Digraph functors which admit
both left and right adjoints

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Abstract

For our purposes, two functors Λ and Γ are said to be adjoint if for any digraphs G and H, there exists a homomorphism of $\Lambda(G)$ to H if and only if there exists a homomorphism of G to $\Gamma(H)$. We investigate the right adjoints characterised by Pultr in [A. Pultr, The right adjoints into the categories of relational systems, In *Reports of the Midwest Category Seminar, IV*, volume 137 of *Lecture Notes in Mathematics*, pages 100–113, Berlin, 1970]. We find necessary conditions for these functors to admit right adjoints themselves. We give many examples where these necessary conditions are satisfied, and the right adjoint indeed exists. Finally, we discuss a connection between these right adjoints and homomorphism dualities.

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1 Introduction

A *digraph functor* is a construction Γ which makes a digraph $\Gamma(G)$ out of a digraph G, such that if there exists a homomorphism of G to H, then there exists a homomorphism of $\Gamma(G)$ to $\Gamma(H)$. Consider two functors Λ and Γ. We say that Λ is a *left adjoint* of Γ and Γ is a *right adjoint* of Λ if the existence of a homomorphism of $\Lambda(G)$ to H is equivalent to the existence of a homomorphism of G to $\Gamma(H)$. Note that the precise categorial definition requires a natural correspondence between the morphisms of $\Lambda(G)$ to H and those of G

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to $\Gamma(H)$, but for our applications it is usually enough to distinguish between the existence and non-existence of a homomorphism between two digraphs. Hence we work in the "thin" category of digraphs, in which there is at most one generic morphism from one digraph to another.

Some significant constructions in graph theory turn out to be functors, and sometimes some of their fundamental properties are related to the fact that they have a left adjoint, a right adjoint, and sometimes both. Thus it is worth the while to characterise the pairs of adjoint functors. However, that objective may be out of reach for the moment. Our purpose is to lay groundwork in that direction.

In a sense, Pultr [6] has already characterised the pairs of adjoint functors. Nevertheless, his characterisation holds in the category of multidigraphs, where morphisms must specify images of vertices and of arcs. Right-left adjunction is preserved in the thin category of digraphs; however, more pairs of adjoint functors exist. In particular, some of the right adjoint functors of Pultr themselves admit right adjoints, while no such adjoints exist in the category of multidigraphs.

Some of the adjoints characterised by Pultr are in fact well-known constructions in graph theory. The left adjoints of Pultr encompass arc subdivisions and standard products, and his right adjoints encompass the shift graph construction and exponentiation.

We will call the left and right functors characterised by Pultr respectively *left Pultr functors* and *central Pultr functors*. One feasible objective seems to be the characterisation of the central Pultr functors which admit right adjoints. In Theorem 2.5, we prove necessary conditions. In Sections 3 to 7, we show that some of the functors satisfying these conditions do indeed admit right adjoints, which leads us to wonder whether these conditions are also sufficient.

The existence of central Pultr functors with right adjoints allows us to define new pairs of adjoint functors by composition. It is not at all clear whether compositions of Pultr functors suffices to define all pairs of digraph adjoint functors.

2 Pultr templates and functors

A *homomorphism* is an arc-preserving map between digraphs. If G, H are digraphs, we write $G \to H$ if there exists a homomorphism of G to H. G and H are called *homomorphically equivalent* if $G \to H$ and $H \to G$.

Definition 2.1. A *Pultr template* is a quadruple $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ where P, Q are digraphs and ϵ_1, ϵ_2 homomorphisms of P to Q.

Definition 2.2. Given a Pultr template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ the *central Pultr functor* $\Gamma_{\mathcal{T}}$ is the following construction: For a digraph H, the vertices of $\Gamma_{\mathcal{T}}(H)$ are the homomorphisms $g : P \to H$, and the arcs of $\Gamma_{\mathcal{T}}(H)$ are the couples (g_1, g_2) such that there exists a homomorphism $h : Q \to H$ with $g_1 = h \circ \epsilon_1, g_2 = h \circ \epsilon_2$.

Where P is a set or a digraph and ϵ is a mapping, by $\epsilon[P]$ we mean the image of this set or digraph (where the mapping is applied elemet-wise).

Definition 2.3. Given a Pultr template, $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ the *left Pultr functor* $\Lambda_{\mathcal{T}}$ is the following construction: For a digraph G, $\Lambda_{\mathcal{T}}(G)$ contains one copy P_u of P for every vertex u of G, and for every arc (u, v) of G, $\Lambda_{\mathcal{T}}(G)$ contains a copy $Q_{u,v}$ of Q with $\epsilon_1[P]$ identified with P_u and $\epsilon_2[P]$ identified with P_v.

Formally, the identification of P and P' is described by defining an equivalence relation in which the corresponding vertices of P and P' are equivalent, and then quotienting the digraph by this relation; see [1] for details.

Theorem 2.4 (Pultr [6]). *For any Pultr template $\mathcal{T}$, $\Lambda_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}$ are left and right adjoints.*

For some templates $\mathcal{T}$, the central Pultr functor $\Gamma_{\mathcal{T}}$ not only admits the left adjoint $\Lambda_{\mathcal{T}}$, but also a right adjoint $\Omega_{\mathcal{T}}$. Not all templates have this property. The following result provides necessary conditions.

Theorem 2.5. *Let $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ be a Pultr template such that $\Gamma_{\mathcal{T}}$ admits a right adjoint $\Omega_{\mathcal{T}}$. Then P and Q are homomorphically equivalent to trees. Moreover, for any tree T, $\Lambda_{\mathcal{T}}(T)$ is homomorphically equivalent to a tree.*

Proof. Suppose that $\Gamma_{\mathcal{T}}$ admits a right adjoint $\Omega_{\mathcal{T}}$. Let $H = \vec{P_0}$, the one-vertex digraph with no arcs. Then a graph G satisfies $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $\Gamma_{\mathcal{T}}(G)$ has no arcs, that is, $Q \nrightarrow G$. On the other hand, $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $G \to \Omega_{\mathcal{T}}(H)$, thus we have

$$Q \nrightarrow G \text{ if and only if } G \to \Omega_{\mathcal{T}}(H).$$

Therefore $(Q, \Omega_{\mathcal{T}}(H))$ is a homomorphism duality pair in the sense of [5]. It is known (by [4], see also [5]) that a digraph Q is the left-hand member of such a duality pair if and only if it is homomorphically equivalent to a tree.

