(\frac{L}{r}\frac{\gamma}{\gamma-1})^{-\gamma}\right]$$

Differentiating w.r.t. c, substitutingc_d and setting equal to 0 gives

$$\left[1 - \left(\frac{\beta}{\beta-\gamma}\right)(1-\gamma)V^{\gamma}{c_d}^{-\gamma}(\frac{L}{r}\frac{\gamma}{\gamma-1})^{-\gamma}\right] = 0$$

$$\left[1 - V^{\gamma}(gc_d\frac{L}{r}\frac{\gamma}{\gamma-1})^{-\gamma}\right] = 0 \qquad [A42]$$

By letting $g^{-\gamma} = \left(\frac{\beta}{\beta-\gamma}\right)(1-\gamma) = \frac{V_{di}}{V_{df}}$ [A43]

We now assume (as for the default option) that the borrower only agrees the optimal coupon c_d with the lender and obtains the required funds when the property value $V = V_{di}$ the entry level threshold thus [A42] becomes

$$V_{di} = c_d g \frac{L}{r} \frac{\gamma}{\gamma - 1} \quad \text{[A44]}$$

Using [A26] again at c_d and V_{di}

$$F(V_{di}, c_d) = V_{di} + \frac{\tau c_d}{r}\left[1 - (\frac{\beta}{\beta - \gamma}) \left(\frac{V_{di}}{V_{df}}\right)^{\gamma}\right]$$

$$= V_{di} + \frac{\tau c_d}{r}\left[1 - (\frac{\beta}{\beta - \gamma})\, g^{\gamma}\right]$$

$$= V_{di} + \frac{\tau c_d}{r}\left[1 - (\frac{\beta}{\beta - \gamma}) \left(\frac{\beta - \gamma}{\beta(1 - \gamma)}\right)\right]$$

$$= V_{di} + \frac{\tau c_d}{r}\left[1 - \frac{1}{(1 - \gamma)}\right]$$

$$= V_{di} + \frac{\tau c_d}{r}\left[\frac{\gamma}{(\gamma - 1)}\right]$$

And from [A44]

$$= V_{di} + \frac{\tau r V_{di}}{gLr} \frac{(\gamma - 1)}{\gamma}\left[\frac{\gamma}{(\gamma - 1)}\right]$$

Giving that at the optimal investment moment

$$F(V_{di}, c_d) = V_{di}(1 + \frac{\tau}{gL})$$

Or when

$$V_{di} = F(V_{di}, c_d)(1 + \frac{\tau}{gL})^{-1}$$

However using standard real option relationship for an entry level investment it can be shown that the optimal investment occurs when $F(V_{di}, c_d)$ is $\frac{\beta}{\beta-1}I$.

Thus $$V_{di} = \frac{\beta}{\beta - 1}\left(1 + \frac{\tau}{gL}\right)^{-1} I$$ [A45] or [8]

Using [A44] $$c_d = r\frac{\gamma - 1}{\gamma}\frac{\beta}{\beta - 1}(gL + \tau)^{-1}I$$ [A46] or [10]

Derivation of a Closed Form Optimal LTV Negotiation at Origination

The LTV ratio is defined as $$LTV_{di} = \frac{D(V_{di}, c_d)}{F(V_{di}, c_d)} = \frac{F(V_{di}, c_d) - E(V_{di}, c_d)}{F(V_{di}, c_d)}$$ [A47]

(From [A26]and [A39] LTV_{di}

$$= \frac{V + \frac{\tau C}{r}\left[1 - (\frac{\beta}{\beta-\gamma})\left(\frac{V}{V_{df}}\right)^{\gamma}\right] - (V - \frac{C(1-\tau)}{r} + \left[\frac{C}{r}\frac{(\tau(1-\phi)-1)}{\gamma-1} - \frac{\phi\tau C}{r}\frac{\beta}{\beta-\gamma}\right]\left(\frac{V}{V_{df}}\right)^{\gamma})}{V + \frac{\tau C}{r}\left[1 - (\frac{\beta}{\beta-\gamma})\left(\frac{V}{V_{df}}\right)^{\gamma}\right]}$$

