Inference Rules in some temporal multi-epistemic propositional logics

Erica Calardo

A thesis submitted in partial fulfilment of the requirement of the Manchester Metropolitan University for the degree of

Doctor of Philosophy

Department of Computing and Mathematics
Manchester Metropolitan University

May 2008

b

Acknowledgement

So many people have supported me during these years that I am really afraid to forget someone. I want to thank Professor Rybakov and all the staff from MMU for their help and guidance, my dearest Sabrina, in which I found a sister, my awesome colleagues Darren, Naresh and Tim and my very good friends Annalisa, Cristina, Quoc, Alex and Dimitri. My Mom, my Dad and my whole family have been supportive and encouraging through all this time, though I guess it has been tough for them. Without Paolo I would have never made it.

Abstract

Multi-modal logics are among the best tools developed so far to analyse human reasoning and agents' interactions. Recently multi-modal logics have found several applications in Artificial Intelligence (AI) and Computer Science (CS) in the attempt to formalise reasoning about the behavior of programs. Modal logics deal with sentences that are qualified by modalities. A modality is any word that could be added to a statement p to modify its *mode of truth*. Temporal logics are obtained by joining tense operators to the classical propositional calculus, giving rise to a language very effective to describe the flow of time. Epistemic logics are suitable to formalize reasoning about agents possessing a certain knowledge. Combinations of temporal and epistemic logics are particularly effective in describing the interaction of agents through the flow of time. Although not yet fully investigated, this approach has found many fruitful applications. These are concerned with the development of systems modelling reasoning about knowledge and space, reasoning under uncertainty, multi-agent reasoning et c.

Despite their power, multi modal languages cannot handle a changing environment. But this is exactly what is required in the case of human reasoning, computation and multi-agent environment. For this purpose, inference rules are a core instrument. So far, the research in this field has investigated many modal and superintuitionistic logics. However, for the case of multi-modal logics, not much is known concerning admissible inference rules.

In our research we extend the investigation to some multi-modal propo-

sitional logics which combine tense and knowledge modalities. As far as we are concerned, these systems have never been investigated before. In particular we start by defining our systems semantically; further we prove such systems to enjoy the effective finite model property and to be decidable with respect to their admissible inference rules. We turn then our attention to the syntactical side and we provide sound and complete axiomatic systems. We conclude our dissertation by introducing the reader to the piece of research we are currently working on. Our original results can be found in [9, 4, 11] (see Appendix A). They have also been presented by the author at some international conferences and schools (see [8, 10, 5, 7, 6] and refer to Appendix B for more details).

Our project concerns philosophy, mathematics, AI and CS. Modern applications of logic in CS and AI often require languages able to represent knowledge about dynamic systems. Multi-modal logics serve these applications in a very efficient way, and we would absorb and develop some of these techniques to represent logical consequences in artificial intelligence and computation.

Contents

List of Figures

Chapter 1

Introduction

Modelling human reasoning and agents' behaviour in a system is nowadays a very active area. There are several ways to approach this research field and multi-modal logics are definitely quite a strong candidate for this purpose. These logics provide a combination of great expressive power and intuitive semantic tools and they have already been successfully applied to both Artificial Intelligence and Computer Science in the attempt to formalise, for instance, reasoning about the behaviour of programs (cf. Goldblatt [27, 26]), social interactions, games and so on. Multi-modal logics can thus be regarded as a very good tool in the analysis of Multi-Agent systems.

Our approach to Multi-Agent reasoning does lay its foundation on multi-modal propositional logic. We aim at defining some temporal multi-epistemic logics, focusing on the aspect of admissible inference rules in these systems.

Our starting point is a basic understanding of the problems and the themes related to Multi-Agent reasoning. This introduction aims at providing the reader with a concise knowledge base in support of the further development of our analysis. We start by explaining the reasons that led us to our choice of multi-modal languages and logics. We proceed by reviewing a few representative and successful attempts to systematise the subject. We turn then our attention to the area of inference rules and we survey the latest results in the area. Finally we summarise both the content and the structure of the following chapters.

1.1 Choosing Multi-Modal Languages

The main feature of modal languages is that they enable the switch from *extensionality* (the expression of *facts, statements* which can be either true or false) to *intensionality.*

Classical Propositional Logic is, in fact, purely *truth-functional*: the truth value of a complex proposition as $p \wedge q$ is completely dependent on the truth values of its components p and q. Let us say, for instance, that the proposition p stands for *it is raining* whereas q means *I take my umbrella.* Then the truth value of the proposition $p \wedge q$ would be *true* if and only if it is true both that it is raining and that I take my umbrella, i.e. if both the propositions share the same truth value *true.*

This approach works fine in any case of *assertive speech*, whenever we utter sentences which state facts, statements linked to each other by means of the classical logical connectives. However as soon as we read what we write or listen to what we say, we realise that not all the sentences we use are *necessarily* so. There are so many sentences that in spite of being both grammatically correct and meaningful are not suitable to be interpreted using a truth-functional approach. The classical example of a sentence whose truth value does not depend *only* on the truth values of its components is provided by Frege [19]. Let us say, for instance, that I *know* that the *morning star* is the planet Venus, although I lack the knowledge of the fact that the *morning star* and the *evening star* are actually the same star, which is not a star, by the way, but the planet Venus. Thus a proposition as *I know that the morning star is the planet Venus* would be true and intuitively a proposition as *I know that the morning star is the evening star* would be false, for I do not possess the latter information. Nevertheless according to the rules of Classical First Order Logic, the expressions *morning star*, *evening star* and *planet Venus* are interchangeable by Leibniz' Law, as they share the same semantics. Therefore it is clear how a sentence as the one provided is not truth-functional at all: its truth value does not depend on the truth values of its component parts. In fact, as soon as we get propositions qualified by modalities as *can, could, might, may, must, know, believe* et c. the truth functionality is no longer applicable. These phrases tell something more than a pure fact: they say something about the *mode of truth* of the sentence itself. Such sentences belong to the realm of *modal languages*.

In order to construct a modal language we usually add to a classical boolean language a set of *modal operators* according to the set and the quality of sentences we want our language to be able to express. Clearly the expressive power of a modal language is much greater than the one of a language which does not contain operators. The modal operator which is traditionally added to the language of Classical Propositional Calculus in order to get a new modal language is the *box* operator $\Box$ (starting from which its dual, the *diamond* operator, can be easily defined). Likewise, multi-modal logics are obtained by adding more than one modal operator to an existing language.

Although traditionally read as expressing necessity and possibility, modal operators may be given potentially endless interpretations. The choice would be then suggested by the context one is to describe. In the case of tense logics, one can interpret the modal proposition $\Box p$ as *it will always be the case that* p, and its dual $\Diamond p$ as *at some point in the future it will be the case that* p. Such language is, therefore, effective whenever a description of the flow of time, towards both future and past is needed. Multi-epistemic languages, on the other hand, are suitable to formalise reasoning about agents not possessing a complete base of information (see Fagin *et al.* [17], Rybakov [56]). However, these languages may suffer of an expressive limitation, for it may be difficult to deal with modifications in the pieces of information each agent possesses as well as to give an account of a changing environment. Adding a dynamic dimension to such languages is therefore almost a necessity. The most natural way partially to improve on the expressive limitation is adding a temporal operator to a multi-epistemic one. Hence we would generate a multi-modal language combining tense and knowledge operators (see Fagin *et al.* [17], Halpern *et al.* [30], C. and Rybakov [9, 4], C. [11]).

1.2 Choosing Multi-Modal Logics

We have seen how versatile multi-modal languages are, but we have not mentioned yet one of the main reasons that led so many researchers to use multi-modal logics as tools to build Multi-Agent systems, to investigate knowledge, to construct models in computer science and so on. The reason is that multi-modal languages can be interpreted in the *Possible Worlds*

Semantics, or Kripke Semantics, which is a highly intuitive tool to study modal logics (cf. Kripke [41, 42, 43]).

Kripke Semantics is based on the idea of dealing with objects interpreted as *possible worlds*. Such objects are linked one to each other by some binary relations. This set of worlds plus the binary relations are a *Kripke-frame*. Let us consider for simplicity the traditional case of the *necessity* operator. Let us suppose we want to interprete a modal sentence as $\Box p$ in this scenario. Then we could say that in some world in the model, the expression is *true* if it is true in *all the worlds accessible by it*. This is to say that a sentence as *it is necessary that I am reading* is true in a world if the fact that I am reading holds true in all the worlds related to it. It is immediately clear how versatile this framework is for many purposes. Let us suppose that we want to interpret some temporal language. Then, we might consider each world as a moment and we can interpret the binary relation as an order on moments. Hence a sentence like *it will rain eventually* is true at a moment if there is another moment which comes *later*, i.e. that is related to the present moment, in which the fact that it is raining holds true[1]. Likewise one could interpret an epistemic language in this scenario. In this case we can interpret the possible worlds as the states of affairs that I consider possible. In this spirit, a sentence as *I know that it is raining in Manchester* is true if I cannot imagine a situation in which Manchester is dry, which is to say that in all the worlds related to mine, in Manchester it is raining cats and dogs.

In order to get an even clearer idea of how such semantic tools work, we

[1]see Prior [50]).

can read a passage from the already cited book written by Fagin *et al.*, which we find particularly effective in explaining how Kripke semantics works:

> The intuitive idea behind the possible-worlds model is that besides the true state of affairs, there are a number of other possible states of affairs or *worlds*. Given his current information, an agent may not be able to tell which of a number of possible worlds describes the actual state of affairs. An agent is then said to *know* a fact ϕ if ϕ is true at all the worlds he considers possible (given his current information). For example, agent 1 may be walking on the streets of San Francisco. Thus, in all the worlds that the agent considers possible, it is sunny in San Francisco. (We are implicitly assuming here that the agent does not consider it possible that he is hallucinating and in fact it is raining heavily in San Francisco.) On the other hand, since the agent has no information about the weather in London, there are worlds he considers possible in which it is sunny in London, and others in which it is raining in London. Thus, this agent knows that it is sunny in San Francisco but he does not know whether it is sunny in London. Intuitively, the fewer worlds an agent considers possible, the less his uncertainty, and the more he knows. If the agent acquires additional information – such as hearing from a reliable source that it is currently sunny in London – then he would no longer consider possible any of the worlds in which it is raining in London[2].

Thus if we combine the intuitive tools of Kripke Semantics with the numerous applications Multi-Agent systems have in different areas, it becomes

[2] Fagin *et al.* [17] p.16

clear the reason why so many researchers are currently devoting their studies to the analysis of this topic.

1.3 Multi-Agent Reasoning

As we have anticipated, multi-modal languages and logics are a very useful and versatile tool to build up the so called Multi-Agent systems. These systems are actually rather more useful than the classic investigations on situations in which only one agent is present in the system:

> When trying to understand and analyze the properties of knowledge, philosophers tended to consider only the single-agent case. But the heart of any analysis of a system is the interaction between agents.. [...] Our agents may be negotiators in a bargaining situation, communicating robots, or even components such as wires or message buffers in a complicated computer system. [...] We are often interested in situations in which *everyone* in the group knows a fact. [3]

As M. Wooldridge clearly states in the Preface to his book *An Introduction to Multiagent Systems* [70]:

> Multiagent systems are systems composed of multiple interacting computing elements, known as agents. Agents are computer systems with two important capabilities. First, they are at least to some extent capable of *autonomous action* - of deciding *for themselves* what they need to do in order to satisfy their design

[3]Fagin *et al.* [17] p.2

> objectives. Second, they are capable of interacting with other agents - not simply by exchanging data, but by engaging in analogues of the kind of social activity that we all engage in every day of our lives: cooperation, coordination, negotiation and the like[4].

Thus, typical agents are computer programs which run on some platform, although they may also be seen as buffers and other devices linked to the realm of Computer Science and Artificial Intelligence. Moreover, one may also see agents as human beings operating and co-operating in a social environment in order to reach a common goal. This approach can hence provide analytical tools to be used in the study of social-economic phenomena (e.g. game theory, economical analysis of markets, local and global social interactions). Being a fairly new and active area, the potential applications one may think of are flexible and potentially infinite.

> [...] Multiagent systems seem to be a natural metaphor for understanding and building a wide range of what we might crudely call *artificial social systems*. The ideas of multiagent systems are not tied to a single application domain, but, like objects before them, seem to find currency in a host of different application domains[5].

As a matter of fact, systems generated by joining operators representing both time and knowledge have already proved themselves to be particularly effective in describing the interaction between agents through the flow of time (see Fagin *et al.* [17], Gabbay *et al.* [21], Halpern *et al.* [30]).

[4] Wooldridge [70], p. xi.
[5] *ivi.*

These systems are based on a language which contains two sets of modalities: one to model the flow of time, the other to describe agents' knowledge. The interaction of such modalities gives a precise account of the dynamic development of agents' knowledge.

The last decades have given birth to many Multi-Agent systems based on multi-epistemic temporal languages. Several systems have also been developed and successfully applied both in the study of human reasoning and in computing (see [17, 26, 28, 71, 30]). These theories are concentrated on the development of systems modelling reasoning about knowledge and space, reasoning under uncertainty or with bounded resources, Multi-Agent reasoning and other aspects of artificial intelligence. Nevertheless we are talking to a relatively young research area and there is still quite a lot of work to be done in the filed.

1.4 A brief historical overview

We have seen that the power of a language with interacting temporal multi-epistemic modalities combined with the tools of Kripke Semantics makes multi-modal logics quite appealing to researchers willing to investigate the field of Multi-Agent reasoning. Thus, in order to understand the subject in a deeper way, we feel that we need some basic notions about the history of the subject. As we have anticipated, we aim at studying temporal multi-epistemic logics which are a combination of Temporal Logics and Epistemic Logics.

The study of Temporal Logics is closely linked to many sciences and we could actually say that tense logics may work as a common background:

> [...] it is obvious that time plays such a fundamental role in our thinking that there is a clear need for precise reasoning about it, such as we see in Physics, formal Linguistics, Computer Science, and Artificial Intelligence. While these enterprises are not necessarily concerned with the same concept of time, they all could go under the heading of Temporal Logic[6].

Nevertheless, throughout the decades, tense logic has usually been considered in a more restricted way, as a branch of modal logic. Arthur Prior (1914-1969) can be considered the founder of modern temporal logic. He found out that it is possible to relate some of the aspects of Diodorean Logic to modal logic and he built up a calculus in which the modal operators were interpreted as representing quantifiers over temporal states or moments (see Prior [50]).

When it comes to talk about multi-modal logics which combine tense and epistemic modalities, it is natural to think about the work of Halpern, Moshe and Vardi (see Halpern et al. [30]). In this paper the authors consider some previous works written by themselves and by others. They introduce a general framework to fit all the previously defined logics in and then they find a complete axiomatisation for several propositional logics which combine tense and knowledge modalities. In particular the authors take their starting point from the work of Sato [64], Spaan [66], Fagin et al. [17] and others. They analyse the work done and manage to fit a significant part of the previously introduced systems into a more general framework. In particular:

[6] Venema [69]

> Logics for knowledge and time were categorized along two major dimensions: the language used and the assumptions made on the underlying distributed system. The properties of knowledge in a system turn out to depend in subtle ways on these assumptions. The assumptions considered in [HV89][7] concern whether agents have *unique initial states*, operate *synchronously* or *asynchronously*, have *perfect recall*, and whether they satisfy a condition called *no learning*. There are 16 possible combinations of these assumptions on the underlying system. Together with 6 choices of language, this gives us 96 logics in all. All the logics considered in the papers mentioned above fit into the framework. [...] Of these 96 logics, 48 involve *linear time* and 48 involve *branching time*. [...] We focus here on the linear time logics and provide axiomatic characterizations of all the linear time logics for which an axiomatization is possible at all (i.e., for those logics for which the set of valid formulas is r.e.).[8]

As we shall see in the further chapters, the language adopted by Halpern, Moshe and Vardi has one more tense operator than the one we adopt. In particular, concerning the tense modalities, they use the operators until $\mathcal{U}$ and next $\bigcirc$, whereas we use only the operator $\Box_{\preccurlyeq}$, to be read as *true from now on*. Since they do not consider the case of branching time, they do not take into account the operator $\forall\bigcirc$, which is suitable to quantify over all the possible future paths. Our case involves a linear time line as well, therefore we do not consider on this particular modality either (although, as we shall see, we describe a case which simulates branching time while keeping the

[7] Halpern [31].
[8] See Halpern et al. [30], page 1.

actual time line linear). Informally speaking, saying that an agent has *perfect recall* is equivalent to assume that such agent may not forget the piece of information given at any time. Conversely an agent which does not have perfect recall is someone who is somehow allowed to forget. In our logics, we consider this latter case, allowing agents to forget as well as to learn. The condition of *no learning*, in fact, implies that agents may not increase their knowledge base throughout the time. If associated with the condition of perfect recall, we get a set of agents with a stable knowledge base, which stays unchanged throughout the flow of time, and this is not the case we want to model. We believe, in fact, that one of the most useful tools provided by multi-modal languages and logics is allowing the description of an environment which may change and affect agents' knowledge bases, mimicking up to some extent what happens to human beings' knowledge bases throughout time.