A similar argument (with $H = \emptyset$) shows that P is also homomorphically equivalent to a tree. More generally, let T be a tree with dual $D(T)$. Then for any digraph G,

$$T \nrightarrow \Gamma_{\mathcal{T}}(G) \iff \Gamma_{\mathcal{T}}(G) \to D(T),$$

that is,

$$\Lambda_{\mathcal{T}}(T) \nrightarrow G \iff G \to \Omega_{\mathcal{T}}(D(T))$$

Therefore $(\Lambda_{\mathcal{T}}(T), \Omega_{\mathcal{T}}(D(T)))$ is a duality pair, whence $\Lambda_{\mathcal{T}}(T)$ is homomorphically equivalent to a tree. $\square$

It is not clear how to characterise the templates $\mathcal{T}$ with the property that for any tree T, the left adjoint $\Lambda_{\mathcal{T}}(T)$ is homomorphically equivalent to a tree. Small examples seem to suggest that templates with this property have a Q that is itself a tree, unless it is disconnected. Also, it remains open whether the converse to Theorem 2.5 holds. In the

rest of this paper we establish partial results in this respect. Our main result is Theorem 7.1 in Section 7, which proves that for templates $\mathcal{T}$ where P is a vertex or an arc, and Q is a tree, $\Gamma_\mathcal{T}$ has a right adjoint $\Omega_\mathcal{T}$. To facilitate the understanding of this main theorem without resorting to extremely formal definitions, we begin by several examples.

3 Example: The arc graph construction

We write $V(G)$ for the vertex set and $A(G)$ for the arc set of a digraph G. If x, y are vertices of G, we sometimes write $x \to y$ for $(x, y) \in A(G)$ (when there is no confusion about G). Note that we also write $G \to H$ for "there exists a homomorphism of G to H", but this notation is consistent, since the thin category of digraphs is itself a digraph. If X, Y are sets of vertices, we write $X \Rightarrow Y$ if $x \to y$ for any $x \in X$ and any $y \in Y$. We abbreviate $X \Rightarrow \{y\}$ to $X \Rightarrow y$ and $\{x\} \Rightarrow Y$ to $x \Rightarrow Y$. Note that for any set X, we have $\emptyset \Rightarrow X \Rightarrow \emptyset$.

The *arc graph* of a digraph G is the digraph $\delta(G)$ constructed as follows: For every arc $x \to y$ of G, $\delta(G)$ contains the vertex (x, y), and for every pair of consecutive arcs $x \to y \to z$ of G, $\delta(G)$ contains the arc $(x, y) \to (y, z)$. The arc graph construction is a well-known method for constructing graphs with large odd girth and large chromatic number (see [3]).

It turns out that the arc graph construction is a central Pultr functor. We have $\delta(G) = \Gamma_\mathcal{T}(G)$ where $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ with $P = \vec{P}_1 = (\{0, 1\}, \{(0, 1)\})$, $Q = \vec{P}_2 = (\{0, 1, 2\}, \{(0, 1), (1, 2)\})$, and ϵ_1, ϵ_2 mapping P to the first and second arc of Q. The right adjoint $\Omega_\mathcal{T}$ of $\Gamma_\mathcal{T} = \delta$ exists; we call it δ_R.

Definition 3.1. For a digraph H, the vertices of $\delta_R(H)$ are all the pairs (R^-, R^+) such that $R^-, R^+ \subseteq V(H)$ and $R^- \Rightarrow R^+$. (Note that $(\emptyset, V(H))$ and $(V(H), \emptyset)$ are vertices of $\delta_R(H)$.) $(R^-, R^+) \to (S^-, S^+)$ is an arc of $\delta_R(H)$ if and only if $R^+ \cap S^- \neq \emptyset$.

Proposition 3.2. *For any digraphs G, H, $\delta(G) \to H$ if and only if $G \to \delta_R(H)$.*

Proof. Let $g : G \to \delta_R(H)$ be a homomorphism, with $g(u) = (g_-(u), g_+(u))$. For any arc $u \to v$ of G, we have $g(u) \to g(v)$, hence $g_+(u) \cap g_-(v) \neq \emptyset$. Define $f : \delta(G) \to H$ by taking $f(u, v)$ to be any element of $g_+(u) \cap g_-(v)$. Whenever $(x, y) \to (y, z)$ in $\delta(G)$, we have $f(x, y) \in g_-(y) \Rightarrow g_+(y) \ni f(y, z)$; hence $f(x, y) \to f(y, z)$ in H. Therefore f is a homomorphism.

Conversely, let $f : \delta(G) \to H$ be a homomorphism. For $u \in V(G)$, put

$$g_-(u) = \{f(x, u) : (x, u) \in A(G)\},$$
$$g_+(u) = \{f(u, y) : (u, y) \in A(G)\}.$$

Whenever $x \to u \to y$ in G, we have $(x, u) \to (u, y)$ in $\delta(G)$, so $f(x, u) \to f(u, y)$ in H because f is a homomorphism. Thus $g_-(u) \Rightarrow g_+(u)$ for any vertex u of G. Hence $g(u) := (g_-(u), g_+(u))$ is a vertex of $\delta_R(H)$. Furthermore, if $u \to v$ in G, then $f(u, v) \in g_+(u) \cap g_-(v)$, and so $g_+(u) \cap g_-(v) \neq \emptyset$. This shows that $g(u) \to g(v)$ in $\delta_R(H)$. Therefore, $g : G \to \delta_R(H)$ defined by $g(u) = (g_-(u), g_+(u))$ is a homomorphism. $\square$

Now, consider the template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ with $P = \vec{P_1} = \big(\{0,1\}, \{(0,1)\}\big)$, $Q = \big(\{0,1,2,3\}, \{(0,1),(2,3),(0,2),(1,3)\}\big)$, and $\epsilon_1(0) = 0, \epsilon_1(1) = 1, \epsilon_2(0) = 2, \epsilon_2(1) = 3$. For any graph G, $\delta(G)$ is a subgraph of $\Gamma_{\mathcal{T}}(G)$, and $\Gamma_{\mathcal{T}}(G) = \delta(G)$ whenever G does not contain a 4-cycle $x \to y \to z \leftarrow w \leftarrow x$. However, $\Gamma_{\mathcal{T}}$ does not have a right adjoint. Indeed, even though Q is homomorphically equivalent to a tree, it is easy to see that for the tree $T = \big(\{0,\dots,5\}, \{(0,1),(1,2),(3,2),(3,4),(4,5)\}\big)$, $\Lambda_{\mathcal{T}}(T)$ is not homomorphically equivalent to a tree. Thus by Theorem 2.5, $\Gamma_{\mathcal{T}}$ does not have a right adjoint.

4 Example: A path template

In [2, 7], (undirected) graph constructions are studied, which turn out to be right adjoints of central Pultr functors for templates $(P, Q, \epsilon_1, \epsilon_2)$ where P is a point, Q is a path of odd length and ϵ_1, ϵ_2 map P to the endpoints of Q. Similar constructions work for directed graphs, we outline one example.

Let Q be the oriented path $0 \leftarrow 1 \to 2 \to 3$. Let $P = \vec{P_0} = (\{0\}, \emptyset)$, and $\epsilon_1(0) = 0$, $\epsilon_2(0) = 3$. For $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$, a right adjoint $\Omega_{\mathcal{T}}$ of $\Gamma_{\mathcal{T}}$ is constructed as follows.

