Unpacking Pedagogy: New Perspectives for Mathematics

Chapter: Learning through digital technologies

Nigel Calder and Tony Brown

Ideas in Theory

How do pedagogical media transform the mathematical phenomena to which they seek to grant access? This chapter examines the ways in which understanding emerges when mathematical phenomena are engaged through digital pedagogical media, the spreadsheet in particular. We specifically consider the insights into teaching and learning processes that a hermeneutic frame presents. In the **Ideas in Practice** section the data, drawn from a study involving 10-year-old children, were illustrative of how their learning was fashioned by the particular affordances of the pedagogical media.

The pre-conceptions that each learner brings to mathematical phenomena, and activity associated with it, are derived from the specific cultural domain that the learner inhabits. Learning is a process of interpretation, where understanding and ‘concepts’ might be seen as states caught in on-going formation, rather than as once-and-for-all fixed realities. Our understandings of the mathematical phenomena, and of who we are, evolve through cyclical engagements with the phenomena and the constant drawing forward of prior experiences and understandings that are consequently formed through that engagement. Here ‘concepts’ are not fixed realities, but rather more elusive, formative processes that become further enriched as the learner views events from fresh, ever-evolving perspectives. The mathematical task, the pedagogical media, the pre-conceptions of the learners, and the dialogue evoked are inextricably linked. It is from their relationship with the learner that understanding emerges. This understanding is their interpretation of the situation through those various filters. Understanding emerges from cycles of interpretation, but this is forever in transition: there may always be another interpretation made from the modified stance (Brown, 2001, Calder, 2008).

Central to this interpretive process is the hermeneutic circle, which combines notions of language and structure in emphasising interpretation through the development of individual explanations (Gadamer, 1989). The learner develops explanations based on their interpretations of the phenomena, where these explanations then encounter resistance from broader discourses, so that understanding evolves and the explanations shift. Here there is always a gap between the interpretation and the explanation, and this provides the space for understanding and learning to occur. It is the circularity between present understanding and explanation, where the explanation gives rise to a change in perspective, which in turn evokes a new understanding. The interpreter's attention moves cyclically from the part to the whole, to the part and so forth, until, perhaps, some manner of resolution or consensus emerges. Within the learning context, the whole can be aligned with the various discourses or schema the learner brings to the situation, and the part with the specificity of the situation they confront (perhaps in the form of a learning activity). The learner's engagement oscillates between their prevailing discourse and the activity. With each of these iterations their perspective alters, and as they re-engage with the activity from these fresh perspectives, their understanding evolves. Ricoeur's (1981) notion of the hermeneutic circle, that guides us here, emphasises the interplay between understanding and the narrative framework within which this understanding is expressed discursively, and which helps to fix it. While these 'fixes' are temporary, they orientate the understanding that follows and the way this comes to be expressed. In seeing understanding as linguistically based, the student dialogue and comment provides the source for the interpretations of their mathematical understanding. In the research that follows the evolving history of the learner is a collaboration of their dialogue and the corresponding action. A hermeneutic viewpoint allows the incorporation of dialogue and actions, as the links between what was being said or written, and the participants' investigative approach, were examined in terms of their interpretation of the mathematical phenomena. As well, the data were hinged to the discourse that constituted its production and analysis. This perspective "begins with the problem of unmediated access to a *transparent* mathematical reality, shifting the emphasis from the critical learner as the site of original presence, to a decentred relational complex process" (Walshaw, 2001, p. 28). Consequently, by varying the pedagogical medium alternative frames are generated, hence rendering the learning experiences and ensuing dialogue in a different manner, and allowing space for the restructuring of mathematical understanding, for alternative ways of knowing (Brown, 2008a, b). This challenges the notion of constructed, abstract concepts being transposed intact across varying contexts.