In a system in which several agents are operating, we can imagine that there is a sort of clock, external to the system itself and that such clock measures time:

> We assume that time is measured on some clock external to the system. [...] This external clock need not measure *real time*. [...] In general, we model the external clock in whatever way makes it easiest for us to analyze the system[9].

Finally, in a *synchronous* system

> [...] we assume that every agent has access to a sort of global clock that ticks at every instant of time, and the clock reading

[9] Fagin *et al.* [17] pp. 112–113

> is part of its state. Thus, in a asynchronous system, each agent always "knows" the time[10].
>
> A standard assumption in many systems is that agents have access to a shared clock, or that actions take place in rounds or steps, and agents know what round it is at all times. Put another way, it is implicitly assumed that the time is common knowledge, so that all the agents are running in synchrony. [...] Indeed, although synchrony is not a necessary assumption when modelling games, it is often assumed by game theorists. When linguists analyze a conversation, it is also typically assumed (albeit implicitly) that the agents share a clock or that the conversation proceeds in structured steps. In computer science, many protocols are designed so that they proceed in rounds (where no agent starts round $m+1$ before all agents finish round m)[11].

We shall also set agents as operating synchronously. By doing so we aim at simulating the human condition: many agents operating on the framework of a shared time line.

1.5 Focusing on Inference Rules

The main results proved by Halpern, Moshe and Vardi is one of greatest interest and, as a matter of fact, it is the starting point of our research. Among the 96 logics they describe, they consider 48 cases (the ones involving linear time, as we have seen) and provide an axiomatisation whenever this is possible. One could then think that another work on multi-modal logics with

[10] Halpern et al. [30]
[11] Fagin *et al.* [17], p.135

epistemic modalities on a linear time framework is somehow unnecessary or at least redundant. But although the subject of combined multi-epistemic tense logics has been fairly widely studied, we think that not enough attention has yet been dedicated to the investigation of admissible inference rules in this specific area. In fact whereas complete axiomatic systems have recently been provided for several multi-epistemic temporal logics, many problems regarding inference rules related to these systems are, as we shall see later in this Chapter, indeed still open. There are excellent works oriented to the study of wide classes of multi-epistemic temporal logics, as we have seen, but very few analysing the area of inference rules applied to such systems. This is the main reason that led us to start our research topic and it is in this area that we give our main contribution.

The results and techniques related to the study of axiomatic systems and their complexity work fine in numerous applications, but it is reasonable, however, to ask whether and how the inference machinery could be enlarged: inference rules are, as a matter of fact, extremely important in derivations.

But why is it so important to focus on the investigation of inference rules? First of all, let us introduce the concept of *inference rule*. An inference rule, or a *logical consequence* or else just an *argument* is a set of formulae called the *premisses* of the argument followed by a formula called the *conclusion*. It is usually displayed as:

$$\begin{array}{c} \mathtt{A}_1 \\ \vdots \\ \mathtt{A}_n \\ \hline \mathtt{B} \end{array}$$

The premisses $\mathtt{A}_1, \ldots, \mathtt{A}_n$ are separated from the conclusion $\mathtt{B}$ by a line, indicating that between the two sets of formulae there is some sort of connection.

This link is the logical entailment: a rule can be read as *given the premisses* $\mathtt{A}_1, \ldots, \mathtt{A}_n$*, the conclusion* $\mathtt{B}$ *may be inferred.* Clearly this does not hold true for every rule in every system. The study of the truth of this type of sentences referred to rules is the core of the research concerning logical consequences. In particular, one may be interested in finding out whether a given rule is *correct* for some logic, which is to say if its conclusion *must* hold true whenever its premisses do so. If such relation between premisses and conclusion holds for a logic, the rule is said to be *valid* or *admissible* for the logic itself. One can say that if a rule is valid for a logic, the truth of the premisses is *transfered* to its conclusion[12].

Intuitively the set of admissible rules for a logic is the widest class of rules which can be implemented in the logic itself without altering its set of theorems: it is the class of all those rules under which the logic is closed. Finding valid rules for a logic is quite important. For instance, a rule which has already been checked as valid for a logic can be immediately used in derivations in order to produce new theorems. Moreover, in modal logics, rules can describe properties of modal frames in some cases in which using formulae may be difficult. A good example is Gabbay's *irreflexive rule* (cf. [22]):

$$\mathbf{ir} := \frac{\neg(p \rightarrow \Diamond p) \rightarrow \mathtt{A}}{\mathtt{A}}$$

(where p does not occur in the formula $\mathtt{A}$). This rule states that each world of a model, where $\mathtt{A}$ is not valid, should be irreflexive. Admissible consequences have been deeply investigated for many modal and superintuitionistic logics (see, for instance, Ghilardi [23, 24, 25], Golovanov et al. [29], Iemhoff [34, 36], Jeřábek [38], Rybakov [53, 54, 55]).

[12] cf Bellissima and Pagli [1].

Moreover, having a wider set of rules available is very useful in order to simplify derivations. Once an inference rule has been proven to be admissible for some logic, it can be used in a derivation with the result of shortening significantly the whole process. But let us focus on the concept of admissible rule. When can we say that an inference rule is admissible, rather than just derivable? Are these two concepts really that different?

The study of classical propositional logic may lead one to think that the study of admissible rules is very important. But as soon as we move to the realm of non classical logics, the situation changes substantially. In fact the definition of admissible rules does not depend on the choice of a specific axiomatic system. We say that the class of admissible rules is the widest class of rules which can be applied to a given logic without altering its set of theorems. This is a very comprehensive notion and should not be confused with the syntactical concept of derivable rules. A rule is derivable, basically, if there is a derivation of its conclusions given its premisses as assumptions in a specific axiomatic system. Therefore it is clear that the collection of derivable rules in some system depends completely on the specific choice of the axiomatic system itself. The question which naturally rises at this point is whether these two concepts, even though characterized in such a different way, do always have the same extension. A negative answer is given by Harrop [32]: there are axiomatic systems which can be actually enlarged by adding rules which are admissible although not syntactically derivable. In particular, according to Harrop, the intuitionistic propositional logic IPC admits rules which are not derivable on the system itself. We say that this logic is not *structurally complete*, which is to say that it is not, in some sense, self-contained[13]. As it usually happens in logic, a negative answer opens a

[13] cf. Rybakov [55]: p.10

new field of study and research. In fact in his *One Hundred and Two Problems in Mathematical Logic*, Friedman has been led by Harrop's observation to ask whether there is a way to establish, given some inference rule, whether it is admissible or not in IPC[14]. This question has been solved in a series of papers by Rybakov (see for instance [51, 52]) and later summarised in a book [55]. Rybakov has also extended his results to many well known modal calculi and a robust mathematical theory has been developed[15].

More specifically, Rybakov has built an algorithm which is able to check wether any inference rule is admissible for IPC or not. Moreover he showed that the Intuitionistic Propositional Calculus itself does not admit any finite basis. This is to say that there is no finite collection of inference rules starting from which one can generate all the possible admissible inference rules. Nevertheless, both de Jongh and Visser have defined a recursively enumerable set of rules which they conjectured to be an infinite basis for IPC's admissible rules. This conjecture has been proved to hold true by Iemhoff [35, 34, 36, 37] in her Phd thesis. Iemhoff's results are based both on Rybakov's and Ghilardi's techniques.

Using the techniques developed by Ghilardi and employed by Iemhoff, in 2005 Jeȓábek provided explicit bases of admissible rules for a representative class of normal modal logics (including the systems K4, GL, S4, Grz, GL.3) (see Jeȓábek [38]). Later on he turned his attention to the problem of complexity when dealing with inference rules [39].

Rybakov has Recently dedicated much of his work to the investigation

[14] see Friedman [20], problem 40

[15] For a more detailed historical account see Rybakov [55], Iemhoff [35].

of admissible inference rules in tense logics. In particular he has examined the case of intransitive temporal linear logic of finite intervals [61], discrete linear temporal logic [58, 60], linear temporal logics [57], linear temporal logics based on integer numbers [62, 63], temporal next-time logic [59], and other types of tense logics.

However, for the case of multi-modal logics, not much is known concerning admissible inference rules, though there have been some attempts to approach the problem (see for instance Golovanov et al. [29, 28]).

In our research we aim at going a little deeper towards this direction, focusing on the aspect of inference rules in some combined multi-modal logics. Most of the material we are going to present in this dissertation has already been published in C. and Rybakov [9, 11] and C. [4] and it has also been presented at several international conferences ([8, 10, 5, 7, 6]). The Reader may refer to the Appendix for a complete collection of our published works.

1.6 Our research: Objectives, Methodology and Overview

Our research started in order to achieve three objectives and to give our contribution to common knowledge in three different ways. (i) We wanted to build some logical systems suitable to model the behaviour of agents operating on a temporal framework. Since these logics should be applicable both to Computer Science and to Artificial Intelligence, they ought to enjoy some specific properties. In particular we wanted to introduce a logic decidable with respect to its theorems. (ii) Moreover, in order to master an infinite number of formulae, we wanted our logics to be generated by a finite number of axiom schemata. (iii) Finally we wanted to give our theoretical contribution to a systematic and complete investigation of the problems related to inference rules applied to multi-modal propositional logic. As we have seen in this introductory chapter, in fact, the investigation of multi-modal logics from the perspective of inference rules has begun quite recently and the results provided are not numerous yet. We would like to give our contribution to this field by providing some decidability results related to multi-epistemic temporal logics.

Our work can be seen as a further step towards the investigation of the wide field of multi-modal logics. Starting from a specific logical system which combines tense and epistemic modalities, we try to give an answer to some of the questions we have been introducing so far. In particular, in our dissertation, we have decided to organize our research in four chapters, where each of them is devoted to the investigation of a specific problem. The first two chapters analyse semantic aspects of the problem, whereas the last section is mostly devoted to a syntactical analysis.

In Chapter 2 we provide a semantic definition of the logic LTK and some other systems. These logics are introduced as the set of all those formulae valid in a specific class of multi-modal Kripke-frames. We make here a substantial use of the so called *Possible World Semantics* or Kripke-semantics. As we shall see, the application of the standard techniques implied in this field is not at all straightforward and it needs to be modified according to our specific needs. As far as we are concerned our logic are, in fact, original and they have never been studied before. After describing the intended models of the most general of our logics, namely LTK, we prove some basic semantic properties. In particular we show that the logic LTK has the *effective finite model property* and it is hence decidable with respect to its theorems. This is much more than showing that a logic enjoys only the *finite model property*. In fact if a logic L has the *finite model property*, then for each formula A which is *not* a theorem of L there is a finite model $\mathcal{M}$ such that: (i) in $\mathcal{M}$ all the theorems of L are true; (ii) the formula A is not true in $\mathcal{M}$. The *effective* finite model property implies something more. It actually adds a very important condition which makes a great difference. In fact a logic L has the *effective finite model property* if it enjoys the finite model property *and* (iii) the size of its finite model $\mathcal{M}$ is computable and bounded by the size of A. This means that for each formula A which is *not* a theorem of L we can build a model $\mathcal{M}$, whose size is finite *and* computable from the size of A, such that it verifies all theorems of L and falsifies A. The two definitions are deeply different. The last one implies that a logic is decidable with respect to its theorems, whereas the former one does not. In fact in order to check whether a formula A is a theorem of a logic L with the *finite model property*, we should check if *none* of the finite models of L falsifies A. This means that one should check an *infinite* number of models and this is not possible in a finite time. On the other hand if L has the *efective finite model property*, one has to check if *none* of the finite models of L *whose size is at most n for some finite n* falsifies A. This is quite different, as the number of models one should check would now be *finite*. We prove that our logic enjoys this last property. Therefore for any formula A in the language of our logic LTK, we can check *in a finite number of steps* whether A is or is not a theorem of LTK.

In Chapter 3 we turn our attention to the topic of inference rules. We construct some special n-characterising models which enable us to show that one of the systems introduced is decidable with respect to inference rules. To fulfil this task, we use several semantic techniques introduced by Rybakov [55], modifying them to suite our case. In fact all the logics presented in [55] are normal 1-modal systems, i.e. systems based on a language containing only one modal operator. The systems we deal with are, on the other hand, *multi*-modal, as their language may contain countably many modal operators. An algorithm designed for a specific 1-modal system cannot straightforwardly be applied to a multi-modal system without being deeply modified. In Chapter 3 we generalise the techniques presented in [55] in order to apply them to the case of our multi-modal systems.

Chapter Chapter 4 is entirely devoted to quest for an axiomatic system modelled to capture all and only the theorems of LTK. We provide a sound and complete axiomatisation for our logic and a generalised version of it. Here we make a substantial use of well known techniques such as the *filtration* one developed by Segerberg [65], finding a way to adapt well known results in modal logic to our specific case. In fact the application of the standard techniques is not straightforward and several difficulties arise whenever one is to prove an axiomatic system to be sound and complete with respect to a class of multi-modal frames. According to Bennett *et al.* [2] and Kurucz [44], if there is no interaction between modalities, a transfer of properties (such as *finite model property*, *decidability*, et c.) from the component simple modal logics to the newly generated multi-modal system does apply. However, as soon as such interaction takes place it is not straightforward anymore to prove that the combined system is conservative with respect to the properties of its components. In some cases the opposite may apply. Nevertheless, despite such difficulties, interaction between modalities is necessary to exploit the power of multi-modal languages. As we shall see after the introduction of the multi-modal language we adopt, it is impossible to express concepts as *learning* and *forgetting* if the interaction between different modalities is not allowed. Let us consider, for instance, two modal operators K_i and $\Diamond$, to be read as meaning *agent i knows ...* and *sometimes in the future it will be true that ...* respectively. Without interaction, one could only express formulae as $\mathsf{K}_i\mathtt{A}$ or $\Diamond\mathtt{A}$, meaning *agent i knows* A and *in the future it will be the case that* A. On the other hand, as soon as the interaction is allowed, one could express the following: $\neg\mathsf{K}_i\mathtt{A} \wedge \Diamond\mathsf{K}_i\mathtt{A}$, to be read as *agent i does not know* A *but in the future it will be the case that he/she will know it.* In the example above it is clear how the concept of *learning* can be expressed by the interaction of two different modal operators. The same case happens for the idea of *forgetting*. An expression as $\mathsf{K}_i\mathtt{A} \wedge \Diamond\neg\mathsf{K}_i\mathtt{A}$ could be interpreted as meaning *the agent i knows* A *but in the future he/she will forget it.* It seems clear that *learning* and *forgetting* are intrinsically temporal: in order to express them one needs both epistemic and temporal modalities and, most important of all, a way to combine such modalities. Being able to express concepts as *learning* and *forgetting* is very important. In a language which is not powerful enough to express these notions, in fact, it would be impossible to handle changing knowledge bases: each agent would just possess a static piece of information. On the contrary we aim at describing the specific dynamic aspects of knowledge bases which may (or may not) change through the flow of time. For this reason we want a language which can deal with *learning* and *forgetting* situations and hence with changing knowledge bases. The axiom schemata we define in Chapter 4 allow the interaction between modalities and they are, therefore, suitable to express both learning and forgetting.