Definition 4.1. For a digraph H, the vertices of $\Omega_{\mathcal{T}}(H)$ are all the pairs (a, A) such that $a \in V(H)$ and $A \subseteq V(H)$. (Note that each $(a, \emptyset)$ with $a \in V(H)$ is a vertex of $\Omega_{\mathcal{T}}(H)$.) $(a, A) \to (b, B)$ is an arc of $\Omega_{\mathcal{T}}(H)$ if and only if $b \in A \Rightarrow B$.

Note that for all $a, b \in V(H)$, $(a, \emptyset)$ is a sink, and (b, B) is a source unless $b \Rightarrow B$.

Proposition 4.2. *For any digraphs G, H, $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $G \to \Omega_{\mathcal{T}}(H)$.*

Proof. Let $g : G \to \Omega_{\mathcal{T}}(H)$ be a homomorphism, with $g(u) = (g_0(u), g_+(u))$. Define $f : \Gamma_{\mathcal{T}}(G) \to H$ by $f(u) = g_0(u)$. Whenever $u \to v$ in $\Gamma_{\mathcal{T}}(G)$, there exist vertices x, y such that $u \leftarrow x \to y \to v$ in G. Since g is a homomorphism, we then have $g(u) \leftarrow g(x) \to g(y) \to g(v)$ in $\Omega_{\mathcal{T}}(H)$. By definition of adjacency in $\Omega_{\mathcal{T}}(H)$, this implies

$$f(u) = g_0(u) \in g_+(x) \Rightarrow g_+(y) \ni g_0(v) = f(v).$$

Therefore $f(u) \to f(v)$, so f is a homomorphism.

Conversely, let $f : \Gamma_{\mathcal{T}}(G) \to H$ be a homomorphism. For $u \in V(G)$, put

$$g_+(u) = \{f(y) \colon (u, y) \in A(G)\}.$$

We define $g : G \to \Omega_{\mathcal{T}}(H)$ by $g(u) = (f(u), g_+(u))$. If (u, v) is an arc of G, then $f(v) \in g_+(u)$, and for every $(u, x), (v, y) \in A(G)$, we have $x \leftarrow u \to v \to y$ in G, hence $x \to y$ in $\Gamma_{\mathcal{T}}(G)$ and $f(x) \to f(y)$ in H. Thus $g_+(u) \Rightarrow g_+(v)$. Therefore $(g(u), g(v))$ is an arc of $\Omega_{\mathcal{T}}(H)$. This shows that g is a homomorphism. $\qquad\square$

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There are often many constructions of a right adjoint of a central Pultr functor, even though they are all homomorphically equivalent. The above construction of $\Omega_{\mathcal{T}}(H)$ is more compact than the construction derived later from the proof of Theorem 7.1, using one less coordinate. For future reference, we provide a second construction in the spirit of the proof of Theorem 7.1.

Definition 4.3. For a digraph H, the vertices of $\Omega'_{\mathcal{T}}(H)$ are all the triples (a, A_1, A_2) such that $a \in V(H)$, $A_1, A_2 \subseteq V(H)$ and $A_1 \Rightarrow A_2$. $(a, A_1, A_2) \to (b, B_1, B_2)$ is an arc of $\Omega'_{\mathcal{T}}(H)$ if and only if $b \in A_1$ and $B_1 \subseteq A_2$.

The following can be proved in a similar fashion to Proposition 4.2.

Proposition 4.4. *For any digraphs G, H, $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $G \to \Omega'_{\mathcal{T}}(H)$.*

We finish this section with a construction that gives the right adjoint $\Omega_{\mathcal{T}}$ for a fairly general family of templates $\mathcal{T}$, in which the respective Q's are oriented paths.

Definition 4.5. Suppose that Q is an oriented path with m arcs and vertex set $\{0, 1, \ldots, m\}$. Let $\mathcal{T} = (\vec{P_0}, Q, \epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2 map the one-vertex graph $\vec{P_0}$ to the end-points of Q. For a digraph H, let the vertex set of $\Omega_{\mathcal{T}}(H)$ be $V = \{(x, X_1, \ldots, X_m) : x \in V(H), X_i \subseteq V(H)$ for each i, $X_m \Rightarrow x\}$. There is an arc in $\Omega_{\mathcal{T}}(H)$ from $(x, X_1, \ldots, X_m)$ to $(y, Y_1, \ldots, Y_m)$ if

(1a) $x \in Y_1$ if $0 \to 1$ in Q,

(1b) $y \in X_1$ if $1 \to 0$ in Q; and

(2) for each $i = 1, \ldots, m - 1$:

    (a) $X_i \subseteq Y_{i+1}$ if $i \to i + 1$ in Q,

    (b) $Y_i \subseteq X_{i+1}$ if $i + 1 \to i$ in Q.

The correctness of this construction, asserted in the following proposition, follows from the proof of Theorem 7.1.

Proposition 4.6. *Let $\mathcal{T}$ and $\Omega_{\mathcal{T}}$ be as in Definition 4.5. Then for any digraphs G, H, we have $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $G \to \Omega_{\mathcal{T}}(H)$.*

5 Examples: Compositions of adjoint functors and multiply exponential constructions

The k-th iterated arc graph construction δ^k can be defined recursively by $\delta^{k+1} = \delta \circ \delta^k$. It can also be defined directly as $\delta^k = \Gamma_{\mathcal{T}}$, where $\mathcal{T} = (\vec{P_k}, \vec{P}_{k+1}, \epsilon_1, \epsilon_2)$, with

$$\vec{P_k} = \big(\{0, 1, \ldots, k\}, \ \{(0, 1), (1, 2), \ldots, (k - 1, k)\}\big),$$
$$\vec{P}_{k+1} = \big(\{0, 1, \ldots, k + 1\}, \ \{(0, 1), (1, 2), \ldots, (k, k + 1)\}\big),$$
$$\epsilon_1(i) = i \quad \text{and} \quad \epsilon_2(i) = i + 1.$$

All iterated arc graph constructions admit right adjoints, defined recursively by $\delta_R^{k+1} = \delta_R \circ \delta_R^k$. In particular this shows the existence of a right Pultr adjoint for templates with arbitrarily large P.

Note that the construction δ_R^k is an exponential construction iterated k times. In this case it turns out that multiply exponential size is necessary: For arbitrarily large integers n, there are graphs G such that $\chi(G) = n$ and $\chi(\delta^k(G)) = \Theta(\log^{(k)}(n))$. For $m = \Theta(\log^{(k)}(n))$, we then have $G \to \Omega(K_m)$, for any right adjoint Ω of δ^k. This means that $\Omega(K_m)$ needs at least n vertices.