Cancelling some terms, and substituting V_{di}, c_d for V, c

$$= \frac{\frac{\tau c_d}{r}\left[(\frac{\beta}{\beta-\gamma})\left(\frac{V_{di}}{V_{df}}\right)^{\gamma}\right] + \frac{c_d}{r} - \left[\frac{c_d}{r}\frac{(\tau(1-\phi)-1)}{\gamma-1} - \frac{\phi\tau c_d}{r}\frac{\beta}{\beta-\gamma}\right]\left(\frac{V_{di}}{V_{df}}\right)^{\gamma})}{V_{di} + \frac{\tau c_d}{r}\left[1 - (\frac{\beta}{\beta-\gamma})\left(\frac{V_{di}}{V_{df}}\right)^{\gamma}\right]}$$

Substituting $V_{di} = gV_{df}$ and using the expression in [A37] for V_{df}

$$= \frac{\frac{\tau c_d}{r}\left[(\frac{\beta}{\beta-\gamma})(g)^{\gamma}\right] + \frac{c_d}{r} - \left[\frac{c_d}{r}\frac{(\tau(1-\phi)-1)}{\gamma-1} - \frac{\phi\tau c_d}{r}\frac{\beta}{\beta-\gamma}\right](g)^{\gamma})}{c_d g\frac{L}{r}\frac{\gamma}{\gamma-1} + \frac{\tau c_d}{r}\left[1 - (\frac{\beta}{\beta-\gamma})(g)^{\gamma}\right]}$$

Using $g^{\gamma} = \frac{\beta-\gamma}{\beta(1-\gamma)}$ and substituting and eliminating gives

$$= \frac{\frac{\tau c_d}{r}\left[\frac{1}{(1-\gamma)}\right] + \frac{c_d}{r} - \left[\frac{c_d}{r}g^{\gamma}\frac{(\tau(1-\phi)-1)}{\gamma-1}\right] + \frac{\phi\tau c_d}{r}\frac{1}{(1-\gamma)}}{c_d g\frac{L}{r}\frac{\gamma}{\gamma-1} + \frac{\tau c_d}{r}\left[1 - \frac{1}{(1-\gamma)}\right]}$$

$$= \frac{\left[\frac{\tau c_d}{r}\left[\frac{1}{(\gamma-1)}\right] - \frac{\phi\tau c_d}{r}\frac{1}{(\gamma-1)}\right] + [\frac{c_d}{r} - \frac{c_d}{r}g^{\gamma}\frac{\tau(1-\phi)}{(\gamma-1)} + \frac{c_d}{r}g^{\gamma}\frac{1}{(\gamma-1)}]}{c_d g\frac{L}{r}\frac{\gamma}{\gamma-1} + \frac{\tau c_d}{r}\left[\frac{\gamma}{(\gamma-1)}\right]}$$

$$= \frac{\left[\frac{\tau c_d}{r}\left[\frac{1}{(\gamma-1)}\right] - \frac{\phi\tau c_d}{r}\frac{1}{(\gamma-1)} - \frac{c_d}{r}g^{\gamma}\frac{\tau(1-\phi)}{(\gamma-1)}\right] + \frac{c_d}{r}\left(1 + g^{\gamma}\frac{1}{(\gamma-1)}\right)}{c_d g\frac{L}{r}\frac{\gamma}{\gamma-1} + \frac{\tau c_d}{r}\left[\frac{\gamma}{(\gamma-1)}\right]}$$

$$= \frac{\frac{c_d}{r}(1-g^{\gamma})\frac{\tau(1-\phi)}{(\gamma-1)} + \frac{c_d}{(\gamma-1)r}(\gamma-1+g^{\gamma})}{c_d g\frac{L}{r}\frac{\gamma}{\gamma-1} + \frac{\tau c_d}{r}\left[\frac{\gamma}{(\gamma-1)}\right]}$$

Eliminating $\frac{c_d}{r}\left[\frac{1}{(\gamma-1)}\right]$, gathering terms and simplifying

$$LTV_{di} = \frac{\tau(1-g^{\gamma})(1-\phi)+\gamma-1+g^{\gamma}}{\gamma gL+\gamma\tau} = \frac{\gamma-[(1-g^{\gamma})(1+\tau(\phi-1))]}{\gamma(gL+\tau)} \quad \text{[A48]or[15]}$$