In this chapter, we are specifically concerned with the learner's preconceptions of the pedagogical media, and how these in conjunction with the affordances and constraints offered by the media itself, promote distinct pathways in the learning process. That is, mathematical activity is inseparable from the pedagogical device as it were, derived as it is from a particular understanding of social organisation, and hence the mathematical ideas developed will inevitably be a function of this device. For example, in some Vygotskian accounts of mathematical learning, tools, such as linguistic constructs, are seen as acting as mediators situating the learning with reference to particular traditions (e.g., Lerman, 2006). Research involving the utilisation of ICT in mathematics education often utilise this frame in accounting for alternative cognitive internalisation through the mediation of cultural tools (e.g., Arzarello, Paola & Robutti, 2006; Marriotti, 2006). Confrey and Kazak (2006) likewise argued that learning in mathematics involves both activity and socio-cultural communication interacting in significant ways. They contend that neither influence is privileged, nor in fact can be separated, as we are simultaneously participants and observers in all enterprise, at all times. In a similar manner, the objectification of understanding can be perceived as being underpinned by the interplay of *typological* meaning (language) and *topological* meaning (visual figures and motor gestures) (Radford, Bardini, & Sabena, 2007).

In the study to be outlined in the following section an analysis was undertaken into the ways participants' pre-conceptions in mathematical thinking were re-organised by engagement through the spreadsheet medium. The manner in which these new perspectives then framed any subsequent re-engagements, and the participants' learning trajectories were influenced by the pedagogical medium, was considered. Likewise, the dialogue evoked by the engagement was examined to ascertain ways it may have led to alternative conceptualisation and understanding. The following section interrogates the learning process using research data viewed through the hermeneutic lens.

Ideas in Practice

The research for this thesis is part of an ongoing research programme exploring how spreadsheets might function as pedagogical media. The participants were drawn from 10-year-old students, attending five schools associated with the University of Waikato at Tauranga campus. They were at the time involved in a collaborative project offering programmes to develop gifted and talented students in their schools. There were twenty-one students (twelve boys and nine girls) who had been identified through a combination of problem-solving assessments and teacher reference. The pupils came from a range of socio-economic backgrounds. The participants were located in a classroom situation that included seven computers with spreadsheets as available software. This was the typical working environment for two of the schools, while the other three schools had three or four computers in each class at this level. For the students from those three schools, the computer access was therefore marginally less constrained than their usual class situation.

For the research project, the students worked on a programme of activities using spreadsheets to investigate mathematical problems, targeted at developing algebraic thinking. They were observed, their conversations were recorded and transcribed, and their investigations were printed out or recorded. There were school group interviews, and interviews with working pairs. They undertook a survey based on opinion and motivational considerations. Some on-going data was also gathered over a longer-term period (eighteen months) with three of the groupings, allowing for some case study styled data to emerge. Observations, and the recording and transcribing of participants' conversations in ensuing *Beach Brilliance* groups, also further enriched the data set and understandings gained. The research questions for this study centred upon the participants' learning experiences, when mathematics phenomena were encountered through the pedagogical medium of the spreadsheet. Allied to this were the understandings that emerged for the students in that learning environment. Hence the study was situated in classroom settings and initially approaches were used to gathering the data that involved observation, description and reporting. The inquiry attended to understandings and meanings, and with context profoundly implicated in meaning, a natural setting was considered most illuminating. However, the intrusion and associated influence of the researcher was inevitable. In their description of situations and occurrences, the researcher is influential in any experience by their presence (Mason, 2002). As such, they become a constituent of the data, but an aim was to minimise the intrusion, and while this presence would exert some influence, any ensuing effect was not the focus of the observations.

Participants were involved in the following procedures:

- Observations
- Activities using spreadsheets, as part of their programme
- Individual assessment tasks
- Interviews
- Questionnaires

Illustration of the hermeneutic circle

The following excerpt from the data illustrates how a hermeneutic circle models the process by which learners come to their understandings. It applied to a localised learning situation drawn from the study, which involved a pair of students investigating the 101 X activity (see Figure 1 below). It demonstrates how their generalisations of the patterns, and their understanding, evolved through interpreting the situation from the perspective of the preconceptions that were brought forth by their underlying discourses in the associated domains. These interpretations were from the perspectives summoned by personal discourses related to school mathematics, language, the pedagogical medium, and other socio-cultural influences. They influenced the manner in which the participants engaged with and then investigated the task, while the interaction with the task and subsequent reflection shifted their existing viewpoint, it repositioned their perspective. The participants then re-engaged with the task from that modified perspective. It was from this cyclical oscillating between the part (the activity) and the whole (their prevailing mathematical discourse), with the associated ongoing interpretations, that their understanding emerged.