For any two central Pultr functors Γ_1, Γ_2, the composition $\Gamma_1 \circ \Gamma_2$ is a central Pultr functor for a suitably defined template. If Ω_1, Ω_2 are right adjoints of Γ_1 and Γ_2 respectively, then $\Gamma_1 \circ \Gamma_2$ admits a right adjoint, namely $\Omega_2 \circ \Omega_1$. For $\Gamma_1 = \delta$ and $\Gamma_2 = \Gamma_{\mathcal{T}}$, with $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ being the template of Section 4, we get $\Gamma_1 \circ \Gamma_2 = \Gamma_{\mathcal{U}}$, with $\mathcal{U} = (\Lambda_{\mathcal{T}}(\vec{P}_1), \Lambda_{\mathcal{T}}(\vec{P}_2), \epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2 map $\Lambda_{\mathcal{T}}(\vec{P}_1)$ respectively to the first and second copy of $\Lambda_{\mathcal{T}}(\vec{P}_1)$ in $\Lambda_{\mathcal{T}}(\vec{P}_2)$. It is not clear whether the doubly exponential construction of a right adjoint $\Omega_{\mathcal{T}} \circ \delta_R$ of $\delta \circ \Gamma_{\mathcal{T}}$ is necessary in this case. Composing in the reverse order, we get $\Gamma_{\mathcal{T}} \circ \delta = \Gamma_{\mathcal{V}}$, where $\mathcal{V} = (\vec{P}_1, T, \epsilon_1, \epsilon_2)$ with $T = (\{0, 1, 2, 3, 4\}, \{(0, 1), (1, 2), (1, 3), (3, 4)\})$ and ϵ_1, ϵ_2 map $\vec{P}_1$ to the arcs $(1, 2)$ and $(3, 4)$ respectively. The proof of Theorem 7.1 gives an exponential construction of a right adjoint of $\Gamma_{\mathcal{V}}$, while the composition $\delta_R \circ \Omega_{\mathcal{T}}$ is doubly exponential.

Finally we note that with multiply exponential constructions, the conditions defining adjacency may become increasingly intricate. Consider the path $P = \vec{P}_2 = 0 \to 1 \to 2$ and let Q be the path $0 \to 1 \to 2 \leftarrow 0' \to 1' \to 2'$. Put $\epsilon_1[P] = 0 \to 1 \to 2$ and $\epsilon_2[P] = 0' \to 1' \to 2'$. For the Pultr template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$, $\Gamma_{\mathcal{T}}$ does admit a right adjoint. We give the following doubly exponential construction for $\Omega_{\mathcal{T}}$.

- The vertices of $\Omega_T(H)$ are quadruples $R = (R^{--}, R^{-+}, R^{+-}, R^{++})$ such that each of the four sets is a set of sets of vertices of H (that is, each $R^{**} \subseteq 2^{V(H)}$) and for any $M \in R^{-+}$ and any $N \in R^{+-}$ we have $M \cap N \neq \emptyset$.
- There is an arc $R \to S$ in $\Omega_{\mathcal{T}}(H)$ if and only if $\bigcup R^{++} \Rightarrow \bigcup S^{--}$, $R^{+-} \cap S^{--} \neq \emptyset$, and $R^{++} \cap S^{-+} \neq \emptyset$.

It can be shown that $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $G \to \Omega_{\mathcal{T}}(H)$. Both conditions "$A$ intersects B" and "every element of A intersects every element of B" are used in the construction of $\Omega_{\mathcal{T}}(H)$. If the converse of Theorem 2.5 holds, increasingly complex relations may be needed to describe right adjoints corresponding to each suitable Pultr template.

6 Example: A tree template
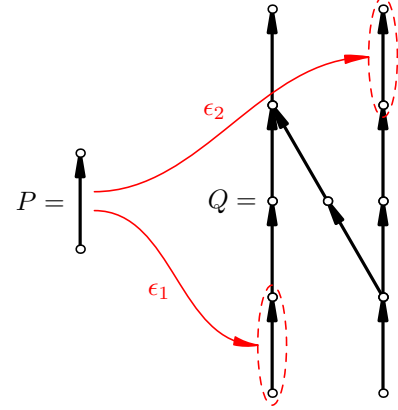
Our final example models the proof of Theorem 7.1.

Definition 6.1. Let $P = \vec{P}_1 = \left(\{0, 1\}, \{(0, 1)\}\right)$; let Q have vertex set $\{0, 1, \ldots, 10\}$ and arcs $0 \to 1 \to 2 \to 3 \to 4, 6 \to 7 \to 8 \to 9 \to 10, 7 \to 5 \to 3$. Let $\epsilon_1 : P \to Q$, $\epsilon_1(0) = 0$, $\epsilon_1(1) = 1$, and $\epsilon_2 : P \to Q$, $\epsilon_2(0) = 9$, $\epsilon_2(1) = 10$. Consider the Pultr template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$, see figure.

For a digraph H, define $\Omega_{\mathcal{T}}(H)$ by:

$$V(\Omega_{\mathcal{T}}(H)) = \Big\{ (S^-, S^+, S^{--}, S^{++}, S^{---}, S^{+++}, S^{-*+++}, S^{--*+++}) \in (2^{V(H)})^8 :$$

$$\text{if } S^+ \neq \emptyset, \text{ then } S^{---} \Rightarrow S^{--*+++} \Big\}$$

and $S \to T$ in $\Omega_{\mathcal{T}}(H)$ if
- $S^+ \cap T^- \neq \emptyset$,
- $S^- \subseteq T^{--}$,
- $S^{--} \subseteq T^{---}$,
- $T^+ \subseteq S^{++}$,
- $T^{++} \subseteq S^{+++}$,
- $S^{-*+++} \subseteq T^{--*+++}$, and
- $S^- = \emptyset$ or $S^{+++} \subseteq T^{-*+++}$.

Proposition 6.2. *Let $\mathcal{T}$ and $\Omega_{\mathcal{T}}$ be as in Definition 6.1. Then for any digraphs G, H we have $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $G \to \Omega_{\mathcal{T}}(H)$.*

Proof. Let $g : G \to \Omega_{\mathcal{T}}(H)$. Define $f : V(\Gamma_{\mathcal{T}}(G)) \to V(H)$ by setting $f(u,v)$ to be any element of the (nonempty) set $g(u)^+ \cap g(v)^-$. If $(u,v) \to (x,y)$ in $\Gamma_{\mathcal{T}}(G)$, then there exists $h : Q \to G$ such that $h(0,1) = (u,v)$, $h(9,10) = (x,y)$. By definition, $f(u,v) \in g(v)^-$. Because h and g are homomorphisms, $g(v) = g(h(1)) \to g(h(2))$ in $\Omega_{\mathcal{T}}(H)$, thus $g(v)^- \subseteq g(h(2))^{--}$. Similarly,

$$f(u,v) \in g(v)^- \subseteq g(h(2))^{--} \subseteq g(h(3))^{---},$$

and

$$f(x,y) \in g(x)^+ \subseteq g(h(8))^{++} \subseteq g(h(7))^{+++} \subseteq g(h(5))^{-*+++} \subseteq g(h(3))^{--*+++};$$

here, for the third inclusion we need to observe that $g(h(7))^- \neq \emptyset$, which follows from the existence of the arc $g(h(6)) \to g(h(7))$. Moreover, $g(h(3)) \to g(h(4))$, so $g(h(3))^+ \neq \emptyset$, and hence $g(h(3))^{---} \Rightarrow g(h(3))^{--*+++}$. Therefore $f(u,v) \to f(x,y)$ in $\Omega_{\mathcal{T}}(H)$ and consequently f is a homomorphism.