Finally, in Chapter 5 we present both our last result and our current research topic. We start by proving that the logic LTK_1 is not *structurally complete*. Intuitively this means that there are inference rules which are not derivable on the axiomatic system which generates LTK_1. These rules, are, nevertheless, admissible for LTK_1. In this Chapter we define an infinite set of rules with this property. Since all admissible rules can be applied in derivations without altering the set of theorems of a logic, the class of admissible and not derivable rules we present here adds new syntactical tools which can be used in derivations.

Moreover we provide Algebraic Semantics for LTK_1. Although Kripke Semantics is widely used in order to deal with modal logics, Algebraic Semantics is historically the first developed. In this Chapter we introduce algebraic tools as well as all those results which link Algebraic to Kripke Semantics. Moreover we translate the results from the previous chapters into the language of this alternative semantic framework.

Finally, we introduce the further work and the piece of research we are currently working on. We start to investigate the problem of finding a finite basis for admissible inference rules. This is to say that we aim at finding a set of rules to *axiomatise* all the inference rules admissible for LTK_1, i.e. the smallest set of rules starting from which one can derive all the admissible rules for LTK_1[16]. This topic, as we shall see, is rather problematic and it is currently an open research field. We introduce the reader to the problems related to such investigation and we show our attempts to solve these problems.

[16] Please refer to Chapter 2 and Chapter 3 for a formal definition of the system LTK_1.

Chapter 2

A Semantic Definition of LTK

Multi-modal logics, as we have seen in the introductory chapter of our dissertation, are clearly a powerful tool to deal with multi-agents contexts. In this chapter we shall introduce in more detail our approach to multi-modal logics. We shall start by introducing a new multi-modal language and we shall proceed by defining semantically the set of modal logics we shall work with in what follows.

Our multi-modal language is a propositional language with some operators. According to our needs, we shall define the meaning of each operator.

2.1 Syntax: The Language $\mathcal{L}^{\mathsf{LTK}}$

A propositional logical language has two components: an alphabet, or signature, which includes all the symbols one is allowed to use and a series of formation rules, which gives precise instructions to build grammatical sentences.

The alphabet of the language $\mathcal{L}^{\mathsf{LTK}}$ includes a countable set of propositional letters $P := \{p_1, \ldots, p_n, \ldots\}$, round brackets $(,)$ and the boolean operations $\{\rightarrow, \bot\}$ as well as a set of modal operators $\{\Box_{\preccurlyeq}, \mathsf{K}_{\mathsf{e}}, \mathsf{K}_1, \ldots, \mathsf{K}_k\}$. Well formed formulae (wff's henceforth) are defined as follows: each propositional letter $p \in P$ is a wff and if $\mathtt{A}$ is a wff, then so are $\Box_{\preccurlyeq}\mathtt{A}$, $\mathsf{K}_{\mathsf{e}}\mathtt{A}$, $\mathsf{K}_i\mathtt{A}$. We assume $\Diamond_{\preccurlyeq}$, $\Diamond_{\mathsf{e}}$ and $\Diamond_i$ to be abbreviations for $\neg\Box_{\preccurlyeq}\neg$, $\neg\mathsf{K}_{\mathsf{e}}\neg$ and $\neg\mathsf{K}_i\neg$ respectively. The boolean operations $\neg, \wedge, \vee$ are defined in the usual way by means of $\rightarrow$ and $\bot$. In paricular $\top := \bot \rightarrow \bot$ (cf. Rybakov [55] and Blackburn *et al.* [3]).

The intended meaning of the modal operators formerly introduced is: (i) $\Box_{\preccurlyeq}\mathtt{A}$: *the fact* $\mathtt{A}$ *is true from now on*; (ii) $\mathsf{K}_{\mathsf{e}}\mathtt{A}$: $\mathtt{A}$ is true everywhere in the *environment*; (iii) $\mathsf{K}_i\mathtt{A}$: *the agent i operating in the system knows* $\mathtt{A}$ *in the current moment* in the sense that all the information points accessible to agent i provide the information $\mathtt{A}$.

Formulae in the language $\mathcal{L}^{\mathsf{LTK}}$ allow occurrences of temporal operators in the scope of the epistemic modalities $\mathsf{K}_1, \ldots, \mathsf{K}_k$, leading to the possibility of expressing formulae such as $\mathsf{K}_i\Diamond_{\preccurlyeq}\mathtt{A}$, interpreted as *agent i knows that eventually it will be the case that* $\mathtt{A}$. Sometimes we may not want to be able to express this kind of expression. In fact, as we shall see later, according to the kind of semantics we shall further introduce, this might generate epistemic paradoxes. Suppose, for instance, that a fact $\mathtt{A}$ is true at some point in the future. According to the standard definition of truth in Kripke Semantics, this implies that *each* agent would know that such event *is bound to happen* eventually. Although this might not create any problem in most situations, it might sound unnatural in other cases. If one is, for instance, interested in describing human behaviour, the assumption that agents know future events may not result of use. In order to provide tools to describe a larger number of situations, we can introduce another version of the language just described. Thus, in order to prevent agents from having pre-knowledge concerning future events, we introduce a weaker language $\mathcal{L}^{-}_{\mathsf{LTK}}$. Let us define a formula $\mathtt{A}$ *local* if and only if it does not contain any occurrence of the modal operator $\Box_{\preccurlyeq}$, i.e. each propositional letter is local and if $\mathtt{A}$ is local, then so are $\mathsf{K_e}\mathtt{A}$ and $\mathsf{K}_i\mathtt{A}$ for each i. Well formed formulae are defined as they are in the former case, with the exception of formulae containing a modal operator K_i for some i: if $\mathtt{A}$ is a wff, then $\mathsf{K}_i\mathtt{A}$ is a wff, provided that $\mathtt{A}$ is *local*.

By the expression $Fma(\mathcal{L}^{\mathsf{LTK}})$ we denote the set of all the wff's on $\mathcal{L}^{\mathsf{LTK}}$ and by the term *formula* we usually refer to a member of $Fma(\mathcal{L}^{\mathsf{LTK}})$ unless otherwise specified. Clearly $Fma(\mathcal{L}^{-}_{\mathsf{LTK}}) \subset Fma(\mathcal{L}^{\mathsf{LTK}})$.

We have suggested that the language $\mathcal{L}^{-}_{\mathsf{LTK}}$ may help to solve the problem of agents having pre-knowledge of future events. But is it really this the solution to all our problems? Let us assume that we want to model a situation in which four people are playing poker. Then we would have four agents and each of these four agents would be in a certain state: each agent would have some cards, know how much the players playing before him had bet, have some information about the usual behaviour of the players and so on. On the other hand, something he definitely would not know is what kind of cards the other players have, otherwise the game would be pretty dull. If we try to formalise this situation using our language, then we could say that each agent in the set $\{a, b, c, d\}$ has access to a set of information, whereas the environment collects all those formulae which are true in the environment. Then if we suppose that the agent a knows that she has certain cards (information $\mathtt{A}$), we can formalise this information as $\mathsf{K}_a\mathtt{A}$ and consequently the formula $\Diamond_{\mathsf{e}}\mathsf{K}_a\mathtt{A}$ would be true at any point in the environment cluster. Therefore we would get $\mathsf{K}_{\mathsf{e}}\Diamond_{\mathsf{e}}\mathsf{K}_a\mathtt{A}$ and so each agent and in particular b would know this piece of information, i.e. $\mathsf{K}_b\Diamond_{\mathsf{e}}\mathsf{K}_a\mathtt{A}$. Clearly this would greatly spoil the game! If we move outside the example of the poker game, we can see that in many situations it is very unlikely that each agent is aware of the knowledge base of other agents. Therefore whenever we deal with such a situation, we may want to use a language which is even more restricted than $\mathcal{L}^{-}_{\mathsf{LTK}}$. Let us define the language $\mathcal{L}^{--}_{\mathsf{LTK}}$ in the following way. We call a formula *agent-local* if it is local and, if we have a set $\{1,\ldots,k\}$ of agents, the modal operator K_{e} does not occur in the scope of any modal operator K_i. This is to say that a propositional letter p is agent-local and if $\mathtt{A}$ and $\mathtt{B}$ are both local and agent-local, then $\neg\mathtt{A}$, $\mathtt{A}\wedge\mathtt{B}$, $\mathtt{A}\vee\mathtt{B}$, $\mathtt{A}\rightarrow\mathtt{B}$ and $\mathsf{K}_i\mathtt{A}$ for each $i\in\{1,\ldots,k\}$ are agent-local too. Going back to our poker example, we can see that the formula $\mathsf{K}_b\Diamond_{\mathsf{e}}\mathsf{K}_a\mathtt{A}$ is no longer gramatical.

We have now defined three different languages which are suitable to formalise and talk about different situations. Our choice would then be made according to the specific state of affairs we want to model. Summarising: (i) the language $\mathcal{L}^{\mathsf{LTK}}$ allows the agents to have pre-knowledge of future events; (ii) the language $\mathcal{L}^{-}_{\mathsf{LTK}}$ forbids agents to have pre-knowledge of future events; (iii) the language $\mathcal{L}^{--}_{\mathsf{LTK}}$ prevents agents both from having pre-knowledge and from having access to the knowledge base of any other agent operating in the same environment at the same moment.

Clearly the set of formulae in any of these three languages is properly included in the set of formulae in the more expressive language, $\mathcal{L}^{\mathsf{LTK}}$ being the most expressive one and $\mathcal{L}^{--}_{\mathsf{LTK}}$ the least, i.e. $Fma(\mathcal{L}^{--}_{\mathsf{LTK}})\subset Fma(\mathcal{L}^{-}_{\mathsf{LTK}})\subset$

$Fma(\mathcal{L}^{\mathsf{LTK}})$.

2.2 Semantics

2.2.1 Key Concepts: Multi-Modal Kripke Semantics

As we have already pointed out in the Introduction, one of the main features of multi-modal logics is that they have been provided with a very intuitive set of semantic tools. Although Algebraic Semantics is historically the first developed, the Possible Worlds Framework, or Kripke Semantics is definitely the type of semantics which has been adopted more widely (cf. [27]). This is due to the fact that these tools have an extremely intuitive interpretation and are flexible enough to be employed in various circumstances without loss of their intuitive counterpart. This is the reason that leads us to introduce the Possible Worlds Semantics first. In later chapters, nevertheless, we shall also use traditional Algebraic Semantics. We shall introduce it in a more mature stage when many results based on Kripke Semantics are already stated and described.

As we have anticipated, Kripke Semantics (cf. Kripke [41, 42, 43]) has a very intuitive interpretation which confers great appeal. The idea behind it is very simple. It takes its origins in the ideas of Leibniz, who stated that there is a plurality of possible worlds, and the actual one is nothing but one of the many possibilities. According to Leibniz, nevertheless, the actual world is definitely the best one among all the possibilities, chosen by God who has the capability of searching and choosing the perfect solution. Nowadays, however, researchers in modal logic tend to bypass these theoretical and metaphysical aspects while keeping the main idea of Leibniz's approach.

For instance, let us suppose that we want to describe any situation which sees several agents interacting one with each other. Let us suppose that such agents are, for instance, playing dice. Then whenever the pair of dice is cast, there are several possible outputs. It is perfectly clear how we can consider each of the possible outputs as a different world. This may be of use for instance if we want to make considerations on probability and so on. Moreover, we may turn our attention to the analysis of agents' knowledge. Any fact p is then *known* by an agent whenever he cannot consider as possible a state of affairs in which p does not hold.

> The intuitive idea behind the possible-worlds model is that besides the true state of affairs, there are a number of other possible states of affairs or *worlds*. Given his current information, an agent may not be able to tell which of a number of possible worlds describes the actual state of affairs. An agent is then said to *know* a fact ϕ if ϕ is true at all the worlds he considers possible (given his current information). For example, agent 1 may be walking on the streets of San Francisco. Thus, in all the worlds that the agent considers possible, it is sunny in San Francisco. (We are implicitly assuming here that the agent does not consider it possible that he is hallucinating and in fact it is raining heavily in San Francisco.) On the other hand, since the agent has no information about the weather in London, there are worlds he considers possible in which it is sunny in London, and others in which it is raining in London. Thus, this agent knows that it is sunny in San Francisco but he does not know whether it is sunny in London. Intuitively, the fewer worlds an agent considers possible, the less his uncertainty, and the more

he knows. If the agent acquires additional information – such as hearing from a reliable source that it is currently sunny in London – then he would no longer consider possible any of the worlds in which it is raining in London[1].

Although we assume the reader to be familiar with *Possible Worlds semantics*, we provide few basic definitions necessary to understand the particular case we shall work with.

Definition 2.2.1 *A k-modal Kripke-frame is a tuple $\mathcal{F} = \langle W, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$ where W is a non-empty set of worlds and each R_j is some binary relation on $W \times W$. Given a frame $\mathcal{F}$, by $W_{\mathcal{F}}$ we denote its base set.*
Given a Kripke-frame $\mathcal{F}$, a Kripke-model (or just a model) $\mathcal{M}$ on $\mathcal{F}$ is a tuple $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where V is a valuation mapping the elements of a set P of propositional letters into the power set of the universe of $\mathcal{F}$, i.e. the valuation V associates to each propositional letter p a set of worlds from $W_{\mathcal{F}}$, intuitively those worlds in which p is true.

In what follows we shall use some symbols in our meta-language, namely the symbols & and $\Rightarrow$ shall be used in order to shorten the english expressions *and* and *implies* respectively. Moreover we shall use the symbols (quantifiers) $\forall$ and $\exists$ as meaning *for all* and there exists.

Definition 2.2.2 *Given a Kripke-frame $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$, for any R_i, an R_i-cluster of worlds is a subset $\mathcal{C}_{\mathsf{R}_i}$ of $W_{\mathcal{F}}$ s.t.: $\forall w \forall z \in \mathcal{C}_{\mathsf{R}_i}$ $(w\mathsf{R}_i z \;\&\; z\mathsf{R}_i w)$ and $\forall z \in W_{\mathcal{F}} \forall w \in \mathcal{C}_{\mathsf{R}_i}$ $((w\mathsf{R}_i z \;\&\; z\mathsf{R}_i w) \Rightarrow z \in \mathcal{C}_{\mathsf{R}_i})$.*
*An R_i-cluster is said to be: **degenerate** if it consists of one single R_i-irreflexive world; **simple** if it consists of a single R_i-reflexive world; **proper***

[1] Fagin *et al.* [17] p.16

if it contains at least two R_i*-reflexive worlds.*

For any R_i*,* $\mathcal{C}_{\mathsf{R}_i}(w)$ *is the* R_i*-cluster s.t.* $w \in \mathcal{C}_{\mathsf{R}_i}(w)$*. Given two* R_i*-clusters* $\mathcal{C}_m$ *and* $\mathcal{C}_j$ *the expression* $\mathcal{C}_m \mathsf{R}_i \mathcal{C}_j$ *is an abbreviation for* $\forall w \in \mathcal{C}_m \forall z \in \mathcal{C}_j (w \mathsf{R}_i z)$*.*

2.2.2 Linear Time and Knowledge structures: $\mathcal{LTK}$-Frames

We use a special kind of multi-modal Kripke frames called $\mathcal{LTK}$-frames, where the prefix $\mathcal{LTK}$ is an acronym for *Linear Time and Knowledge.* These structures aim at modelling a set of agents operating in a temporal framework.