101 times table

Investigate the pattern formed by the 101 times table by:

Predicting what the answer will be when you multiply numbers by 101

What if you try some 2 and 3 digit numbers? Are you still able to predict?

Make some rules that help you predict when you have a 1, 2, or 3-digit number. Do they work?

What if we used decimals?

Figure 1: 101 times table task.

They begin the task:

Clare: *Investigate the pattern formed by the 101 times table. When you multiply numbers by 101.*
Diane: *Times tables - so we just go like 2 x that and 3 x that.*

Their initial engagement and interpretations were filtered by their preconceptions associated with school mathematics. “Times table” is imbued with connotations for each of them drawn from their previous experiences. The linking of the term to “multiply numbers” and “ 2 X that and 3 X that …” brings to the fore interpretations of what the task might involve. These position their initial perspectives. Their preconceptions regarding the pedagogical medium were also influential. It was from the viewpoint evoked by these preconceptions that they engaged with the task. They entered the following:

101
202

Clare: *Yeah but couldn’t we just go times 2 or 101 times.*
Diane: *Yeah just do that.*
Clare: *You go equals, 101 times 2. Then you click in there. Oh man we did it. Now what are we going to go up to?*

Their engagement with the task, and the dialogue this evoked, were influenced by their understanding of the situation, the mathematical processes involved (e.g. the patterns) and the pedagogical medium. This interaction has shaped their underlying perspectives in these areas and they re-engaged with the task from these fresh perspectives.

They re-entered the data with a change to the format to give the following:

A	B	C
101	1	101
101	2	202
101	3	303
101	4	404
101	5	505
…	…	…

Diane: *What we did was, we got 101. We went into A1 then we typed in 101. Then we typed in B1, and then we typed in equals A1 then the times sign then two. Then we put enter and we dragged that little box down the side to the bottom to get all the answers. That gives you the answers when you multiply numbers by 101. We multiplied two by 101. You get 202.*

Clare: *So you get the number, zero, then the number again. The next thing is to try other numbers. Like two zero, twenty.*

They articulated an informal conjecture for a generalised form of the pattern, based on the visual pattern revealed by the spreadsheet structure, in conjunction with other affordances of the medium (e.g., instant feedback), and their mathematical preconceptions. They investigated the situation further from this fresh perspective.

Diane: *So if we do two-digit numbers can we still predict?*

Clare: *So we'll do like ten times 101. That's a thousand and ten.*

Diane: *Shall we try like 306.*

Clare: *No, we'll try thirteen, an unlucky number. That'll be 13, zero, 13.*

They enter 13 then drag down:

101	13	1313
101	14	1414

101	15	1515
101	16	1616 etc

.

Diane: Wow!!
Clare: Cool
Diane: Now putting our thinking caps on.

They had anticipated an outcome of 13, zero, 13 (13013) when 13 was entered, consistent with their emerging informal conjecture, yet the output was unexpected (1313). There was a difference between the *expected* and the *actual* output, initiating reflection and a reorientation of their thinking.

Clare: We had the number by itself then we saw that it was the double. So with two-digits you get a double number. What if we had three-digit numbers?
Diane: Lets try 100. That should add two zeros. Yeah see. OK now. Now copy down a bit.

101	100	10100
101	101	10201
101	102	10302
101	103	10403
101	104	10504
101	105	10605
101	106	10706
101	107	10807

Clare: Wow, there's a pattern. You see you add one to the number like 102 becomes 103 then you add on the last two numbers [02, which makes the 103, 10302. So 102 was transformed to 10302].

Their engagement with the task has evoked a shift in their interpretation of the situation. The alternating of their attention from the whole (their underlying perceptions) and the part (the task), as filtered by the pedagogical medium and their interaction, was modifying the viewpoint from which they engaged and the approach they engaged the task with. It was from their interpretations of this interplay of influences that their understanding was emerging. This cyclical oscillation from the part to the whole continued with their viewpoint refining with each iteration.