For any $u \in V(G)$, let u^+ be the set of all arcs outgoing from u in G, let u^{++} be the set of all arcs outgoing from outneighbours of u, etc. In this way, for instance, u^{+++} will be the set of images of the arc $(9,10)$ of Q, under any homomorphism h of $Q[7,8,9,10]$, the subtree of Q on $\{7,8,9,10\}$, to G, with the property that $h(7) = u$. Analogously, let

$$u^{-*+++} = \{h(9,10) : h : Q[5,6,\dots,10] \to G, \ h(5) = u\},$$
$$u^{--*+++} = \{h(9,10) : h : Q[4,5,\dots,10] \to G, \ h(4) = u\}.$$

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Now let $f : \Gamma_{\mathcal{T}}(G) \to H$. Define $g : V(G) \to V(\Omega_{\mathcal{T}}(H))$ by

$$g(u) = (f[u^-], f[u^+], f[u^{--}], \ldots, f[u^{--*+++}]).$$

If $u^+ \neq \emptyset, a_1 \in u^{---}, a_2 \in u^{--*+++}$, then we observe that there is a homomorphism $h : Q \to G$ such that $h(3) = u$, $h(0,1) = a_1$, $h(3,4) \in u^+$, and $h(9,10) = a_2$. Thus $a_1 \to a_2$ in $\Gamma_{\mathcal{T}}(G)$, and hence $f(a_1) \to f(a_2)$ in H. Therefore whenever $g(u)^+ \neq \emptyset$, we have $g(u)^{---} \Rightarrow g(u)^{--*+++}$, so each $g(u)$ is indeed a vertex of $\Omega_{\mathcal{T}}(H)$.

If $u \to v$ in G, then $f(u,v) \in g(u)^+ \cap g(v)^- \neq \emptyset$. Checking the inclusions in the definition of an arc of $\Omega_{\mathcal{T}}(H)$ is then relatively easy. We can conclude that $g(u) \to g(v)$ in $\Omega_{\mathcal{T}}(H)$, so g is a homomorphism. $\qquad\square$

7 Right Pultr adjoints for tree templates

In this section we consider templates $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$, where $P = \vec{P}_0$ (a single vertex) or $P = \vec{P}_1$ (a single arc), and Q is a tree. We prove the following:

Theorem 7.1. *Let $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ be a Pultr template such that $P = \vec{P}_0$ or $P = \vec{P}_1$ and Q is a tree. Then there exists a functor $\Omega_{\mathcal{T}}$ such that for any digraphs G, H we have $\Gamma_{\mathcal{T}}(G) \to H$ if and only if $G \to \Omega_{\mathcal{T}}(H)$.*

As a matter of fact, many different non-isomorphic constructions are possible (and we shall hint at some variations here as well), but they are all homomorphically equivalent.

Proof of Theorem 7.1. There is a pathologically trivial case where $\epsilon_1 = \epsilon_2$, which we do not consider in the following exposition.

The subtrees of Q. Every vertex of $\Omega_{\mathcal{T}}(H)$ will be a vector of subsets of $V(H)$, indexed by some rooted subtrees of Q. Namely, the rooted subtrees are determined as follows:

The images $\epsilon_1[P]$ and $\epsilon_2[P]$ in Q are connected by a path; call this path $\tilde{Q}$. The path $\tilde{Q}$ contains one vertex from each of $\epsilon_1[P]$ and $\epsilon_2[P]$. Thus in the template of Definition 4.1, $\tilde{Q} = Q$; in the arc graph construction, $\tilde{Q}$ is just the vertex 1; whereas for the $\mathcal{T}$ of Section 6, $\tilde{Q}$ is the path $1 \to 2 \to 3 \leftarrow 5 \leftarrow 7 \to 8 \to 9$. We choose any vertex m on the path $\tilde{Q}$ and call it the *middle vertex*. (The discretion in our choice of the middle vertex is one of the causes for the existence of many different right adjoints.)

If $P = \vec{P}_0$ and $\epsilon_1[P]$ and $\epsilon_2[P]$ are joined by an arc, either $\epsilon_1[P]$ or $\epsilon_2[P]$ can be taken to be the middle vertex. If $P = \vec{P}_1$ and the images $\epsilon_1[P]$ and $\epsilon_2[P]$ intersect (like in the arc graph template), the middle vertex is the vertex they intersect in.

Now, every non-leaf u of Q is a cut vertex. Consider each subtree induced by the vertex u and all the vertices of some component of $Q - u$ (obtained from Q by removing the vertex u); take only those components that do not contain the middle vertex. Let $\mathcal{S}_u$ be

the set of all such subtrees, each rooted in u. Finally, let $\mathcal{S}' = \bigcup\{\mathcal{S}_u : u$ is a non-leaf of $Q\}$. Notice that the root of each $T \in \mathcal{S}'$ has degree 1 in T.

Some of the trees in $\mathcal{S}'$ contain either $\epsilon_1[P]$ or $\epsilon_2[P]$; call this image the *P-arc* or *P-vertex* of that tree (depending on whether $P = \vec{P}_1$ or $P = \vec{P}_0$). Trees from $\mathcal{S}'$ with no P-arc or P-vertex are called *pendent subtrees* because they appear to hang from the path $\tilde{Q}$. Each $\mathcal{S}_u$ contains at most one non-pendent tree T_u unless u is the middle vertex m; then $\mathcal{S}_m$ may contain two non-pendent subtrees: $T_{m,1}$ containing $\epsilon_1[P]$ and $T_{m,2}$ containing $\epsilon_2[P]$.

Finally, let $\mathcal{S}$ consist of the trees in $\mathcal{S}'$, taken up to isomorphism. An isomorphism is meant to preserve not only the vertices and arcs, but the root and the P-vertex or P-arc as well.[1]

The vertices of $\Omega_{\mathcal{T}}(H)$. A vertex of $\Omega_{\mathcal{T}}(H)$ is
- any vector $(R^\bullet, R^T : T \in \mathcal{S})$ if $P = \vec{P}_0$,
- any vector $(R^T : T \in \mathcal{S})$ if $P = \vec{P}_1$,

where $R^\bullet \in V(H)$, $R^T \in \{0,1\}$ if T is a pendent subtree, and $R^T \subseteq V(H)$ otherwise, if it satisfies the following condition (remember that m is the middle vertex of Q):

$$\text{If } R^T = 1 \text{ for every pendent } T \in \mathcal{S}_m, \text{ then } R^{T_{m,1}} \Rightarrow R^{T_{m,2}}. \tag{1}$$

Recall that $A \Rightarrow B$ means that $a \to b$ for any vertex $a \in A$ and any vertex $b \in B$; here the arc is meant to exist in H.