Definition 2.2.3 *An* $\mathcal{LTK}$*-frame (*Linear Time and Knowledge frame*) is a k+2-modal Kripke-frame* $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$*, where* $W_{\mathcal{F}}$ *is the disjoint union of certain non empty sets* $\mathcal{C}_n$*, for* $n \in \mathbb{N}$*:* $W_{\mathcal{F}} := \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$*. The binary relations* $\mathsf{R}_{\preccurlyeq}$*,* R_{e}*, and* R_j *are as follows:*

(i) $\mathsf{R}_{\preccurlyeq}$ *is the linear, reflexive and transitive relation on* $W_{\mathcal{F}}$ *such that:* $\forall v \forall z \in W_{\mathcal{F}} (v \mathsf{R}_{\preccurlyeq} z \;\; \textit{iff} \;\; \exists i, j \in \mathbb{N} \; ((v \in \mathcal{C}_i) \;\&\; (z \in \mathcal{C}_j) \;\&\; (i \leq j)))$

(ii) R_{e} *is a universal relation on any* $\mathcal{C}_i \in W_{\mathcal{F}}$*:* $\forall v \forall z \in W_{\mathcal{F}} (v \mathsf{R}_{\mathsf{e}} z \;\Leftrightarrow\; \exists i \in \mathbb{N} \; (v \in \mathcal{C}_i \;\&\; z \in \mathcal{C}_i))$*;*

(iii) each R_j *is some equivalence relation on each* $\mathcal{C}_i$*.*

Each world can be interpreted as a single information point. The linear temporal relation $\mathsf{R}_{\preccurlyeq}$ links such information points so that, given two worlds v and z, the expression $v \mathsf{R}_{\preccurlyeq} z$ means either that v and z are both available at a moment n, or that z will be available in the future with respect to v. Hence two information points are concurrent if they belong to the same $\mathsf{R}_{\preccurlyeq}$-cluster (time-cluster) and an $\mathsf{R}_{\preccurlyeq}$-cluster can be seen as a moment in the time line. Although time is usually perceived as continuous, it may as well

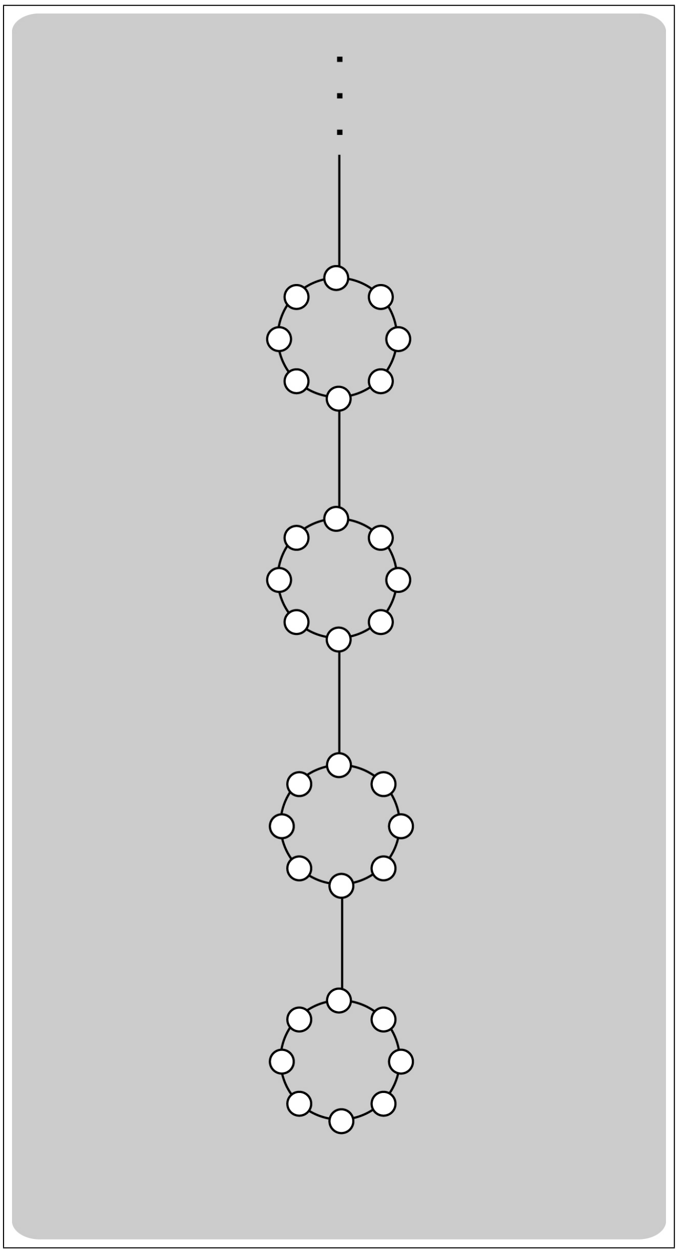

Figure 2.1: Scheme of the structure of an $\mathfrak{LTK}$-frame: here each big circle represents both a moment in the time line and an environmental cluster, whereas each small circle is intended to represent a single information point.

be thought as discrete.

> [...] Although we typically think of time as being continuous, assuming that time is discrete is quite natural. Computers proceed in discrete time steps, after all. Even when analyzing situations involving human agents, we can often usefully imagine that the relevant actions are performed at discrete time instances [...][2].

In this context the property of discreteness means that given any two distinct points in the time line, there might be only a finite number of moments between them (though each moment may contain an infinite number of information points). Therefore the relation $\mathsf{R}_{\preccurlyeq}$ is discrete with respect to time-clusters. This is actually the way in which computers work. Moreover, the temporal line has a first point starting from which it proceeds towards the future. The most important assumption is to consider the flow of time as linear and hence not branching. If we assumed the time to be branching we might have different possible future paths: among these, only one would become actual. Conversely, if the time line is assumed as linear, there is only one possible path towards the future: the actual one. This implies that we may not quantify over possible, although not actual, temporal paths. In other words, what is relevant is only the actual path the world goes through. Such strong theoretical deterministic assumption may be practically justified by the observation that, in analogy to the human situation, all the agents operating in the system are not aware of the prefixed unicity of their temporal path and they act as heading to a not-determined future.

The relation R_{e} is defined at each moment in the time line and it links all the information points belonging to the same *environment* or *network*.

[2] Fagin *et al.* [17] page 112.

Thus, we define an *environment* as the collection of all the concurrent information points which are accessible to a set of agents at a given time. Hence the environment an agent operates in is nothing but the collection of the information points potentially available. It represent the informations network each agents operates within. Each agent (see below) may have access to all, some or none of such information points, nevertheless these points form the environment this agent *lives and operates in.* Notice that in the semantics we have just described a moment and an environment do coincide. In this specific semantics, in fact, only one environment is possible at each moment and hence time-clusters and environment-clusters do coincide. We remind the reader to Chapter 4, Section 4 for a more general semantic definition and further discussion. We shall introduce, in fact, some generalised Kripke-frames which allow different environments to occur at the same moment.

The relation R_i links all the information points accessible by agent i in a given environment. Any information point provides the agents with some information.

It can be easily noticed that $\mathcal{LTK}$-frames are a suitable tool to interpret the language $\mathcal{L}^{-}_{\mathsf{LTK}}$ and $\mathcal{L}^{--}_{\mathsf{LTK}}$ as well as $\mathcal{L}^{\mathsf{LTK}}$. The only difference would be that in the case of $\mathcal{L}^{-}_{\mathsf{LTK}}$, all the facts available to the agents are *local* and therefore do not concern any future event, whereas in the case of $\mathcal{L}^{--}_{\mathsf{LTK}}$ the piece of information available to the agents is *agent local.* Nevertheless at each world a certain number of statements about the future could be true, but this piece of information would not be available to the agents.

We have now described the intended interpretation of our semantics. Let us suppose, for instance, that one is to describe a conference with several parallel sessions using our semantic tools. The situation would then be interpreted as follows:

- the base set of the model would be the set of all the sessions (information points);

- a moment n would be the collection of all those lectures given at the time n;

- any person attending the conference would be an agent operating in the system;

- any agent has access to some information point at any moment.

According to our semantic definition, $\mathcal{LTK}$-frames enjoy some properties and among these, the most important and peculiar ones are the following:

PM.1: $v\mathsf{R}_{\mathsf{e}}z \;\Rightarrow\; (v\mathsf{R}_{\preccurlyeq}z \;\&\; z\mathsf{R}_{\preccurlyeq}v)$ i.e. the information points available in the same environment are concurrent

PM.2: $v\mathsf{R}_{i}z \;\Rightarrow\; v\mathsf{R}_{\mathsf{e}}z$ i.e. the information points available to agent i must be in the same environment (hence at the same moment)

PM.3: $(v\mathsf{R}_{\preccurlyeq}z \;\&\; z\mathsf{R}_{\preccurlyeq}v) \;\Rightarrow\; v\mathsf{R}_{\mathsf{e}}z$ i.e. concurrent information points are in the same environment

A model $\mathcal{M}$ on $\mathcal{F}$ is a pair $\langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is an $\mathcal{LTK}$-frame and V is a map (valuation) which associates to each propositional letter $p \in P$ a set of worlds from the base set of $\mathcal{F}$. The valuation V can be extended in the standard way from the set P onto all the well formed formulae built up on P. In particular, $\forall v \in W_{\mathcal{F}}$,

(i) $(\mathcal{F}, v) \Vdash_V p \;\Leftrightarrow\; v \in V(p)$;

(ii) $(\mathcal{F}, v) \Vdash_V \Box_{\preccurlyeq}\mathtt{A} \;\Leftrightarrow\; \forall z \in W_{\mathcal{F}}\ (v\mathsf{R}_{\preccurlyeq}z \;\Rightarrow\; (\mathcal{F}, z) \Vdash_V \mathtt{A})$;

(iii) $(\mathcal{F}, v) \Vdash_V \mathsf{K}_{\mathsf{e}}\mathtt{A} \;\Leftrightarrow\; \forall z \in W_{\mathcal{F}}\ (v\mathsf{R}_{\mathsf{e}}z \;\Rightarrow\; (\mathcal{F}, z) \Vdash_V \mathtt{A})$;

(iv) For each j, $(\mathcal{F}, v) \Vdash_V \mathsf{K}_j\mathtt{A} \;\Leftrightarrow\; \forall z \in W_{\mathcal{F}}\ (v\mathsf{R}_j z \;\Rightarrow\; (\mathcal{F}, z) \Vdash_V \mathtt{A})$.

If $\mathcal{M} = \langle \mathcal{F}, V\rangle$ is a model on a frame $\mathcal{F}$, a formula $\mathtt{A}$ is said to be *true in the model* $\mathcal{M}$ *at the world* v if $(\mathcal{F}, v) \Vdash_V \mathtt{A}$; $\mathtt{A}$ is *true in the model* $\mathcal{M}$, notation $\mathcal{F} \Vdash_V \mathtt{A}$, if $\forall v \in W_{\mathcal{F}},\ (\mathcal{F}, v) \Vdash_V \mathtt{A}$; $\mathtt{A}$ is *valid in the frame* $\mathcal{F}$, notation $\mathcal{F} \Vdash \mathtt{A}$, if, for any valuation V for $\mathcal{F}$ (that is for any model $\mathcal{M}_{\mathcal{F}}$ on $\mathcal{F}$), $\mathcal{F} \Vdash_V \mathtt{A}$. Given a class of frames $\mathbb{F}$, $\mathtt{A}$ *is valid on* $\mathbb{F}$ (and we say $\mathtt{A}$ to be $\mathbb{F}$-valid) if $\forall \mathcal{F} \in \mathbb{F},\ \mathcal{F} \Vdash \mathtt{A}$.

Definition 2.2.4 *Let* $\mathcal{LTK}$ *be the class of all* $\mathcal{LTK}$*-frames. The logic* LTK *is the set of all* $\mathcal{LTK}$*-valid formulae:* $\mathsf{LTK} := \{\mathtt{A} \in Fma(\mathcal{L}^{\mathsf{LTK}}) \mid \mathcal{F} \Vdash \mathtt{A}\ \&\ \mathcal{F} \in \mathcal{LTK}\}$*. If* $\mathtt{A}$ *belongs to* LTK*, then* $\mathtt{A}$ *is a theorem of* LTK*. Likewise* $\mathsf{LTK}^- := \{\mathtt{A} \in Fma(\mathcal{L}^-_{\mathsf{LTK}}) \mid \mathcal{F} \Vdash \mathtt{A}\ \&\ \mathcal{F} \in \mathcal{LTK}\}$ *and* $\mathsf{LTK}^{--} := \{\mathtt{A} \in Fma(\mathcal{L}^{--}_{\mathsf{LTK}}) \mid \mathcal{F} \Vdash \mathtt{A}\ \&\ \mathcal{F} \in \mathcal{LTK}\}$.

2.3 *Effective finite model property* for LTK

The first question we shall give an answer to is whether LTK has the *effective finite model property (efmp)*. If so, the same property would clearly be enjoyed by LTK^- and LTK^{--} too.

Showing that a logic has the *effective finite model property (efmp)* is quite different from showing that it enjoys only the *finite model property*. In fact if a logic L has the *finite model property*, then for each formula $\mathtt{A}$ which is *not* a theorem of L there is a finite model $\langle \mathcal{F}, V\rangle$ such that:

(i) in $\langle \mathcal{F}, V\rangle$ all the theorems of L are true, i.e. for each formula $\mathtt{B} \in \mathsf{L}$, $\mathcal{F} \Vdash_V \mathtt{B}$;

(ii) the formula $\mathtt{A}$ is not true in $\langle \mathcal{F}, V \rangle$, i.e. $\mathcal{F} \not\Vdash_V \mathtt{A}$.

The *effective* finite model property implies something more. It actually adds a very important condition which makes the difference. In fact if a logic L has the *effective finite model property*, then for each formula $\mathtt{A}$ which is *not* a theorem of L there is a finite model $\langle \mathcal{F}, V \rangle$ such that:

(i) in $\langle \mathcal{F}, V \rangle$ all the theorems of L are true, i.e. for each formula $\mathtt{B} \in \mathsf{L}$, $\mathcal{F} \Vdash_V \mathtt{B}$;

(ii) the formula $\mathtt{A}$ is not true in $\langle \mathcal{F}, V \rangle$, i.e. $\mathcal{F} \not\Vdash_V \mathtt{A}$.

(iii) the size of $\langle \mathcal{F}, V \rangle$ is computable and bounded by the size of $\mathtt{A}$, i.e. $\|W_{\mathcal{F}}\| \leq f(\|\mathtt{A}\|)$, where f is computable.

This means that for each formula $\mathtt{A}$ which is *not* a theorem of L we can build a model $\langle \mathcal{F}, V \rangle$ whose size is finite *and* computable from the size of $\mathtt{A}$; this model verifies all theorems of L and falsifies $\mathtt{A}$. The two definitions are deeply different. The last one implies that a logic is decidable with respect to its theorems, whereas the former one does not. In fact in order to check whether a formula $\mathtt{A}$ is a theorem of a logic L with the *finite model property*, we should check if *none* of the finite models of L falsifies $\mathtt{A}$. This means that one should check an *infinite* number of models and this is not possible in a finite time. On the other hand if L has the *efective finite model property*, we should check if *none* of the finite models of L *whose size is at most n for some computable and finite n* falsifies $\mathtt{A}$. This is quite different, as the number of models one should check would now be *finite*.

We shall prove below that LTK has the *efmp* and hence it is decidable. Thus for any formula $\mathtt{A}$ in the language of our logic LTK, we can check *in a finite number of steps* whether $\mathtt{A}$ is or is not a theorem of LTK. The result we show in this Chapter may be found in C. and Rybakov [9].