Diane: *Yeah, it's like you add one to the hundred and sort of split the number. Try going further.*

They dragged the columns down to 119 giving:

101	108	10908
101	109	11009
101	110	11110
101	111	11211
101	112	11312
…	…	…
101	118	11918
101	119	12019

Clare: *You see the pattern carries on. It works.*
Diane: *Look, there's another pattern as you go down. The second and third digit go 1,2, 3, up to 18, 19, 20 and the last two go 0, 1, 2, 3, up to 19. Its like you're counting on. Try a few more.*

101	120	12120
101	121	12221
101	122	12322
101	123	12423

Clare: *Right our rule is add one to the number then add on the last two digits. Like 123 goes 124 then 23 gets added on the end 12423-see.*
Diane: *OK lets try 200. That should be 20100*

They enter 200, getting:

101	200	20200

Oh…it's added on a 2 not a one.

This unexpected outcome evoked a tension with their emerging generalisation, instigating reflection and renegotiation of their perspective. The direction of their investigative process shifts slightly; they propose a new sub-goal or direction to their approach and investigate further.

Clare: *Maybe its doubled it to get 202 then got the two zeros from multiplying by 100. Try another 200 one.*

They enter 250 then 251 with the following output:

101	250	25250
101	251	25351

Diane: *No, it is adding two now-see 250 plus 2 is 252 then the 50 at the end [25250]. Where's that 2 coming from? Is it cause it starts with 2 and the others started with 1 [the first digit is a two as compared to the earlier examples where the first digit was a one]. See if it adds three when we use 300s.*

They enter in the following:

101	300	30300
101	350	35350

Diane: *Yes! Now 351 should be 354 and 51, so 35451. Lets see.*

The enter 351

101	351	35451

Clare: *OK then will you add 4 for the 400s? Lets see.*

They enter some numbers in the four hundreds getting the following output:

101	400	40400
101	456	46056
101	499	50399

Clare: *That last ones a bit weird, going up to a 5*
Diane: *Its adding 4 though. See, 499 plus 4 is 503 and then the 99 at the end. Now how do we put this. It adds the first number to the number then puts the last two digits at the end. We'll put some more 400s in to see. 490 should be 49490 and 491, 49591. Try.*

They entered those two numbers and then dragged down to get the following:

101	490	49490
101	491	49591
101	492	49692
101	493	49793
101	494	49894
101	495	49995
101	496	50096

The participants have negotiated a lingering consensus of the situation: one borne of their evolving interpretations as they engaged the task from their preconceptions in the associated domains. The ensuing interaction and reflection evoked subsequent shifts in their perspective. They then re-engaged with the task from these modifying perspectives. Each iteration of the hermeneutic circle transformed their interpretation of the situation, with the spreadsheet medium influential to their approach, interpretations, and inevitably their consensus of meaning. The mathematical understanding that emerged was inevitably a function of the pedagogical medium employed, in this case the spreadsheet, and the interplay of their interactions as framed by their underlying discourses.

The situation we hold, our positional viewpoint, influences the sense we make of unfamiliar phenomena. Likewise, the interpretations made by the participants, the researcher, and the readers were influenced by the space they occupied at that particular juncture and might have varied in different times. The layering of these local hermeneutic situations informs the macro position, but each retains specificity to its evolution. The data were also indicative of the complexity of influences entailed in a local hermeneutic circle. While the learner, the mathematical task, the pedagogical medium, and the learner’s discourses in those and related domains have primacy in the evolution of interpretation and understanding, discourses to do with power, advocacy, and expectation were pervasive. The particular examples employed, the inter-relationships of the group, and the manner in which their contributions are fashioned and expressed, all influenced the interpretations of and within the process in subtle ways. In the broader picture even these understated flavourings are borne of underlying discourses; that is, everything is brought forth from its interpretative lineage.

Reshaping generalisations

The next episode was part of the same investigation, but with a different pair of pupils, as they began to explore what happens to decimals. Ant predicted that if they multiplied 1.4 by 101, they would get 14.14

Bev: *I get it, cos if you go 14 you'll get fourteen, fourteen.*
Ant: *We'll just make sure.*

They entered 1.4, expecting to get 14.14 as the output.

Bev: *141.4, it should be 1, 4 (after the decimal point, that is 14.14).*

This created a visual perturbation. They began to rationalise this gap between the expected output (14.14) and the actual output (141.4). This visual perturbation caused a reshaping of their conjecture or informal generalisation. In doing so they drew on their current understandings of decimals and multiplication, but also had to amend that position to reconcile the visual perturbation the pedagogical medium has evoked. Again they used a visual lens to do so.