The arcs of $\Omega_{\mathcal{T}}(H)$. Let R, S be vertices of $\Omega_{\mathcal{T}}(H)$. Thus $R = (R^\bullet, R^T : T \in \mathcal{S})$, $S = (S^\bullet, S^T : T \in \mathcal{S})$, or $R = (R^T : T \in \mathcal{S})$, $S = (S^T : T \in \mathcal{S})$. Then $R \to S$ in $\Omega_{\mathcal{T}}(H)$ if and only if all of the following conditions are satisfied:

- If $P = \vec{P}_0$: For an arc e of the path $\tilde{Q}$ with vertices a, b such that $a = \epsilon_i[P]$ for some $i \in \{1,2\}$, let T_b be the non-pendent tree rooted in b that contains a. If $e = (a, b)$, then we have the condition

$$\text{if } R^T = 1 \text{ for every pendent } T \in \mathcal{S}_a, \text{ then } R^\bullet \in S^{T_b}; \tag{2}$$

if $e = (b, a)$, then we have the condition

$$\text{if } S^T = 1 \text{ for every pendent } T \in \mathcal{S}_a, \text{ then } S^\bullet \in R^{T_b}. \tag{3}$$

- If $P = \vec{P}_1$: Let $\epsilon_1[P]$ consist of vertices a, b, and let $\epsilon_2[P]$ consist of vertices c, d, so that b and c would be "closer" to the middle vertex m; that is, b, c are vertices of the path $\tilde{Q}$. (If the images $\epsilon_1[P]$ and $\epsilon_2[P]$ intersect, then $b = c = m$.) The sets $\mathcal{S}_a$ and $\mathcal{S}_d$ contain only pendent trees. Let T_b be the non-pendent tree in $\mathcal{S}_b$ that contains a, b,

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[1]In fact, we may take these trees up to homomorphic equivalence, with homomorphisms preserving arcs, the root and the P-vertex or P-arc.

and let T_c be the non-pendent tree in $\mathcal{S}_c$ that contains c, d. If $a \to b$ in Q, put $A = R$ and $B = S$; if on the other hand $b \to a$ in Q, put $A = S$ and $B = R$. Analogously, if $c \to d$ in Q, put $C = R$ and $D = S$; if $d \to c$ in Q, put $C = S$ and $D = R$. Then we have the following three conditions:

$$\text{If } A^T = 1 \text{ for every } T \in \mathcal{S}_a, \text{ then } B^{T_b} \neq \emptyset. \tag{4}$$

$$\text{If } D^T = 1 \text{ for every } T \in \mathcal{S}_d, \text{ then } C^{T_c} \neq \emptyset. \tag{5}$$

$$\text{If } A^T = 1 \text{ for every } T \in \mathcal{S}_a \text{ and } D^T = 1 \text{ for every } T \in \mathcal{S}_d, \text{ then } B^{T_b} \cap C^{T_c} \neq \emptyset. \tag{6}$$

- For any arc e of the path $\tilde{Q}$ not covered by conditions (2)–(3), let a, b be the vertices of e so that every tree in $\mathcal{S}_a$ is a subtree of T_b, a non-pendent tree in $\mathcal{S}_b$. If $e = (a, b)$, then we have the condition

$$\text{if } R^T = 1 \text{ for every pendent } T \in \mathcal{S}_a, \text{ then } R^{T_a} \subseteq S^{T_b}; \tag{7}$$

if $e = (b, a)$, then we have the condition

$$\text{if } S^T = 1 \text{ for every pendent } T \in \mathcal{S}_a, \text{ then } S^{T_a} \subseteq R^{T_b}. \tag{8}$$

- For any other arc e of Q, let a, b be the vertices of e and let $T' \in \mathcal{S}_b$ so that every tree in $\mathcal{S}_a$ is a subtree of T'. If $e = (a, b)$, then we have the condition

$$\text{if } R^T = 1 \text{ for every } T \in \mathcal{S}_a, \text{ then } S^{T'} = 1; \tag{9}$$

if $e = (b, a)$, then we have the condition

$$\text{if } S^T = 1 \text{ for every } T \in \mathcal{S}_a, \text{ then } R^{T'} = 1. \tag{10}$$

Thus we might say that it is really tough to be an arc.

Please take another look at the example in Section 6. Which vertex of Q did we choose to be the middle vertex? You should be warned that in the example, we combined the roles of the non-pendent tree and the pendent tree $\vec{P}_1$ rooted in the tail, as well as the roles of the non-pendent tree and the pendent tree $\vec{P}_1$ rooted in the head. This has led to a construction not isomorphic but homomorphically equivalent to the recipe given here.

The homomorphisms. Let $f : \Gamma_{\mathcal{T}}(G) \to H$ be a homomorphism. We will define a mapping $g : V(G) \to V(\Omega_{\mathcal{T}}(H))$. For a vertex u of G and a subtree $T \in \mathcal{S}$ rooted in t, define $g(u)^T$ in the following way:
If T is pendent, let

$$g(u)^T = \begin{cases} 1 & \text{if there exists a homomorphism } h : T \to G \text{ such that } h(t) = u, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

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If T is non-pendent, let p be the P-vertex or the P-arc of T and let

$$g(u)^T = \left\{ f(h(p)) \colon h \colon T \to G \text{ is a homomorphism such that } h(t) = u \right\}. \tag{12}$$

Moreover, if $P = \vec{P}_0$, set

$$g(u)^\bullet = f(u). \tag{13}$$

First we need to verify that $g(u)$ is indeed a vertex of $\Omega_\mathcal{T}(H)$. Thus we need to verify the condition (1). Suppose that $u \in V(G)$ and that $g(u)^T = 1$ for every pendent $T \in \mathcal{S}_m$. Hence by definition there exist homomorphisms $h_T \colon T \to G$ with $h_T(m) = u$. Let $x_1 \in g(u)^{T_{m,1}}$ and $x_2 \in g(u)^{T_{m,2}}$. Then by (12) there are homomorphisms $h_i \colon T_{m,i} \to G$ with $h_i(m) = u$ and $x_i = f(h(\epsilon_i[P]))$ for $i = 1, 2$. Since all the trees in $\mathcal{S}_m$ share only the vertex m, we can define a homomorphism $h \colon Q \to G$ by putting $h(m) = u$, $h(a) = h_i(a)$ if a is a vertex of a non-pendent $T_{m,i}$, and $h(a) = h_T(a)$ if a is a vertex of a pendent $T \in \mathcal{S}_m$. Thus by definition $h(\epsilon_1[P]) \to h(\epsilon_2[P])$ in $\Gamma_\mathcal{T}(G)$. Because f is a homomorphism, $x_1 = f(h(\epsilon_1[P])) \to f(h(\epsilon_2[P])) = x_2$. Therefore $g(u)^{T_{m,1}} \Rrightarrow g(u)^{T_{m,2}}$ as we were supposed to prove.

We aim to show that g is a homomorphism, that is, that whenever $u \to v$ in G, then $g(u) \to g(v)$ in $\Omega_\mathcal{T}(H)$. Thus we need to check all the conditions (2)–(10). The conditions are all similar in nature and they are all proved by stitching together homomorphisms in a fashion similar to that of the previous paragraph.