Definition 2.3.1 *Given a Kripke-frame $\mathcal{F} = \langle W, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$ and a world*

w in $W_{\mathcal{F}}$, $w^{\mathsf{R}_i \leq} := \{z \mid w\mathsf{R}_i z\}$ and $w^{\mathsf{R}_i <} := \{z \mid w\mathsf{R}_i z \;\&\; \neg(z\mathsf{R}_i w)\}$. Given a R_i-cluster $\mathcal{C}$, $\mathcal{C}^{\mathsf{R}_i \leq} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{R}_i\mathcal{C}_j\}$ and $\mathcal{C}^{\mathsf{R}_i <} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{R}_i\mathcal{C}_j \;\&\; \neg(\mathcal{C}_j\mathsf{R}_i\mathcal{C})\}$ (In what follows we shall always use the expression $w^{\preccurlyeq}$ and $\mathcal{C}^{\preccurlyeq}$ as abbreviations for $w^{\mathsf{R}_{\preccurlyeq}\leq}$ and $\mathcal{C}^{\mathsf{R}_{\preccurlyeq}\leq}$ respectively. We shall also use $w^{<}$ and $\mathcal{C}^{<}$ instead of $w^{\mathsf{R}_{\preccurlyeq}<}$ and $\mathcal{C}^{\mathsf{R}_{\preccurlyeq}<}$).

Theorem 2.3.2 (C. and Rybakov [9]) *The logic* LTK *has the* efmp *and hence it is decidable with respect to its theorems.*

Proof. Take a formula $\mathtt{A}$ such that $\mathtt{A} \notin \mathsf{LTK}$; then there are an $\mathcal{LTK}$-frame $\mathcal{F}_1 := \langle W_{\mathcal{F}_1}, \mathsf{R}^1_{\preccurlyeq}, \mathsf{R}^1_{\mathsf{e}}, \mathsf{R}^1_1, \ldots, \mathsf{R}^1_k\rangle$, a model $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1\rangle$ and a world $w \in W_{\mathcal{F}_1}$ such that $(\mathcal{F}_1, w) \not\Vdash_{V_1} \mathtt{A}$. Notice that $\mathcal{F}_1$ is infinite by definition. Starting from this fact, our proof follows 4 steps:

Step 1. We make a filtration on *each* time cluster, in order to get a new model $\langle \mathcal{F}_2, V_2\rangle$ which contains only time-clusters with a finite number of worlds. We show that in $\langle \mathcal{F}_2, V_2\rangle$ the formula $\mathtt{A}$ is still false.

Step 2. We proceed by reducing the number of time-clusters, so that we can deal with a finite frame. The resulting model $\langle \mathcal{F}_3, V_3\rangle$ is based on a special finite frame which we shall define as a *reflexive* LTK-*balloon.* Moreover in the model $\langle \mathcal{F}_3, V_3\rangle$ the formula $\mathtt{A}$ is still false.

Step 3. We construct a new model by deleting time-clusters from $\langle \mathcal{F}_3, V_3\rangle$. The resulting model $\langle \mathcal{F}_4, V_4\rangle$ has a finite and computable base set. Again, in $\langle \mathcal{F}_4, V_4\rangle$ the formula $\mathtt{A}$ is false.

Step 4. We show that since any LTK-balloon (and hence also $\langle \mathcal{F}_4, V_4\rangle$) is the p-morphic image of some $\mathcal{LTK}$-frame, in $\langle \mathcal{F}_4, V_4\rangle$ all the theorems of LTK hold true whereas the formula $\mathtt{A}$ does not.

Hence by the end of the fourth step we have a model such that:

(i) all the theorems of LTK are true;

(ii) the formula $\mathtt{A}$ (which is not a theorem of LTK) is false;

(iii) the size of the model is computable from the size of $\mathtt{A}$.

Hence the logic LTK has the *efmp* and it is decidable with respect to its theorems.

Take a formula $\mathtt{A}$ such that $\mathtt{A} \notin \mathsf{LTK}$; then there are an $\mathcal{LTK}$-frame $\mathcal{F}_1 := \langle W_{\mathcal{F}_1}, \mathsf{R}^1_{\preccurlyeq}, \mathsf{R}^1_{\mathsf{e}}, \mathsf{R}^1_1, \ldots, \mathsf{R}^1_k \rangle$, a model $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1 \rangle$ and a world $w \in W_{\mathcal{F}_1}$ such that $(\mathcal{F}_1, w) \nVdash_{V_1} \mathtt{A}$.

Step 1: Filtering each single time-cluster We start by reducing the number of worlds belonging to each $\mathsf{R}^1_{\preccurlyeq}$-cluster $\mathcal{C}$ of worlds from $W_{\mathcal{F}_1}$ using the standard filtration technique, briefly sketched below. Let $Sub(\mathtt{A})$ be the set of all the sub-formulae of $\mathtt{A}$. Define the equivalence relation $\approx$ on $W_{\mathcal{F}_1}$ as follows:
$\forall w \forall z \in W_{\mathcal{F}_1}(w \approx z \quad \Leftrightarrow \quad w\mathsf{R}^1_{\preccurlyeq}z \;\&\; z\mathsf{R}^1_{\preccurlyeq}w \;\&\; \forall \mathtt{B} \in Sub(\mathtt{A})\ ((\mathcal{F}_1, w) \Vdash_{V_1} \mathtt{B} \Leftrightarrow (\mathcal{F}_1, z) \Vdash_{V_1} \mathtt{B})))$ (Recall that the condition $w\mathsf{R}^1_{\preccurlyeq}z \;\&\; z\mathsf{R}^1_{\preccurlyeq}w$ is equivalent to $\exists i(w \in \mathcal{C}_i \;\&\; z \in \mathcal{C}_i)$, that is the worlds w and z belong to the same time-cluster and hence $w\mathsf{R}^1_{\mathsf{e}}z$).

Next, define the quotient set of the original model: $\forall w \in W_{\mathcal{F}_1}\ [w] := \{z \mid w \approx z\}$, $\forall n \in \mathbb{N}\ [\mathcal{C}_n] := \{[w] \mid w \in \mathcal{C}_n\}$. Let $\mathcal{F}_2 := \langle W_{\mathcal{F}_2}, \mathsf{R}^2_{\preccurlyeq}, \mathsf{R}^2_{\mathsf{e}}, \mathsf{R}^2_1, \ldots, \mathsf{R}^2_k \rangle$ be a frame where:

(i) $W_{\mathcal{F}_2} := \bigcup_{n \in \mathbb{N}} [\mathcal{C}_n]$;

(ii) $[w]\mathsf{R}^2_{\preccurlyeq}[z] \Leftrightarrow w\mathsf{R}^1_{\preccurlyeq}z$;

(iii) $[w]\mathsf{R}^2_{\mathsf{e}}[z] \Leftrightarrow w\mathsf{R}^1_{\mathsf{e}}z$;

(iv) $\forall i, 1 \leq i \leq k\ [w]\mathsf{R}^2_i[z] \Leftrightarrow ([w] \in [\mathcal{C}_n] \;\&\; [z] \in [\mathcal{C}_n] \;\&\; \forall \mathsf{K}_i\mathtt{B} \in Sub(\mathtt{A})((\mathcal{F}_1, w) \Vdash_{V_1} \mathsf{K}_i\mathtt{B} \Leftrightarrow (\mathcal{F}_1, z) \Vdash_{V_1} \mathsf{K}_i\mathtt{B}))$.

Let $\mathcal{M}_2 := \langle \mathcal{F}_2, V_2 \rangle$ be a model on $\mathcal{F}_2$ where V_2 is defined as:

$$Dom(V_2) := Prop(Sub(\mathtt{A}))$$

$$\forall p \in Sub(\mathtt{A}) \quad V_2(p) := \{[w] \mid w \in V_1(p)\}$$

Since the model described is the result of a filtration, the standard filtration-lemma holds:

Lemma 2.3.3 *For any formula* $\mathtt{B} \in Sub(\mathtt{A})$*, for any world* $w \in W_1$*,* $(\mathcal{F}_1, w) \Vdash_{V_1} \mathtt{B} \Leftrightarrow (\mathcal{F}_2, [w]) \Vdash_{V_2} \mathtt{B}$*.*

Corollary 2.3.4 $\mathcal{F}_2 \nVdash_{V_2} \mathtt{A}$*.*

Thus the model $\mathcal{M}_2$ refutes $\mathtt{A}$ as well. Moreover, each $\mathsf{R}^2_{\preccurlyeq}$-cluster contains a finite number of worlds, bounded by the size of $\mathtt{A}$, namely $\|\mathcal{C}\| \leq 2^{\|Sub(\mathtt{A})\|}$ for each $\mathsf{R}^2_{\preccurlyeq}$-cluster $\mathcal{C}$. The result of this operation gives still an $\mathcal{LTK}$-frame, i.e. an $\mathsf{R}^2_{\preccurlyeq}$-linear sequence of R_{e}-clusters. The difference is that now each $\mathsf{R}^2_{\preccurlyeq}$-cluster contains a finite number of worlds.

Step 2: Reducing the number of time-clusters We shall reduce, now, the number of time-clusters (i.e. $\mathsf{R}^2_{\preccurlyeq}$-clusters) to a finite one. We need few preliminary facts and definitions.

Definition 2.3.5 *Let* $\mathcal{M}_1 := \langle W_1, \mathsf{R}_1, V_1 \rangle$ *and* $\mathcal{M}_2 := \langle W_2, \mathsf{R}_2, V_2 \rangle$ *be two Kipke-models and* f *be a one-to-one mapping of* W_1 onto W_2*. Then* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are* isomorphic*, in symbols* $\mathcal{M}_1 \cong \mathcal{M}_2$*, if and only if:*

(i) V_1 *and* V_2 *share the same domain;*

(ii) $\forall w, v \in W_1(w\mathsf{R}_1 v \Leftrightarrow f(w)\mathsf{R}_2 f(v))$;

(iii) $\forall w, v \in W_1(w \Vdash_{V_1} p \Leftrightarrow f(w) \Vdash_{V_2} p)$.

Evidently, the following holds:

Proposition 2.3.6 *There is only a finite, computable from the size of* $\mathtt{A}$, *number of non-isomorphic w.r.t.* $Sub(\mathtt{A})$ *time-clusters* $\mathcal{C}$ *from* $W_{\mathcal{F}_2}$.

Definition 2.3.7 *Given an* $\mathcal{LTK}$*-frame* $\mathcal{S} := \langle W_{\mathcal{S}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_1, \dots, \mathsf{R}_k\rangle$ *and a model* $\mathcal{M} := \langle \mathcal{S}, V\rangle$, *an* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_s$ *is a* stabilising cluster *if and only if for any* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_i \in \mathcal{C}_s^{\preccurlyeq}$, *for any* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_j \in \mathcal{C}_s^{\preccurlyeq}$ *there is an* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_k \in \mathcal{C}_i^{\preccurlyeq}$ *such that* $\mathcal{C}_j \cong \mathcal{C}_k$, *i.e. the sets* $\mathcal{C}_s^{\preccurlyeq}$ *and* $\mathcal{C}_i^{\preccurlyeq}$ *coincide up to isomorphism between* $\mathsf{R}_{\preccurlyeq}$*-clusters.*

Lemma 2.3.8 *The model* $\mathcal{M}_2$ *has a stabilising* $\mathsf{R}^2_{\preccurlyeq}$*-cluster* $\mathcal{C}_s$.

PROOF. By Proposition 2.3.6 the number of non-isomorphic $\mathsf{R}^2_{\preccurlyeq}$-clusters $\mathcal{C}$ is finite. Moreover, we have that for any pair of $\mathsf{R}^2_{\preccurlyeq}$-clusters $\mathcal{C}_i$, $\mathcal{C}_j$ from $W_{\mathcal{F}_2}$, $\mathcal{C}_i\mathsf{R}^2_{\preccurlyeq}\mathcal{C}_j \Rightarrow \mathcal{C}_i^{\preccurlyeq} \supseteq \mathcal{C}_j^{\preccurlyeq}$. Consider the sequence of all the time-clusters $\mathcal{C}_1, \mathcal{C}_2, \dots$. We construct a subsequence $\mathcal{C}'_n$ of the sequence $\mathcal{C}_n$, $n \in \mathbb{N}$ as follows. Take $\mathcal{C}_1$; if $\mathcal{C}_1$ is a stabilising cluster, then we stop, and the subsequence is chosen. Otherwise, assume that a subsequence $\mathcal{C}'_1, \dots, \mathcal{C}'_n$ is chosen. If $\mathcal{C}'_n$ is not a stabilising cluster, then there is a cluster $\mathcal{C}_k$, where, up to isomorphism, $\mathcal{C}'^{\preccurlyeq}_n \supset \mathcal{C}_k^{\preccurlyeq}$. Take the $\mathsf{R}^2_{\preccurlyeq}$-smallest $\mathcal{C}_k$ with this property and set $\mathcal{C}'_{(n+1)} := \mathcal{C}_k$. Since $\mathcal{C}'^{\preccurlyeq}_n \supset \mathcal{C}'^{\preccurlyeq}_{(n+1)}$, this procedure must terminate, and it stops at a stabilising cluster. ∎

Lemma 2.3.9 *If* $\mathcal{C}_s$ *is a stabilising cluster, then, for all the* $\mathsf{R}^2_{\preccurlyeq}$*-clusters* $\mathcal{C}_i$, $\mathcal{C}_j$ *of worlds from* $W_{\mathcal{F}_2}$ *such that* $\mathcal{C}_s\mathsf{R}^2_{\preccurlyeq}\mathcal{C}_i$ *and* $\mathcal{C}_s\mathsf{R}^2_{\preccurlyeq}\mathcal{C}_j$, *if* $\mathcal{C}_i$ *is isomorphic to* $\mathcal{C}_j$ *by a mapping* f, *then*

$\forall \mathtt{B} \in Sub(\mathtt{A}),\ \forall w \in \mathcal{C}_i\ (\mathcal{F}_2, w) \Vdash_{V_2} \mathtt{B} \quad \Leftrightarrow \quad (\mathcal{F}_2, f(w)) \Vdash_{V_2} \mathtt{B}$.

PROOF. It may be given by an easy induction on the length of B. Both the basis of the induction and the inductive steps regarding the boolean operations and the modal operators $\mathsf{K_e}$ and $\mathsf{K_a}$ are evident. Hence, we turn our attention only to the case B is $\Box_{\preccurlyeq}\mathtt{D}$, $(\mathcal{F}_2, w) \Vdash_{V_2} \Box_{\preccurlyeq}\mathtt{D}$ and $w^{\preccurlyeq} \subset f(w)^{\preccurlyeq}$. It follows that $\forall z \in w^{\preccurlyeq}$, $(\mathcal{F}_2, z) \Vdash_{V_2} \mathtt{D}$. Since for any $\mathcal{C}_k \in \cup \mathcal{C}_j^{\preccurlyeq}$ there is a $\mathcal{C}'_k$ *in* $\cup\, \mathcal{C}_i^{\preccurlyeq}$ such that $\mathcal{C}_k \cong \mathcal{C}'_k$ and $\mathcal{C}'_k \Vdash_{V_2} \mathtt{D}$, it follows that $\forall v \in f(w)^{\preccurlyeq}$, $v \Vdash_{V_2} \mathtt{D}$ by inductive hypothesis and hence $(\mathcal{F}_2, f(w)) \Vdash_{V_2} \Box_{\preccurlyeq}\mathtt{D}$. ∎

Consider the set $\mathcal{C}_s^{\preccurlyeq}$. We want to reduce the number of its elements to a finite one. Firstly, we make a partition of this set into equivalence classes. We take each time-cluster of worlds from $\mathcal{C}_s^{\preccurlyeq}$ and we define its equivalence class w.r.t. isomorphic time-clusters $[\mathcal{C}]_{\cong} := \{\mathcal{C}_j \mid \mathcal{C}_s \mathsf{R}^2_{\preccurlyeq} \mathcal{C}_j \ \&\ \mathcal{C} \cong \mathcal{C}_j\}$. Then the class $[\mathcal{C}]_{\cong}$ contains all those clusters which are both isomorphic to $\mathcal{C}$ and such that they are *after* the stabilising cluster $\mathcal{C}_s$. Clearly in $\mathcal{M}^2$ there is only a finite number m of such equivalence classes, as the domain of V_2 contains a finite number of propositional letters, the ones occurring in $Sub(\mathtt{A})$. Take and fix for any such equivalence class $[\mathcal{C}_j]_{\cong}$ a representative element $\mathsf{Rep}(\mathcal{C}_j) \in [\mathcal{C}_j]_{\cong}$ and set $\mathsf{REP} := \bigcup_{1 \le j \le m} \mathsf{Rep}(\mathcal{C}_j)$. Let us introduce a new frame $\mathsf{St} := \langle W_{\mathsf{St}}, \mathsf{R}^{\mathsf{St}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{St}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{St}}_1, \dots, \mathsf{R}^{\mathsf{St}}_k \rangle$ where:

(i) $W_{\mathsf{St}} := \bigcup_{\mathcal{C} \in \mathsf{REP}} \mathcal{C}$

(ii) $\mathsf{R}^{\mathsf{St}}_{\preccurlyeq} := W_{\mathsf{St}} \times W_{\mathsf{St}}$

(iii) $\mathsf{R}^{\mathsf{St}}_{\mathsf{e}} := \mathsf{R}^2_{\mathsf{e}} \restriction W_{\mathsf{St}}$ (i.e. $\mathsf{R}^{\mathsf{St}}_{\mathsf{e}}$ is the restriction of $\mathsf{R}^2_{\mathsf{e}}$ on W_{St}.)