Ant: *We're doing decimals so its 141.4.*
Bev: *So it puts down the decimal [point] with the first number then it puts the 1 on, and then it puts in the point single number whatever.*
Ant: *It takes away the decimal to make the number a teen. Fourteen.*
Bev: *141.*
Ant: *Yeah. It takes away the decimal [14 – researcher's insertions] and then it adds a one to the end [141], and then it puts the decimal in with the four [141.4].*

Bev recognised that this was more a visual description of this particular case rather than a generalisation. There was still a tension with her prevailing discourse.

Bev: *No it doesn't, not always, maybe. It might depend which number it is.*
Ant: *Try 21 or 2.1. See what that does.*

According to Ant's conjecture from above, they would be expecting to take away the decimal point (21), add a one to the end (211), and then re-insert the decimal point and the one (211.1). However the output is 212.1, which created another visual perturbation to be reconciled.

Bev: *No it doesn't.*
Ant: *Two, where's the point? One two point one.*
Bev: *Oh yeah, so its like, the first number equals…*

They tried to formulate a more generalised conjecture.

Ant: *Takes away the decimal and puts that number down, then puts the first number behind the second number. Aw, how are we going to write this?*

Bev proffered a definition that they negotiated the meaning of, then situated within their emerging conjecture.

Bev: *It doubles the first numbers.*
Ant: *Takes away the decimal, doubles the first number, then puts the decimal back in.*
Bev: *How does it get here?*

They then entered 2.4 and made predictions regarding the output in light of their newer conjecture.

Ant: *Twenty-four, twenty-four with the decimal in here.*
Bev: *It will be doubled; twenty four, twenty four but the last number has a point in it, a decimal.*

The pupils' predictions were confirmed, and they negotiated the final form of their generalisation. They were still generalising in visual rather than procedural terms, and Bev suggested a name for their theory, double number decimals, that they both have a shared sense of understanding of. This mutual comprehension had emerged through the process; the investigative trajectory they had negotiated their way through. As with an example of a group in the previous chapter, the pupils had associated the term "double numbers" with the visual repetition of the digits e.g. 2424, rather than an operational meaning of actually doubling the number e.g. 24 X 2 = 48. This accentuated the visual interpretation they were applying in their dialogue. The investigative trajectory was influenced by the pedagogical medium through which the pupils engaged with the mathematical activity. More specifically, the questions evoked, the path they took, and the conjectures they formed and tested were fashioned by visual perturbations: the tension arising in their prevailing discourse by the difference between the expected and actual output.

Reconciling technical aspects and alternative forms

The following episode arose from another group's engagement with Rice Mate, a task associated with the doubling of grains of rice for each square of a chessboard. The learning pathway evolved differently from the previous group, but the unexpected output that was generated from engaging with the task, permitted alternative approaches to be considered and explored. This re-envisioning fashioned their understanding in this regard. The tension that arose when there was a gap between their expected output and the actual output promoted the restructuring of their perspective and they approached the task in a slightly modified manner. The recursion of their attending to the task, and interpretation through modified perspectives, allowed the evolution of understanding of technical and conceptual elements of their activity. They began by considering the first square of the chessboard and negotiating a way to double the number of grains of rice in subsequent squares:

Tony: *OK, A1*2.*

The following output was generated:

A1*2
A1*2
A1*2
A1*2
…etc.

The output was unexpected and related to a technical or formatting aspect. Their mathematical preconceptions probably enabled them to envisage a sequence of numbers doubling from one in some form, but the screen output being different and unexpected led them to re-evaluate the manner in which they engaged the exploration of the task. Their alternating engagements with the task, then reflection on the output through their mathematical and spreadsheet preconceptions was facilitating the evolution of their approach to the task, and the emergence of the technical aspects required to enable that approach.