Let us start with condition (2). Suppose that $P = \vec{P}_0$, $e = (a, b)$ is an arc of $\tilde{Q}$, $a = \epsilon_i[P]$. Let T_b be the non-pendent tree rooted in b that contains a. If $g(u)^T = 1$ for every pendent $T \in \mathcal{S}_a$, then by (11) there exist homomorphisms $h_T \colon T \to G$ with $h_T(a) = u$. Since T_b consists of the union of the trees $T \in \mathcal{S}_a$ and the arc (a, b), we may define $h \colon T_b \to G$ by putting $h(a) = u$, $h(b) = v$, and $h(x) = h_T(x)$ for any other vertex x of T_b, where T is the unique tree $T \in \mathcal{S}_a$ that contains x. By our assumption $u \to v$ in G, so h is a homomorphism. Then by (12) we have $g(u)^\bullet = f(u) = f(h(a)) \in g(v)^{T_b}$, which verifies (2).

Condition (3) is analogous to (2).

Suppose now that $P = \vec{P}_1$, $\epsilon_1[P] = (a, b)$, $\epsilon_2[P] = (c, d)$, b, c are vertices of the path $\tilde{Q}$. Let T_b be the non-pendent tree rooted in b that contains a and let T_c be the non-pendent tree rooted in c that contains d. If $g(u)^T = 1$ for every $T \in \mathcal{S}_a$, then by (11) there exist homomorphisms $h_T \colon T \to G$ with $h_T(a) = u$. As above, we can define a homomorphism $h \colon T_b \to G$ by putting $h(a) = u$, $h(b) = v$, and $h(x) = h_T(x)$ with T being the corresponding tree in $\mathcal{S}_a$. Since (a, b) is the P-arc of T_b, by (12) we have $f(u, v) \in g(v)^{T_b}$; hence $g(v)^{T_b} \neq \emptyset$, which verifies (4). By an analogous argument we can show that if $g(v)^T = 1$ for every $T \in \mathcal{S}_d$, then $f(u, v) \in g(u)^{T_c} \neq \emptyset$, which verifies (5). Therefore if both $g(u)^T = 1$ for every $T \in \mathcal{S}_a$ and $g(v)^T = 1$ for every $T \in \mathcal{S}_d$, then $f(u, v) \in g(v)^{T_b} \cap g(u)^{T_c} \neq \emptyset$, which verifies condition (6). If $\epsilon_1[P] = (b, a)$ and/or $\epsilon_2[P] = (d, c)$, the proof is analogous.

For condition (7), let $e = (a, b)$ be an arc of Q, let T_b be the non-pendent tree rooted in b that contains a, let T_a be the non-pendent tree rooted in a, and suppose that $g(u)^T = 1$ for every pendent $T \in \mathcal{S}_a$; thus by (11) for every such T there exists a homomorphism $h_T \colon T \to G$ such that $h_T(a) = u$. Whenever $x \in g(u)^{T_a}$, by (12) there exists a homomorphism

$h_{T_a} : T_a \to G$ such that $h(a) = u$ and $x = f(h_{T_a}(p))$, where p is the P-vertex or the P-arc of T_a (thus also the P-vertex or the P-arc of T_b). Note that T_b is the union of the arc (a, b) and of all the trees in $\mathcal{S}_a$; so there is a homomorphism $h : T_b \to G$ such that $h(b) = v$ that coincides with h_T on each subtree $T \in \mathcal{S}_a$, in particular, $h(p) = x$. Hence, by (12), $x \in g(v)^{T_b}$. Therefore $g(u)^{T_a} \subseteq g(v)^{T_b}$ as we are supposed to show.

Condition (8) is analogous to condition (7), and the remaining two conditions are very similar as well. Thus we have shown that if $\Gamma_{\mathcal{T}}(G) \to H$, then $G \to \Omega_{\mathcal{T}}(H)$.

To show the opposite implication, suppose that $g : G \to \Omega_{\mathcal{T}}(H)$ is a homomorphism. First, if $P = \vec{P}_0$, then vertices of $\Gamma_{\mathcal{T}}(G)$ are essentially the same as vertices of G. For $u \in V(G)$, put

$$f(u) = g(u)^\bullet. \tag{14}$$

If on the other hand $P = \vec{P}_1$, then vertices of $\Gamma_{\mathcal{T}}(G)$ are essentially the same as arcs of G. If (u, v) is an arc of G, then $g(u) \to g(v)$ in $\Omega_{\mathcal{T}}(H)$. Let a, b, c, d, T_b, T_c and A, B, C, D be as in conditions (4)–(5) with $R = g(u)$, $S = g(v)$. If the hypothesis of condition (6) is satisfied, define $f(u, v)$ to be an arbitrary vertex in $B^{T_b} \cap C^{T_c}$. If only the hypothesis of condition (4) is satisfied, define $f(u, v)$ to be an arbitrary vertex in B^{T_b}; if only the hypothesis of condition (5) is satisfied, define $f(u, v)$ to be an arbitrary vertex in C^{T_c}. Otherwise let $f(u, v)$ be an arbitrary vertex of H. (Note that if H has no vertices, then $\Omega_{\mathcal{T}}(H)$ has no arcs; thus if $G \to \Omega_{\mathcal{T}}(H)$, then $\vec{P}_1 \nrightarrow G$, and so $\Gamma_{\mathcal{T}}(G)$ has no vertices.)

Claim 1. *For any pendent $T' \in \mathcal{S}$ rooted in some b and any homomorphism $h : T' \to G$ we have $g(h(b))^{T'} = 1$.*

Proof. By induction on the number of arcs of T'. If T' has one arc, without loss of generality we may assume it is $a \to b$. Then $\mathcal{S}_a = \emptyset$ and we have $g(h(a)) \to g(h(b))$ in $\Omega_{\mathcal{T}}(H)$; hence by (9), $g(h(b))^{T'} = 1$. (If $a \leftarrow b$, then condition (10) applies.)

If T' has more than one arc, then still the root b has degree 1 in T', so there is a unique arc (w.l.o.g.) $a \to b$. By induction, for every $T \in \mathcal{S}_a$ we have $g(h(a))^T = 1$. Since $g(h(a)) \to g(h(b))$ in $\Omega_{\mathcal{T}}(H)$, by (9) we have $g(h(b))^{T'} = 1$. (Again, if $a \leftarrow b$, then condition (10) applies.) □

Claim 2. *Let b be any vertex of the path $\tilde{Q}$ in Q. Let T_b be a non-pendent tree rooted in b and let $i \in \{1, 2\}$ be such that T_b contains $p_i = \epsilon_i[P]$. Let $h : T_b \to G$ be a homomorphism. Then $f(h(p_i)) \in g(h(b))^{T_b}$.*

Proof. The proof is by induction on the distance of b from the P-arc or P-vertex of Q. First, if $P = \vec{P}_0$ and there is an arc $p_i \to b$ or $p_i \leftarrow b$, then by Claim 1 applied to all the pendent trees in $\mathcal{S}_{p_i}$ and the corresponding restrictions of h we get $g(h(p_i))^T = 1$ for every $T \in \mathcal{S}_{p_i}$. If $p_i \to b$ in T_b, then $h(p_i) \to h(b)$ in G and $g(h(p_i)) \to g(h(b))$ in $\Omega_{\mathcal{T}}(H)$. Hence $f(h(p_i)) = g(h(p_i))^\bullet \in g(h(b))^{T_b}$ by (2). If, on the other hand, $p_i \leftarrow b$ in T_b, then condition (3) applies analogously.