(iv) for $1 \le \mathsf{a} \le k$, $\mathsf{R}^{\mathsf{St}}_{\mathsf{a}} := \mathsf{R}^2_{\mathsf{a}} \restriction W_{\mathsf{St}}$

Evidently, the frame St is nothing but an $\mathsf{R}^{\mathsf{St}}_{\preccurlyeq}$-cluster of $\mathsf{R}^{\mathsf{St}}_{\mathsf{e}}$-clusters. We consider, now, the linear part of $\mathcal{M}_2$ up to the stabilising cluster $\mathcal{C}_s$ and we define a subframe $\mathcal{F}_l \sqsubseteq \mathcal{F}_2$, $\mathcal{F}_l := \langle W_l, \mathsf{R}^l_{\preccurlyeq}, \mathsf{R}^l_{\mathsf{e}}, \mathsf{R}^l_{\mathsf{a}} \rangle$, where $W_{\mathcal{F}_l} := W_{\mathcal{F}_2} - \bigcup \mathcal{C}_s^{\preccurlyeq}$. The $\mathcal{LTK}$-frame $\mathcal{F}_3 := \langle W_{\mathcal{F}_3}, \mathsf{R}^3_{\preccurlyeq}, \mathsf{R}^3_{\mathsf{e}}, \mathsf{R}^3_1, \dots, \mathsf{R}^3_k \rangle$ has the following structure (see Figure 2.2):

(i) $W_{\mathcal{F}_3} := W_{\mathsf{St}} \cup W_{\mathcal{F}_l}$

(ii) $\mathsf{R}^3_{\preccurlyeq} := \mathsf{R}^{\mathsf{St}}_{\preccurlyeq} \cup \mathsf{R}^l_{\preccurlyeq} \cup \{\langle w, z\rangle \mid w \in W_{\mathcal{F}_l} \;\&\; z \in W_{\mathsf{St}}\}$

(iii) $\mathsf{R}^3_{\mathsf{e}} := \mathsf{R}^{\mathsf{St}}_{\mathsf{e}} \cup \mathsf{R}^l_{\mathsf{e}}$

(iv) for $1 \leq \mathsf{a} \leq k$, $\mathsf{R}^3_{\mathsf{a}} := \mathsf{R}^{\mathsf{St}}_{\mathsf{a}} \cup \mathsf{R}^l_{\mathsf{a}}$

The kind of frame formerly described, is what we call a *reflexive* LTK-*balloon*, a graphic representation of which may be found depicted in Figure 2.2. Let $\mathcal{M}_{\mathcal{F}_3} := \langle \mathcal{F}_3, V_3\rangle$ be the model in which V_3 is the restriction of V_2 on $W_{\mathcal{F}_3}$.

Lemma 2.3.10 *For any formula* $\mathtt{B} \in Sub(\mathtt{A})$*, for any world* $w \in W_{\mathcal{F}_3}$*,* $(\mathcal{F}_3, w) \Vdash_{V_3} \mathtt{B} \Leftrightarrow (\mathcal{F}_2, w) \Vdash_{V_2} \mathtt{B}$*.*

Proof. The proof can be given by induction on the length of $\mathtt{B}$. We consider only the case in which $\mathtt{B}$ is $\Box_{\preccurlyeq}\mathtt{D}$, $(\mathcal{F}_3, w) \Vdash_{V_3} \Box_{\preccurlyeq}\mathtt{D}$ and $w \in W_{\mathsf{St}}$. This means that $\mathtt{D}$ is true at all those worlds $z \in W_{\mathcal{F}_3}$ s.t. $w\mathsf{R}^3_{\preccurlyeq}z$, i.e. all the worlds belonging to W_{St} (recall that $\mathsf{R}^{\mathsf{St}}_{\preccurlyeq}$ is an equivalence relation on W_{St}). Notice that the world w belongs to $\mathcal{F}_2$ and, by construction of $\mathcal{F}_3$, $w \in \cup\mathcal{C}^{\preccurlyeq}_s$ where $\mathcal{C}_s$ is the $\mathsf{R}^2_{\preccurlyeq}$-deepest stabilising cluster in $\mathcal{F}_2$. By Inductive Hypothesis we have that $(\mathcal{F}_2, z) \Vdash_{V_2} \mathtt{D}$ for any z belonging both to $W_{\mathcal{F}_3}$ and to $W_{\mathcal{F}_2}$. Consider a world $v \in W_{\mathcal{F}_2}$ such that $v \in \mathcal{C}^{\preccurlyeq}_s$. We can have two cases: either v belongs to W_{St} or v does not. In the former case $(\mathcal{F}_2, v) \Vdash_{V_2} \mathtt{D}$ holds by Inductive Hypothesis, while in the latter, since v belongs to an $\mathsf{R}^2_{\preccurlyeq}$-cluster isomorphic to an $\mathsf{R}^{\mathsf{St}}_{\mathsf{e}}$-cluster from W_{St}, $(\mathcal{F}_2, v) \Vdash_{V_2} \mathtt{D}$ holds by Lemma 2.3.9. Therefore $(\mathcal{F}_2, w) \Vdash_{V_2} \Box_{\preccurlyeq}\mathtt{D}$. ∎

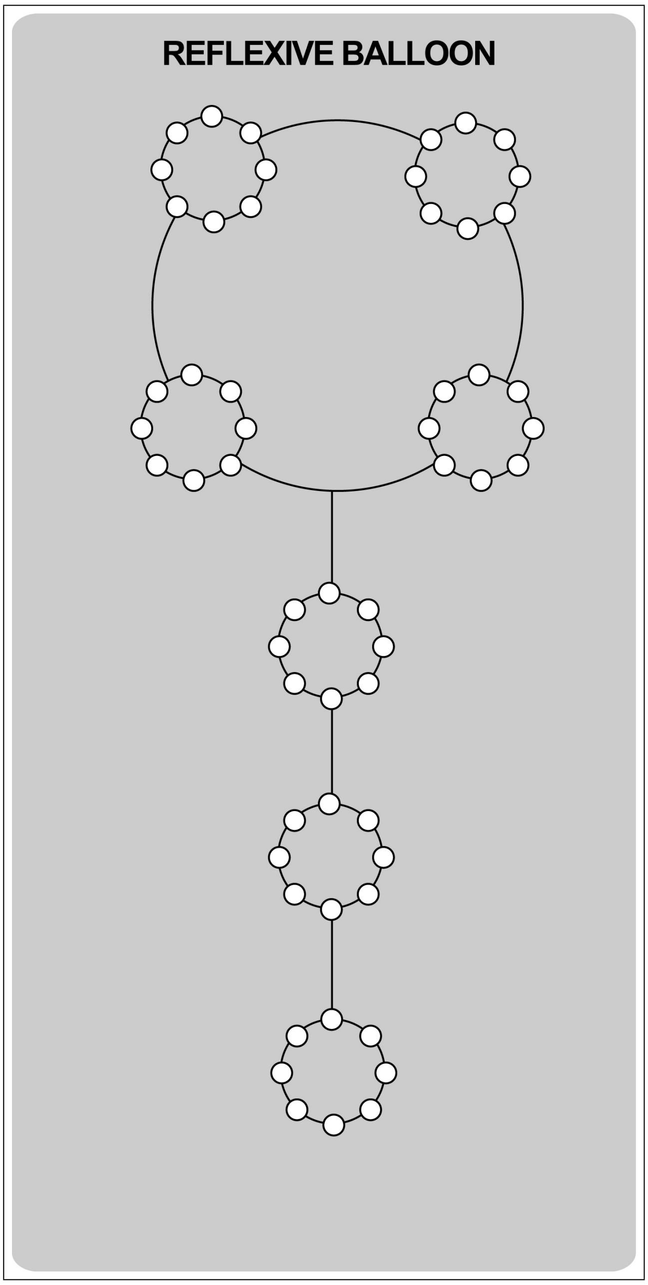

Figure 2.2: Scheme of the structure of the frame $\mathfrak{F}_3$, a case of reflexive LTK-balloon.

Step 3: Getting a finite and computable number of time-clusters
The base set of $\mathcal{M}_{\mathcal{F}_3}$ contains a finite number of worlds, but, since we do not know how many they are, we need to contract it again and we shall do so by dropping some clusters.

STEP 4. For each $\mathtt{B} \in Sub(\mathtt{A})$, we consider the $\mathsf{R}^3_{\preccurlyeq}$-maximal $\mathsf{R}^3_{\mathsf{e}}$-cluster $\mathcal{C}$ of worlds from $W_{\mathcal{F}_3}$ such that $\exists w \in \mathcal{C}$, $(\mathcal{F}_3, w) \Vdash_{V_3} \mathtt{B}$ and we denote it by $\mathcal{C}_{\mathtt{B}}$. Likewise, by $\mathcal{C}_{\neg\mathtt{B}}$ we denote the $\mathsf{R}^3_{\preccurlyeq}$-maximal $\mathsf{R}^3_{\mathsf{e}}$-cluster containing a world z refuting $\mathtt{B}$. Then we introduce a new frame $\mathcal{F}_4 := \langle W_{\mathcal{F}_4}, \mathsf{R}^4_{\preccurlyeq}, \mathsf{R}^4_{\mathsf{e}}, \mathsf{R}^4_1, \dots, \mathsf{R}^4_k \rangle$ where:

$$W_{\mathcal{F}_4} := \bigcup_{\mathtt{B} \in Sub(\mathtt{A})} \mathcal{C}_{\mathtt{B}} \cup \bigcup_{\mathtt{B} \in Sub(\mathtt{A})} \mathcal{C}_{\neg\mathtt{B}} \ \cup \ W_{\mathsf{St}}$$

and all the binary relations are the restriction of the ones from $\mathcal{F}_3$ on $W_{\mathcal{F}_4}$. Let $\mathcal{M}_4 := \langle W_{\mathcal{F}_4}, V_4 \rangle$ be a model on $\mathcal{F}_4$ where V_4 is nothing but the restriction of V_3 on $W_{\mathcal{F}_4}$. Clearly the frame $\mathcal{F}_4$ is still a case of reflexive LTK-balloon.

Lemma 2.3.11 *For any formula* $\mathtt{B} \in Sub(\mathtt{A})$*, for any world* $w \in W_4$*,* $(\mathcal{F}_4, w) \Vdash_{V_4} \mathtt{B} \Leftrightarrow (\mathcal{F}_3, w) \Vdash_{V_3} \mathtt{B}$.

PROOF. We conduct an easy induction on the length of $\mathtt{B}$, and we illustrate only the case $\mathtt{B}$ is $\Box_{\preccurlyeq}\mathtt{D}$, $(\mathcal{F}_4, w) \Vdash_{V_4} \Box_{\preccurlyeq}\mathtt{D}$ and $w \notin W_{\mathsf{St}}$. Suppose $(\mathcal{F}_3, w) \nVdash_{V_3} \Box_{\preccurlyeq}\mathtt{D}$. Then there is a world $z \in W_{\mathcal{F}_3}$ such that $w\mathsf{R}^3_{\preccurlyeq}z$, $(\mathcal{F}_3, z) \nVdash_{V_3} \mathtt{D}$ and $z \notin W_{\mathcal{F}_4}$. By construction of $\mathcal{M}_{\mathcal{F}_4}$, there must be an $\mathsf{R}^4_{\preccurlyeq}$-maximal $\mathsf{R}^4_{\mathsf{e}}$-cluster $\mathcal{C}_{\neg\mathtt{D}}$ in $\mathcal{F}_4$ such that there exists a world $v \in \mathcal{C}_{\neg\mathtt{D}}$, $(\mathcal{F}_4, v) \nVdash_{V_4} \mathtt{D}$. Since $\mathcal{C}_{\neg\mathtt{D}}$ is $\mathsf{R}^4_{\preccurlyeq}$-maximal, we also have $w\mathsf{R}^4_{\preccurlyeq}v$. This is a contradiction, hence $(\mathcal{F}_3, w) \Vdash_{V_3} \Box_{\preccurlyeq}\mathtt{D}$. ∎

Now the number of worlds from $W_{\mathcal{F}_4}$ is finite and it is $f(\|\mathtt{A}\|)$, where f

is a computable function and $f(\|\mathtt{A}\|) \leq \left(2^{\|Sub(\mathtt{A})\|}(2\|Sub(\mathtt{A})\| + 2^{2^{\|Sub(\mathtt{A})\|}})\right)$.

Step 4: Defining a p-morphic image. Our final step is to show that $\mathcal{F}_4$ is the p-morphic image of an $\mathcal{LTK}$-frame and hence $\forall \mathtt{B} \in \mathsf{LTK}$, $\mathcal{F}_4 \Vdash \mathtt{B}$. After doing so, we shall be able to show a more general result, namely that any reflexive LTK-balloon may be *unraveled* in order to get an $\mathcal{LTK}$-frame which is a *p-morphic* image of the original frame. Let us start by defining a *p-morphism* (*pseudo-epimorphism*):

Definition 2.3.12 *Let f be a mapping of a frame $\mathcal{S}_1 := \langle S_1, \mathsf{R}_1\rangle$ into a frame $\mathcal{S}_2 := \langle S_2, \mathsf{R}_2\rangle$. The mapping f is called a* p-morphism *if:*

(i) $\forall w, v \in S_1(w\mathsf{R}_1 v \Rightarrow f(w)\mathsf{R}_2 f(v))$

(ii) $\forall w, v \in S_1(f(w)\mathsf{R}_2 f(v) \Rightarrow \exists t \in S_1(w\mathsf{R}_1 t \;\&\; f(t) = f(v)))$

The frame $\mathcal{S}_2$ is also said to be a p-morphic image *of $\mathcal{S}_1$.*

Definition 2.3.13 *Given two models $\mathcal{M}_1 := \langle \mathcal{S}_1, V_1\rangle$ and $\mathcal{M}_2 := \langle \mathcal{S}_2, V_2\rangle$, a mapping f is a* p-morphism *of $\mathcal{M}_1$ into $\mathcal{M}_2$ if and only if:*

(i) f is a p-morphism of $\mathcal{S}_1$ into $\mathcal{S}_2$;

(ii) the valuations V_1 and V_2 share the same domain;

(iii) $\forall p \in Dom(V_1), \forall w \in W_1(w \Vdash_{V_1} p \Leftrightarrow f(w) \Vdash_{V_2} p)$.