Tony: *In A1 we want 1 and then you go something like =A1*2 then you go fill down and it times everything by 2. So 1 by 2, then 2 by 2, then 4 by 2, then 8 by 2, 16 by 2.*
Fran: *To double it? Times 2 more than the one before.*
Tony: *The amount of rice for each year will be in each cell.*
Fran: *What's the first thing we need to start off with? =A1*2.*
Tony: *We have 1 in cell 1 [for one grain of rice], and then we add the formula in cell A2 now.*
Fran: *And then fill down.*
Tony: *Got it. Go right down to find out.*

They have now entered:

	A
1	1
2	=A1*2

They *Fill Down* from cell A2 to produce the sequence of numbers they anticipated would give them the number of grains of rice for each square of the chessboard. They encountered something unexpected with the following output generated:

	A
1	1
2	2
3	4
4	8
…	…
…	…
26	33554432
27	67108864
28	1.34E+08
29	2.68E+08
…	…
64	9.22E+18

Fran: *Ok, that isn't supposed to happen.*
Tony: *9.22E +18, that makes a lot of sense.*

The output was unforeseen and in a form they weren't familiar with (scientific form). There was a tension between the expected and actual output causing them to reflect, adjust their position, and re-interpret. These pupils initially sought a technical solution to resolve their visual perturbation. They looked for a way to reformat the spreadsheet to alleviate their dubious perceptual position.

Fran: *Oh, make bigger cells.*
Tony: *You can make the cell bigger. Pick it up and move it over.*
Fran: *That should be enough.*
Tony: *It still doesn't work.*

Still perturbed by what the spreadsheet displayed, they sought my intervention, so the notion of scientific form was discussed with them. They indicated that they had a better perception of the idea and proceeded with the task. Tony considered the output 2.25E+15:

Tony: *When you get past the 5 you will need a lot of zeros. We'll need thirteen more.*

They continued with the task, maintaining the numbers in scientific form as they negotiated a way to sum the column of numerical values. This they managed, drawing on their prior understanding of the technical process required. This generated:

1.84467E+19

Tony: *Yeah!!!! It worked.*
Fran: *We got it!*
Tony: *Wow. It's a really, really big number.*

Drawing on their freshly modified perspective, they considered how it might appear in decimal notation. Their shared understanding required further negotiation, however.

Tony: *How many zeros.*
Fran: *19.*
Tony: *Did you count these numbers here?*
Fran: *No.*
Tony: *You need to count from the decimal point to the end and then add the zeros.*

They continued with the task, but carried forward their modified perspective; a perspective moderated through iterations of engagement and interpretation, but initiated by the visual perturbation. Their learning trajectory was shaped, via interpretation and engagement, by the various associated socio-cultural filters including the spreadsheet environment. Their preconceptions were mediated by the pedagogical medium and their understanding and explanations as evidenced by their subsequent interactions had incorporated those modified perceptions.

The particular ways actual learning trajectories might evolve

One of the key aspects of the engagement that was influenced by the spreadsheet as pedagogical medium was the initial engagement with the tasks. Across a range of activities the students, sometimes after a brief familiarisation of the problem, moved immediately to engagement within the spreadsheet environment. Usually this was to generate tables or columns of data, often through the use of formulas and the *Fill Down* function. This initial engagement allowed them to experiment with the intentions of the tasks and to familiarise themselves with the situation. They more readily moved from initial exploration, through prediction and verification, to the generalisation phase. Often, they immediately looked to generalise a formula to model the situation. The visual, tabular structure coupled with the speed of response facilitated their observation of patterns. Their language reflected this and frequently contained the language of generalisation.

The influence of this initial engagement permeated the subsequent ongoing interaction. The distinctive nature of this engagement framed the ongoing interactions, interpretations and explanations as the students envisioned their investigation through that particular lens. The actual learning trajectories were shaped by that initial engagement of creating formulas or columns and tables of data to model the mathematical situation. Digital technologies are generally more conducive to the modelling of mathematical situations than pencil-and-paper media, and the data were illustrative of the spreadsheet enhancing this aspect. The capacity to manipulate large amounts of data quickly, coupled with the potential for symbolic, numerical, and visual representations enabled the students to produce models that could be observed simultaneously, with the links and relationships between them explored in an interactive manner. As well, when the students were required to relate different representations to each other, they had to engage in activity such as dialogue, interpretation, and explanation that enhanced understanding.