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Next, let $P = \vec{P}_1$ and let $p_i = (a, b)$. The non-pendent tree T_b consists of the arc (a, b) and all the pendent trees in $\mathcal{S}_a$. Again, by Claim 1 we have $g(h(a))^T = 1$ for every pendent $T \in \mathcal{S}_a$. Thus by the definition of f we have $f(h(p_i)) = f(h(a, b)) \in g(h(b))^{T_b}$. The case $p_i = (b, a)$ is analogous.

Finally, let T_b contain the arc (a, b) and suppose that a is not the P-vertex of T_b and (a, b) is not the P-arc of T_b. Let T_a be the non-pendent subtree of T_b rooted in a. By induction, $f(h(p_i)) \in g(h(a))^{T_a}$. For every pendent $T \in \mathcal{S}_a$, an application of Claim 1 to the restriction of h to T shows that $g(h(a))^T = 1$. As $a \rightarrow b$ in $\mathcal{T}_b$, we have $h(a) \rightarrow h(b)$ in G and $g(h(a)) \rightarrow g(h(b))$ in $\Omega_\mathcal{T}(H)$. Hence $g(h(a))^{T_a} \subseteq g(h(b))^{T_b}$ by (7). Therefore $f(h(p_i)) \in g(h(b))^{T_b}$. Just like before, the case $b \rightarrow a$ is analogous; condition (8) applies. $\qquad \square$

We aim to show that f is a homomorphism. Let $y \rightarrow z$ in $\Gamma_\mathcal{T}(G)$; y and z may be vertices or arcs of G, depending on whether $P = \vec{P}_0$ or $P = \vec{P}_1$. Then there exists a homomorphism $h : Q \rightarrow G$ such that $y = (h \circ \epsilon_1)[P]$ and $z = (h \circ \epsilon_2)[P]$. Put $p_1 = \epsilon_1[P]$ and $p_2 = \epsilon_2[P]$, so that $y = h(p_1)$ and $z = h(p_2)$.

For $i = 1, 2$, let h_i be the restriction of h to the non-pendent tree $T_{m,i}$ rooted in the middle vertex m of Q. By Claim 2, $f(y) = f(h_1(p_1)) \in g(h_1(m))^{T_{m,1}} = g(h(m))^{T_{m,1}}$ and $f(z) = f(h_2(p_2)) \in g(h_2(m))^{T_{m,2}} = g(h(m))^{T_{m,2}}$. By (1), $g(h(m))^{T_{m,1}} \Rightarrow g(h(m))^{T_{m,1}}$. Therefore $f(y) \rightarrow f(z)$ in H. Indeed, f is a homomorphism of $\Gamma_\mathcal{T}(G)$ to H. This concludes the proof of Theorem 7.1. $\qquad \square$

8 Pultr functors and homomorphism dualities

In our final section, we return to the connection between Pultr functors and homomorphism dualities, which enabled us to prove Theorem 2.5.

Definition 8.1. Let $\mathcal{F}$ be a set of digraphs and H a digraph. We say that $\mathcal{F}$ is a *complete set of obstructions* for H or that $(\mathcal{F}, H)$ is a *homomorphism duality* if for any digraph G,

$$G \rightarrow H \quad \Leftrightarrow \quad \forall F \in \mathcal{F} : F \nrightarrow G.$$

We also say that H has *tree duality* if it admits a complete set of obstructions all of whose elements are trees, and H has *finite duality* if it admits a finite complete set of obstructions.

Connections between left and central Pultr functors and homomorphism dualities were the topic of our paper [1]. Here we consider their relationship to the right adjoints.

Theorem 8.2. *Let $\mathcal{T}$ be a Pultr template. Suppose that $(\mathcal{F}, H)$ is a homomorphism duality.*

(1) If there exists a digraph K such that

$$\text{for any digraph } G, \quad \Gamma_\mathcal{T}(G) \rightarrow H \quad \Leftrightarrow \quad G \rightarrow K, \tag{15}$$

then $\{\Lambda_\mathcal{T}(F) : F \in \mathcal{F}\}$ is a complete set of obstructions for K.

Assume, moreover, that the template $\mathcal{T}$ satisfies the necessary conditions for the existence of $\Omega_{\mathcal{T}}$ given by Theorem 2.5.

(2) If H has tree duality and K satisfies (15), *then K also has tree duality.*

(3) If H has finite duality, then there does exist a digraph K that satisfies (15). *Moreover, K also has finite duality.*

Proof. (1) For any digraph G,

$$G \to K \Leftrightarrow \Gamma_{\mathcal{T}}(G) \to H$$
$$\Leftrightarrow \forall F \in \mathcal{F} : F \nrightarrow \Gamma_{\mathcal{T}}(G)$$
$$\Leftrightarrow \forall F \in \mathcal{F} : \Lambda_{\mathcal{T}}(F) \nrightarrow G.$$

(2) Let $\mathcal{F}$ be a complete set of obstructions for H consisting entirely of trees. Then $\Lambda_{\mathcal{T}}(F)$ is homomorphically equivalent to a tree $\Lambda'(F)$ for each $F \in \mathcal{F}$. Since for any digraph G we have $\Lambda_{\mathcal{T}}(F) \to G$ iff $\Lambda'(F) \to G$, the set $\mathcal{F}' = \{\Lambda'(F) \colon F \in \mathcal{F}\}$ is a complete set of tree obstructions for K.

(3) Let $\mathcal{F}' = \{\Lambda'(F) \colon F \in \mathcal{F}\}$ as above. By [5], there exists a digraph K such that $(\mathcal{F}', K)$ is a homomorphism duality. By the above equivalence, K satisfies (15). □

Thus for *all* the templates $\mathcal{T}$ satisfying the necessary hypotheses in Theorem 2.5, the central Pultr functor $\Gamma_{\mathcal{T}}$ admits a partial right adjoint $\Omega_{\mathcal{T}}$, defined at least on some subclass $\mathcal{D}$ of the class of all digraphs: namely, we can take the class of all digraphs with finite duality.

Furthermore, combining Theorem 7.1, compositions and sporadic examples as in Section 5, we get a wide class of templates satisfying the necessary hypotheses in Theorem 2.5 for which the central Pultr functor admits a right adjoint. This gives reason to think that the converse of Theorem 2.5 might hold. While known constructions of finite duals of trees are rather complex, a right adjoint of a central Pultr functor is even more general. This fact justifies the complexity of the construction given in Section 7.

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