Theorem 2.3.14 *If f is a p-morphism of a Kripke-model $\mathcal{M}_1 := \langle W_1, \mathsf{R}_1, V_1\rangle$ onto a Kripke-model $\mathcal{M}_2 := \langle W_2, \mathsf{R}_2, V_2\rangle$, then for any formula $\mathtt{A}$ which is built up on letters from the domain of the valuation V_1, $\forall w \in W_1(w \Vdash_{V_1} \mathtt{A} \Leftrightarrow f(w) \Vdash_{V_2} \mathtt{A})$.*

Let $\mathcal{C}^{\mathsf{St}}_1, \ldots, \mathcal{C}^{\mathsf{St}}_i$ be an enumeration of all the $\mathsf{R}^4_{\mathsf{e}}$-clusters of worlds from W_{St} and let $\mathcal{F}_5 := \langle W_{\mathcal{F}_5}, \mathsf{R}^5_{\preccurlyeq}, \mathsf{R}^5_{\mathsf{e}}, \mathsf{R}^5_1, \ldots, \mathsf{R}^5_k\rangle$ be a frame such that:

(i) $W_{\mathcal{F}_5} := \bigcup_{1\leq j\leq i} \mathcal{C}^{\mathsf{St}}_j$

(ii) $\forall w \forall z \in W_{\mathcal{F}_5}\ (w\mathsf{R}^5_{\preccurlyeq} z \Leftrightarrow (w \in \mathcal{C}^{\mathsf{St}}_j\ \&\ z \in \mathcal{C}^{\mathsf{St}}_k\ \&\ j \leq k))$

(iii) $\forall w \forall z \in W_{\mathcal{F}_5}\ (w\mathsf{R}^5_{\mathsf{e}} z \quad \Leftrightarrow \quad w\mathsf{R}^4_{\mathsf{e}} z)$

(iv) for $1 \leq \mathsf{a} \leq k$, $\forall w \forall z \in W_{\mathcal{F}_5}\ (w\mathsf{R}^5_{\mathsf{a}} z \quad \Leftrightarrow \quad w\mathsf{R}^4_{\mathsf{a}} z)$

Let $\mathcal{F}_\infty = \langle W_{\mathcal{F}_\infty}, \mathsf{R}^\infty_{\preccurlyeq}, \mathsf{R}^\infty_{\mathsf{e}}, \mathsf{R}^\infty_1, \ldots, \mathsf{R}^\infty_k \rangle$ be an $\mathcal{LTK}$-frame consisting of an infinite repetition of $\mathcal{F}_5$ and let $\mathcal{F}_6 = \langle W_{\mathcal{F}_6}, \mathsf{R}^6_{\preccurlyeq}, \mathsf{R}^6_{\mathsf{e}}, \mathsf{R}^6_1, \ldots, \mathsf{R}^6_k \rangle$ be a subframe of $\mathcal{F}_4$ such that $W_{\mathcal{F}_6} = W_{\mathcal{F}_4} - \bigcup \mathcal{C}^<_s$ (recall that $\mathcal{C}_s$ is the stabilising cluster of $\mathcal{F}_4$). Let $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$ be an $\mathcal{LTK}$-frame such that:

(i) $W = W_{\mathcal{F}_\infty} \cup W_6$

(ii) $\mathsf{R}_{\preccurlyeq} = \mathsf{R}^\infty_{\preccurlyeq} \cup \mathsf{R}^6_{\preccurlyeq} \cup \{\langle w, z \rangle \mid w \in W_{\mathcal{F}_6}\ \&\ z \in W_{\mathcal{F}_\infty}\}$

(iii) $\mathsf{R}_{\mathsf{e}} = \mathsf{R}^\infty_{\mathsf{e}} \cup \mathsf{R}^6_{\mathsf{e}}$

(iv) for $1 \leq \mathsf{a} \leq k$, $\mathsf{R}_{\mathsf{a}} = \mathsf{R}^\infty_{\mathsf{a}} \cup \mathsf{R}^6_{\mathsf{a}}$.

It is easy to see that $\mathcal{F}_4$ is a p-morphic image of $\mathcal{F}$. ∎

Notice that in this proof we have examined only the general case in which the formula `A` is not valid in an *infinite* $\mathcal{LTK}$-frame. If such frame is a *finite* one, we do not need to go through steps 2, 3 and 5.

Corollary 2.3.15 *Any reflexive* LTK*-balloon has a p-morphic image which is an infinite* LTK*-frame.*

We can then conclude that for any formula `A`, if `A` is not a theorem of LTK, then there is a finite reflexive LTK-balloon which refutes `A` and whose size is computable and bounded by `A`.

2.4 Kripke Semantics: On the Ontology of Possible Worlds

Attempts to give a semantical explanation of quantified modal logics start around the mid Forties with the work by Rudolf Carnap [12]. During the Fifties, new semantics for modalities in the predicative case are developed by Kanger [40], Montague [45] and Kripke, but it is only during the Sixties that such theories become systematic and general, thank to the work by Jakko Hintikka [33] and Saul Kripke.

Kripke's work in 1963 [42] is the most influential piece of research in the field. It presents a general semantics for quantified modal logics which extends the Tarskian semantics for classical logic. This is one of the main reasons of its great success: it allows us to deal with modal logics with the same techniques developed in model theory applied to extensional logics (cf. Corsi [13, 14]).

The key concept in this new semantical approach is the one of *possible worlds*. Although in computational and multi-agent approaches, possible worlds are generally considered only as descriptive tools to talk about precise states of a machine or social situations, there is, nevertheless, a debate which is going on in the philosophical community on the ontological status to be given to possible worlds. The ontological status of such objects is, in fact, quite controversial. Although we are little concerned with philosophical and metaphysical issues in this dissertation, we have nevertheless decided to provide the reader with a very brief introduction to the subject, just to give the flavour of the problems related to modal logic in other fields. Logic is, in fact, an intrinsically multi-disciplinary subject and being so it can be analysed using the tools of various disciplines together (mathematics, computer science, philosophy).

We would like to present two different positions about the ontological status of possible worlds. First we shall see Kripke's and then Plantinga's. This will show the core of the discussion on such matters and it will also give a deeper understanding of some of the expressions one may find, or has already found, in our dissertation.

As a start, we shall point out one of the most famous objections to Possible Worlds Semantics. The point is that whenever we want to interpret the *diamond* modal operator $\Diamond$ (in its historical acception as meaning possibility) in the possible worlds framework, we should read an expression as $\Diamond p$ as *there is a possible world in which p holds true* and there is, therefore, an existential quantification over possible worlds. There is no problem as long as we aim at talking about actual situations. The scenario changes whenever we aim at talking about a situation which is merely possible. In this case we are forced to accept the existence of possible objects which are not actual: we should accept in our ontology not only *real* things, but *possibilia* as well.

Melvin Fitting [18] underlines that the same situation happens in the semantics for Classical Propositional Calculus: we take under consideration, in fact, all the possibilities, i.e. all the lines of a truth table, and such possibilities cannot happen simultaneously.

Kripke's answer is that the problem comes from a wrong use of the term *possible world*: if we clarify this notion giving it a fixed and formal meaning, one realises that there is nothing wrong in considering possible worlds for philosophical or technical purposes as abstract entities (cf Kripke [43]). Possible worlds should not be considered as far away planets which come into existence in some other dimension. In order to avoid this kind of confusion, Kripke states that it would be better to change terminology: instead of speaking of possible worlds, it would be better to talk about *possible states or stories of the world*.

In order to understand his acception of possible worlds, Kripke asks us to consider two common six faced dice: dice A and dice B. If we cast them, we would get 36 possible combinations for the couple: for each dice there are six possible outputs. These 36 combinations are literally 36 possible worlds: we completely ignore the rest of the world but the two dice and the faces they show. Only one of these 36 worlds is the actual one, but it is interesting, however, to consider also the others whenever we ask counterfactual questions. When we talk about the 36 possible combinations there is no need to assume the existence of other 35 entities in some other dimension which correspond to the physical object we face. A possible world is not a far away country we visit or see through a telescope. A possible world is given by its descriptive conditions. Possible worlds are stipulated, not found (cf. Kripke [43]). Kripke provides us with this definition of possible worlds in order to correct a frequent philosophical mistake which is caused by a wrong use of the terms involved. It is a classical problem of quantified modal logics and modal logics in general.

Alvin Plantinga [47, 46] has a similar position concerning the ontological status of possible worlds. He defines them as *states of affairs possible and maximal*. He thinks, moreover, that talking about possible worlds is *necessary* and not a mere speculation. There are, in fact, linguistic ambiguities which can be *cured* and understood only if we refer to the possible worlds semantics.

In order to understand Plantinga's view point, it is necessary to introduce the *de dicto* versus *de re* distinction, which is typical of the quantified approach to modal logic.

A modal sentence is a *de dicto* modality whenever a modal property is associated to a *dictum* or sentence, as in the phrase *it is necessary that all men are mortal*, where the necessity operator is applied to the sentence *all men are mortal*. On the other hand we call a sentence a *de re* modality if the modal property is given to an object, as in the phrase *all men are necessarily mortal*, in which the property *being necessarily mortal* is applied to all mankind. It is clear that such a distinction is lost whenever we lose the expressive power of predicate logics in order to analyse the case of propositional calculus. A classical example to explain the necessity of possible worlds is provided by Thomas Aquinas[3]. In his *Summa contra Gentiles* Thomas Aquinas considers the problem of God's pre-knowledge. God can, according to the philosopher, see the action which is taking place. This is coherent with human freedom. In fact consider the truth value of the following sentence:

(1) *If I see someone sitting, he is necessarily sitting.*

This is clearly true if read in the *de dicto* way:

[3] Thomas Aquinas, *Summa de veritate catholicae fidei contra Gentiles* [1259-1264], Roma: edizione Leonina, 1918-1930, voll. XIII-XV.

(2) *It is necessary that if I see someone sitting, that person is sitting.*

which is to say:

(2*) *In every possible world if I see someone sitting, that person is sitting.*

The sentence nevertheless ceases to hold true as soon as we apply the *de re* reading:

(3) *If I see someone sitting, such person has the necessary property of being sat*

i.e. (3*) *If I see someone sitting, in every possible world that person is sitting*
which is clearly false.

It would not be possible, according to Plantinga, to understand such a distinction if we cease to use the possible worlds framework. Just another example to understand such distinction is provided by Fitting[4]. Consider the sentence *The number of planets is necessarily odd.* A *de re* reading would suggest that the number of planets in the solar system is odd in every possible world. Any person without radically deterministic philosophical views would then disagree with it being true. On the other hand its *de re* interpretation proves to be true: *in every possible world it is true that in the actual world the number of planets is odd.*

[4] Fitting [18], p. 86

Leaving planets and men sitting necessarily or not and moving to something more useful in our everyday life, Thomason [67] shows how such a distinction may help in remove the ambiguity in some english expression as *any* and *some*. Consider the following couple of sentences:

a. *Everyone* can come along with us.

b. *Anyone* can come along with us.

In fact the sentence a. could be read as *It is possible that all come with us*, i.e. $\Diamond\forall x Come(x, us)$, whereas b. would be *All can possibly come with us*, i.e. $\forall x \Diamond Come(x, us)$. As soon as we formalise them, we realise how the syntactic difference of the two is actually linked to a different scope of the universal quantifier and this can help in understanding the difference in the use of *any* and *some* in the english language.

If we consider the modal operators as quantifiers over possible worlds, the distinction we are talking about becomes the problem of the swap of the two types of quantifiers.

Going back to Plantinga's position, we have stated that he accepts the necessity of the use of the possible worlds. The problems start when he also states to be an *actualist*, someone who does not accept anything which is merely possible. Possible worlds must, therefore, exist in some way. But which way? A possible world is according to Plantinga an *abstract object* which is capable of action and causal relations. For instance *being for Socrates shorter than Plato* is a state of affairs and it is actual only if Socrates is actually shorter than Plato. On the other hand *being for Socrates able to play the violin* is a state of affairs which *exists* but is not actual. We find this last statement a contradiction with Plantinga's actualism.

From our point of view, we find hard to believe in the actual existence of possible worlds and whenever we shall use such expression in what follows we shall refer to its mathematical acception, i.e. as an object in a model theoretical structure. We shall not make any ontological assumption, although we are prone to consider possible worlds as the interpretation provided by Kripke: we believe them to be mere counterfactual situations which are useful to describe some aspects of the world.

Chapter 3

Admissible Rules in LTK_1: Decidability

We shall now start the analysis of one of the main topics of our dissertaion, i.e. the investigation of inference rules in the logic we formerly defined. An inference rule, or a *logical consequence* or else just an *argument*, is a set of formulae called the *premisses* of the argument followed by a formula called the *conclusion*. It is usually displayed as:

$$\frac{\begin{array}{c}\mathtt{A}_1\\ \vdots \\ \mathtt{A}_n\end{array}}{\mathtt{B}}$$

The premisses $\mathtt{A}_1, \ldots, \mathtt{A}_n$ are separated from the conclusion $\mathtt{B}$ by a line, indicating that between the two sets of formulae there is some sort of connection. This link is the logical entailment: a rule can be read as *given the premisses* $\mathtt{A}_1, \ldots, \mathtt{A}_n$*, the conclusion* $\mathtt{B}$ *may be inferred.* Clearly this does not hold true for every rule in every system. The study of the truth of this type of sentence referring to rules is the core of the research concerning logical consequences. In particular, one may be interested in finding out whether a given rule is *correct* for some logic, which is to say if its conclusion *must* hold true whenever its premisses do so. If such relation between premisses and conclusion holds for a logic, the rule is said to be *valid* or *admissible* for the logic itself. One can say that if a rule is valid for a logic, the truth of the premisses is *transfered* to its conclusion[1]. In the light of what we have just said, one can state that in order for a rule to be valid for a logic, it is necessary to fulfill the following two conditions:

(i) If the premisses are theorems, so is the conclusion;

(ii) The validity of the conclusion depends only on the *logical structure* of the premisses .

It is well known, in fact, that there might be arguments with *true* premisses and a *true* conclusion which are, nevertheless, not valid. In order to catch the intuition behind these concepts, let us consider few examples in a natural language taken from common situations. Consider the following argument:

Some women are philosophers
Some philosophers are lecturers
Therefore some lecturers are women.

In this example although both premisses and conclusion are true in the model represented by England in 2008, the argument is not valid. On the other hand, we can also have valid arguments with false conclusions:

All men are honest
All politicians are men

[1] cf Bellissima and Pagli [1].

Therefore all politicians are honest[2].

In spite of the falsity of its conclusion the argument is, nevertheless, valid. It is its logical skeleton which matters here, not the content of what it is said. This argument stays true no matter what terms we (uniformely) substitue to it.

But one could still argue why it is so important to find valid arguments. Being able to recognise admissible inference rules is actually a very useful tool at this stage of our research. Rules are, in fact, the *dynamic engine* of a logic. Theorems are static in a way: one can find a formula which is actually a theorem in a given logic and then the game is over, for nothing new can be generated by this information. On the other hand, whenever a rule is checked and recognised as admissible, it is immediately available to us in order to find other new theorems. We have not defined our logic syntactically yet. What we have at this point is just a decidable set of formulae. Finding admissible rules is, therefore, of crucial importance.

In this chapter we carry out the work started in Chapter 2 concerning decidability. In Chapter 2, in fact, we found an algorithm to check whether a given formula is a theorem of the logic LTK or not. In the same spirit the task we have now is to build a new algorithm whose goal is to check whether a given rule in the language of LTK_1 is admissible (or valid) for the logic LTK_1[3]. The logic LTK_1 is nothing but a particular case of LTK, namely a system in which only one agent is operating on a temporal framework. Hence, a Kripke-frame for this logic is a tuple of type $\langle W, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}} \rangle$ where

[2]This is a typical example of a *barbara* syllogism, according to the Mediaeval Aristotelian terminology, in which both premisses and conclusion are in affermative universal form.