The spreadsheet environment was also influential in the generation of sub-goals as the students' learning trajectories unfolded. As they alternated between attending to the activities from the perspective of their underlying perceptions, and then reflecting on this engagement with consequential modification of their evolving perspectives, they set sub-goals that plotted their ongoing interaction. These were frequently reset in response to the output generated within the spreadsheet environment. Sub-goals were generated at times because of opportunities afforded by the particular pedagogical medium. As well as those attributes that facilitated the modelling process, the facility to test immediately and reflect on emerging informal conjectures gave potential for the sub-goals the students set being shaped by the medium. The data demonstrated how the students' interpretations of the situations they encountered were influenced by the visual, tabular structure. It allowed more direct comparison of adjacent columns and enabled them more easily to perceive relationships between numerical values on which to base their new sub-goal, often linked to an emerging informal conjecture. It enhanced their ability to perceive relationships and recognise patterns in the data. Seeing the pattern evoked questions. On occasion the students pondered why the pattern was there, and what was underpinning a particular visual sequence.

While investigating in this environment, the students learnt to pose questions and sub-goals but also were encouraged to create personal explanations, explanations that were often visually referenced probably due to the pedagogical medium. It also gave opportunity through its various affordances for the students to explore powerful ideas and to explore concepts that they might not otherwise be exposed to. At times the learning trajectory evolved in unexpected ways (Calder, 2007). When the output varied, sometimes markedly, from what was expected, it caused tension that often led to the resetting of the sub-goal and substantial shifts in the way the student interpreted or engaged the situation. This aspect and other affordances including the interactive nature of the environment also appeared to stimulate discussion. The students wanted to verbally articulate the rapidly generated output and discuss the connections they could see, not least when it was unexpected. This aspect of surprise provoked curiosity and intrigue, which allied with the interactive and visual nature of the experience, in the students' general view made the learning 'more fun and interesting'. This, in turn, enhanced the motivational aspects of working through the spreadsheet medium, a feature that emerged in the interview, survey, and observational data.

The learner's propensity and comfort to move beyond known procedures in recognisable situations, is indicative of their willingness to try fresh strategies in their approach to investigation and problem solving, By implication, problem solving contains an element of the unknown that requires unraveling and addressing through the application of strategies in new situations or in an unfamiliar manner. This requires a degree of creativity and a willingness to take conceptual or procedural risks of a mathematical nature. It is risk taking in a positive, creative sense as compared to risky behaviour. The data were indicative of the spreadsheet environment affording learning behaviours and responses that facilitated the learner's willingness to take risks while operating within an investigative cycle. This seemed to allow the students to pose informal conjectures, to explore then reflect on them, before, perhaps after several investigative iterations, either validating or rejecting them. The offering and investigation of informal conjectures fostered mathematical thinking. These affordances were evident in the spreadsheet environment, but in some instances were characteristic of other digital pedagogical media.

The speed of response to input, when using the spreadsheet, indicated their suitability for facilitating mathematical reasoning. When the students observed a pattern or graph rapidly, they developed the freedom to explore variations and, perhaps with teacher intervention, learned to make conjectures, and then pose questions themselves. This facility to immediately test predictions, reflect on outcomes, then make further conjectures, not only enhanced the students' ability to solve problems and communicate mathematically, but developed their logic and reasoning as the students investigated variations, or the application of procedures.

The evolution of research perspectives

The students in this study engaged with the tasks through their preconceptions derived from earlier experiences. Seeing the output of their mathematising in the visual, tabular form of the spreadsheet modified those preconceptions as they made interpretations of their interaction. In the following brief excerpt, two pupils were investigating the 101 X activity (see Figure 1).

They had produced the following output:

1	101
2	202
3	303
4	404

Tim: *So it's the number, then a zero, and then the number again*
Carl: *Yeah, yeah. 5 will be 505, 55 would be 55055. Drag down.*

...	...
13	1313
14	1414
15	1515
16	1616
17	1717
...	...

Carl: *What? It's just repeating.*
Tim: *Like doubles, so 18 would be eighteen, eighteen and 55 would be fifty-five, fifty-five.*

They continue refining their generalisation through the modification of their perceptions as they interpret the outcome of their engagement and adjust their perspective. Their generalisations are based on the number and positioning of the digits. They have used a form of visual reasoning to generalise the pattern (Presmeg, 1986). They then re-engaged with the activity from a fresh perspective with the interpretation and understanding evolving in this ongoing manner. The broader discourse of mathematics (in this case visual reasoning) was likewise transformed (albeit slightly) by this engagement. The boundaries of mathematics per se were extended, or existing positions enriched, by that engagement. Other pupils commented in the interviews on the way the spreadsheet environment assisted their interpretation, e.g.

> *Chris: Columns make it easier – they separated the numbers and stopped you getting muddled. It keeps it in order, helps with ordering and patterns.*

This cultural formation of mathematics evolved as the mathematics phenomenon was engaged with the subsequent interpretations influencing the way mathematics was perceived.

The individual engagements of the students were also influential on the researcher perspectives and interpretations of the data, and the research methods that were employed. The analysis of the initial data revealed this emerging story around the affordance of the spreadsheet environment to structure the output visually. This analysis of the data, in conjunction with other constitutive influences e.g. the research literature, modified the approach to a more interpretive perspective. Research methods were employed that would give alternative insights into these visual interpretations as the pupils attention shifted alternately from preconception to interaction. Viewing the data through this lens gave further insights into the investigation of the research questions, in particular, the ways understanding emerged for the pupils, and the ways the pedagogical medium of the spreadsheet influenced their understanding. Mathematics education research was modified simultaneously as research practice, drawn from existing prevailing discourses in mathematics education research, was engaged for the research process, and then modified personal perceptions of mathematics education research. The individual transformational research trajectory resonates and modifies mathematics education research per se. In this case, the collegial dialogue, writing papers and presenting at conferences, and writing articles for journals, that indicated this visual, tabular structure and its influence on the research process employed to productively interpret the situations, has extended to some small extent the boundaries of mathematics education research.

In rejoinder, the mathematising at an individual level, the cultural formation of mathematics, the individual research process, and the evolution of mathematics education research, are all inextricably linked, they are mutually influential of each other. They all evolve through cycles of interpretation. Mathematics is an evolving set of perceptions, seeming to become more complex on its peripheries, yet more refined in its core identities, with each iteration of interaction, reflection and interpretation by its users. The elements of mathematics that are engaged transform the perceptions of the person interacting with the mathematics, but likewise those elements are transformed by their engagement with mathematicians, learners, or researchers, even if only by a minuscule amount. The boundaries of mathematics are expanding or becoming more refined through that interaction. The socio-cultural formation of mathematics can also be envisaged as a hermeneutic process, one where iterations of engagement, reflection, interpretation, then re-engagement from modified perspectives fashion those emerging theories.

The reorganisation of mathematical thinking and understanding

The spreadsheet environment reshaped the students' approaches and the manner in which they traversed their actual learning trajectories, by the particular nature of their experiences while working within that environment. It allowed them to engage in alternative processes and to envisage their interpretations and explanations from fresh perspectives. The mathematising facilitated by the medium was transformed by the visual, interactive nature of the investigative process. They used visual elements in their reasoning, while their explanations were punctuated with visual referents, such as the position and visual pattern of the digits. As such, the generalisations that emerged were couched in visual terms. They interpreted and explained their reasoning in alternative ways. There was a visual perspective to their mathematical thinking, while the visual tabular structure enhanced the possibility of seeing relationships in ways that might otherwise have been unattainable or inaccessible. Coupled with other affordances, such as the increased speed of the feedback, this visual dimension expanded the boundaries of what constituted mathematical knowledge, and gave students access to ideas earlier than teachers' usual expectation. It allowed a shift in focus from calculation techniques to a focus on mathematical thinking and understanding. Modeling the situations with various representations, and the capacity to think mathematically and generalise enhanced by the simultaneous viewing and translation between these alternative forms, also fostered the reorganisation of the learners' thinking.

On a broader level, the impact of digital technologies on society and investigative processes in general offers scope for the changing of the nature of some elements of mathematics and mathematical thinking. While there is recognition in some quarters of the mathematics community, that some evolution has already occurred (for instance, the emergence of visual reasoning as a 'legitimate' form of mathematising) there is certainly no consensus within that community regarding this aspect, nor orchestrated intention to explore the boundaries of such possibilities. In the domain of mathematics education, digital technologies are given greater privilege, although their potential use in the classroom is still only partially realised. Modeling is one aspect of mathematics education that might be given greater primacy in both the content and pedagogical areas. The nature and immediacy of feedback, which was featured in the analysis, enables the successive refinement of informal conjectures and solutions.

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