[3]it is clear how this result can be easily extended to the case of ${\mathsf{LTK}_1}^-$ and ${\mathsf{LTK}_1}^{--}$.

the relation R_a is intended to represent all the worlds visible from the agent's point of view.

Scheme and Methodology. The results presented in this Chapter may be found in C. [4]. The techniques employed here are largely taken by Rybakov's methods as presented in [55].

In Section 3.1 we introduce some basic semantic definitions and operations on Kripke-frames. This is propaedeutical to the construction which follows. We define special Kripke models called *n-characterising*. This construction will prove to be the core of the whole chapter. After doing so, we prove that the model defined *is* n-characterising for LTK_1.

In Section 3.2 we prove several technical lemmas, in particular we show that our n-characterising model has a very interesting property: for each world in its base set there is a formula which is true at that world and only at that world. In other words, each point in this model is definable. This is done by means of Jankov Formulae, i.e. special formulae on $\mathcal{L}^{\mathsf{LTK}}$ intended to be true at a single world in the model we work with. After introducing these formulae, we show that the definition is a correct one and therefore each world in the model is definable.

Finally, in Section 3.3, we present the main results. We show that an inference rule $\mathbf{r}$ is admissible in LTK_1 if and only if it is valid in all the frames of a special kind, whose size is computable and bounded by the size of $\mathbf{r}$. As it will be clear later, these frames are a variant of the *reflexive balloon* as described in Chapter 2. Hence, we prove that LTK_1 is decidable w.r.t. inference rules.

3.1 Construction of $Ch_{\mathsf{LTK}_1}(n)$

In this section we shall construct special countable n-characterizing models for the logic LTK_1 (see Definition 3.1.1) based on the tecniques presented by Rybakov [55]. This construction is the ground on which we shall base our main result. The intuition behind it is to construct a model in which only and all the theorems of LTK_1 built up on a finite number n of propositional letters hold true. More technically:

Definition 3.1.1 (n-Characterising Model) *Given a logic L, a Kripke-model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ is an n-characterizing model for L iff:*

(i) $\mathsf{Dom}(V) := \{p_1, \ldots, p_n\}$

(ii) for any formula $\mathtt{A}$ *built up from* $\mathsf{Dom}(V)$*,* $\mathcal{F} \Vdash_V \mathtt{A} \;\Leftrightarrow\; \mathtt{A} \in \mathsf{L}$*.*

Let us introduce few definitions and basic operations on Kripke-frames and models. Given an $\mathcal{LTK}$-frame $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}} \rangle$, a world w (or an $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}$) from $W_{\mathcal{F}}$ has $\mathsf{R}_{\preccurlyeq}$-depth n, in symbols $depth_{\mathsf{R}_{\preccurlyeq}}(w) = n$, if the number of $\mathsf{R}_{\preccurlyeq}$-clusters in $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)^{\preccurlyeq}$ is n (in what follows, we shall always use the expression *depth* instead of $\mathsf{R}_{\preccurlyeq}$-*depth* or $depth_{\mathsf{R}_{\preccurlyeq}}$). The expression $Sl_n(\mathcal{F})$ denotes the n-slice of $\mathcal{F}$, i.e. the family of all the elements of depth n from $W_{\mathcal{F}}$. $S_n(\mathcal{F})$ is the set of all the elements from $W_{\mathcal{F}}$ with depth at most n. Given a model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ and a world $w \in W_{\mathcal{F}}$, by $Val_V(w)$ we shall denote the set $\{p_i \mid w \Vdash_V p_i\}$. For any valuation V, $\mathsf{Dom}(V)$ denotes the domain of V.

A very well known truth preserving operation on Kripke-structures and models in the *disjoint union* of different frames. Since we shall use it in the following construction, we define it as follows:

Definition 3.1.2 (Disjoint Union) *Let $\mathcal{F}_i = \langle W_{\mathcal{F}_i}, \mathsf{R}^i_1, \ldots, \mathsf{R}^i_k \rangle$, for $i \in I$, be a family of k-modal Kripke-frames with pairwise disjoint base sets, i.e.*

$W_{\mathcal{F}_i} \cap W_{\mathcal{F}_j} = \emptyset$ for each $i, j \in I$. The disjoint union of $\mathcal{F}_i$ is the frame:

$$\bigsqcup_{i \in I} \mathcal{F}_i = \langle \bigcup_{i \in I} W_{\mathcal{F}_i}, \bigcup_{i \in I} \mathsf{R}_1^i, \ldots, \bigcup_{i \in I} \mathsf{R}_k^i \rangle$$

Given a family $\mathcal{M}_i = \langle \mathcal{F}_i, V_i \rangle$ of Kripke-models on the family of frames $\mathcal{F}_i$, the disjoint union of $\mathcal{M}_i$ is the model:

$$\bigsqcup_{i \in I} \mathcal{M}_i = \langle \bigsqcup_{i \in I} \mathcal{F}_i, \bigcup_{i \in I} V_i \rangle$$

Theorem 3.1.3 *(i) Let $\bigsqcup_{i \in I} \mathcal{F}_i = \langle \bigcup_{i \in I} W_{\mathcal{F}_i}, \bigcup_{i \in I} \mathsf{R}_1^i, \ldots, \bigcup_{i \in I} \mathsf{R}_k^i \rangle$ be the disjoint union of some Kripke-frames $\mathcal{F}_i, i \in I$. Then for any formula $\mathtt{A}$ $(\bigsqcup_{i \in I} \mathcal{F}_i \Vdash \mathtt{A} \Leftrightarrow \forall i \in I (\mathcal{F}_i \Vdash \mathtt{A}))$;*

(i) Let $\bigsqcup_{i \in I} \mathcal{M}_i = \langle \bigsqcup_{i \in I} \mathcal{F}_i, V \rangle$ be the disjoint union of models on a family of frames $\mathcal{F}_i$, $i \in I$. Then for any formula $\mathtt{A}$ built up on propositional letters from V, $(\bigsqcup_{i \in I} \mathcal{F}_i \Vdash_V \mathtt{A} \Leftrightarrow \forall i \in I (\mathcal{F}_i \Vdash_{V_i} \mathtt{A}))$.

We are now ready to start our construction of an effective n-characterising model for $\mathsf{LTK_1}$.

Step 1: the first slice of $Ch_{\mathsf{LTK_1}}(n)$. Let $\mathbb{F}$ be a class of finite $\mathcal{LTK}$-frames (i.e. $\mathcal{LTK}$-frames whose base sets are finite) such that, for any frame $\mathcal{F} \in \mathbb{F}$, $\forall w \forall z \in W_{\mathcal{F}} (w \mathsf{R}_{\preccurlyeq} z \ \& \ w \mathsf{R}_{\mathsf{e}} z)$. Let $\mathbb{C}(\mathbb{F})_n$ be the class of all the possible different, non isomorphic models $\mathcal{C} := \langle \mathcal{F}, V \rangle$, where:

(i) $\mathcal{F} \in \mathbb{F}$;

(ii) $\mathsf{Dom}(V) = \{p_1, \ldots, p_n\}$;

(iii) $\forall w \forall z \in W_{\mathcal{F}} \Big(\big((Val_V(w) = Val_V(z)) \& (\{Val_V(w') \mid w \mathsf{R}_{\mathsf{a}} w'\} = \{Val_V(z') \mid z \mathsf{R}_{\mathsf{a}} z'\}) \big) \Rightarrow (w = z) \Big)$.

It is easy to notice that the size of $\mathbb{C}(\mathbb{F})_n$ is computable and bounded by n.

Let $\mathcal{P}(\mathbb{C}(\mathbb{F})_n)$ be the set of all the subsets of $\mathbb{C}(\mathbb{F})_n$.

Given a set $\mathbb{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_j\}$ from $\mathcal{P}(\mathbb{C}(\mathbb{F})_n)$, for each $\mathcal{C}_i \in \mathbb{S}$, we display the model $\mathcal{C}_i$ as $\mathcal{C}_i := \langle W_i, \mathsf{R}^i_{\preccurlyeq}, \mathsf{R}^i_{\mathsf{e}}, \mathsf{R}^i_{\mathsf{a}}, V_i\rangle$.

For any set $\mathbb{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_j\}$ from $\mathcal{P}(\mathbb{C}(\mathbb{F})_n)$, $\mathcal{T}_{\mathbb{S}}$ is the Kripke-model $\mathcal{T}_{\mathbb{S}} := \langle W_{\mathbb{S}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}, V\rangle$, where:

(i) $W_{\mathbb{S}} := \bigcup_{1\leq i\leq j} W_i$

(ii) $\mathsf{R}_{\preccurlyeq} := W_{\mathbb{S}} \times W_{\mathbb{S}}$

(iii) $\mathsf{R}_{\mathsf{e}} := \bigcup_{1\leq i\leq j} \mathsf{R}^i_{\mathsf{e}}$

(iv) $\mathsf{R}_{\mathsf{a}} := \bigcup_{1\leq i\leq j} \mathsf{R}^i_{\mathsf{a}}$

(v) $\mathsf{Dom}(V) := \{p_1, \ldots, p_n\}$

(vi) $\forall p \in \mathsf{Dom}(V)(V(p) := \bigcup_{1\leq i\leq j} V_i(p))$

Since the temporal relation $\mathsf{R}_{\preccurlyeq}$ is universal, each $\mathcal{T}_{\mathbb{S}}$ is an $\mathsf{R}_{\preccurlyeq}$-cluster of R_{e}-clusters.

Let $S_1(Ch_{\mathsf{LTK}_1}(n)) := \bigsqcup_{\mathbb{S}\in\mathcal{P}(\mathbb{C}(\mathbb{F})_n)} \mathcal{T}_{\mathbb{S}}$.

Hence the first slice contains a finite number of pairwise disjoint models, where each model is an $\mathsf{R}_{\preccurlyeq}$-cluster of R_{e}-clusters.

Step 2: the second slice. Consider any $\mathcal{T}_{\mathbb{S}}$ from $S_1(Ch_{\mathsf{LTK}_1}(n))$, and any R_{e}-cluster $\mathcal{C}_i$ from $\mathbb{C}(\mathbb{F})_n$ s.t. $\forall \mathcal{C} \in \mathcal{T}_{\mathbb{S}}$, $\mathcal{C}_i$ is not isomorphic to a submodel of $\mathcal{C}$.

For any $\mathcal{T}_{\mathbb{S}_i}$ from $S_1(Ch_{\mathsf{LTK_1}}(n))$ we adjoin a copy of each $\mathcal{C}_j \in \mathbb{C}(\mathbb{F})$ provided that $\forall \mathcal{C}_k \in \mathcal{T}_{\mathbb{S}_i}$, $\mathcal{C}_j \not\cong \mathcal{C}_k$. We set $\mathcal{C}_j$ to be an immediate $\mathsf{R}_{\preccurlyeq}$=predecessor of $\mathcal{T}_{\mathbb{S}_i}$. The resulting model is defined as $S_2(Ch_{\mathsf{LTK_1}}(n))$.

Step 3: the $i+1$th slice. Suppose we have already constructed the model $S_i(Ch_{\mathsf{LTK_1}}(n))$ for $i \geq 2$ such that its frame is a frame for $\mathsf{LTK_1}$ and given two different $\mathsf{R}_{\preccurlyeq}$-clusters $\mathcal{C}_j$, $\mathcal{C}_k$ from this frame, if $\mathcal{C}_j$ is an immediate $\mathsf{R}_{\preccurlyeq}$-predecessor of $\mathcal{C}_k$, then $\mathcal{C}_j$ is not isomorphic to a submodel of $\mathcal{C}_k$. To construct $Sl_{i+1}(Ch_{\mathsf{LTK_1}}(n))$ we add $\mathsf{R_e}$-clusters from $\mathbb{C}(\mathbb{F})_n$ in the following way. We take each $\mathsf{R_e}$-cluster $\mathcal{C}$ of depth i and we add as its immediate $\mathsf{R}_{\preccurlyeq}$-predecessors all the possible different $\mathsf{R_e}$-clusters $\mathcal{C}_j$ from $\mathbb{C}(\mathbb{F})_n$, but only provided that $\mathcal{C}_j$ is not isomorphic to a submodel of $\mathcal{C}$.
Let $S_{i+1}(Ch_{\mathsf{LTK_1}}(n))$ be the model resulting from all such additions. The frame of the resulting model is again a frame for $\mathsf{LTK_1}$.

Step 4: the final model. The final model $Ch_{\mathsf{LTK_1}}(n) := \langle W_{Ch(n)}, \mathsf{R}_{\preccurlyeq}, \mathsf{R_e}, \mathsf{R_a}, V\rangle$ is given by

$$\bigcup_{i\in\mathbb{N}} S_i(Ch_{\mathsf{LTK_1}}(n))$$

We call $Ch(n)$ the frame on which $Ch_{\mathsf{LTK_1}}(n)$ is based.

Clearly what we need now is to prove that the model defined is really n-characterising for $\mathsf{LTK_1}$. To do so, we prove the following lemma:

Lemma 3.1.4 *The model $Ch_{\mathsf{LTK_1}}(n) = \langle Ch(n), V\rangle$ is n-characterizing for $\mathsf{LTK_1}$.*

PROOF. Since $Ch(n) \Vdash \mathsf{LTK_1}$ by construction, the claim $\mathtt{A} \in \mathsf{LTK_1} \Rightarrow Ch(n) \Vdash_V \mathtt{A}$, for any formula $\mathtt{A}$ built up from the propositional letters $p_1, \ldots, p_n$, follows immediately.

Suppose there is a formula $\mathtt{A}$ built up from $p_1, \ldots, p_n$ s.t. $\mathtt{A} \notin \mathsf{LTK_1}$. In order to prove that $\mathtt{A}$ is not true in $Ch_{\mathsf{LTK}_1}(n)$, we will construct a model refuting $\mathtt{A}$, which is isomorphic to an open submodel of $Ch_{\mathsf{LTK}_1}(n)$.

By Theorem 2.3.2, there is a finite $\mathsf{LTK_1}$-reflexive balloon $\mathcal{F}_1 = \langle W_{\mathcal{F}_1}, \mathsf{R}^1_{\preccurlyeq}, \mathsf{R}^1_{\mathsf{e}}, \mathsf{R}^1_{\mathsf{a}} \rangle$ (whose size is computable and bounded by the size of $\mathtt{A}$) and a model $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1 \rangle$ such that $\mathcal{F}_1 \not\Vdash_{V_1} \mathtt{A}$. For any $\mathsf{R}^1_{\mathsf{e}}$-cluster $\mathcal{C}$ from $W_{\mathcal{F}_1}$, $\forall w, z \in \cup\mathcal{C}$, if the following two conditions hold:

(i) $Val_{V_1}(w) = Val_{V_1}(z)$

(ii) $\{Val_{V_1}(w') \mid w\mathsf{R}^1_{\mathsf{a}}w'\} = \{Val_{V_1}(z') \mid z\mathsf{R}^1_{\mathsf{a}}z'\}$

then we delete either w or z. The resulting model $\mathcal{M}_2 := \langle \mathcal{F}_2, V_2 \rangle$ is a p-morphic image of $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1 \rangle$, thus it still refutes $\mathtt{A}$.

Let St_1 be the set of $\mathsf{R}^1_{\mathsf{e}}$-clusters of depth 1 from $\mathcal{F}_1$, and let St_2 be the set of $\mathsf{R}^2_{\mathsf{e}}$-clusters of depth 1 from $\mathcal{F}_2$ (cf. Section 2.3 and Figure 2.2). We delete $\mathsf{R}^2_{\mathsf{e}}$-clusters from St_2 as follows: for any $\mathcal{C}_1$, $\mathcal{C}_2$ from St_2 s.t. $\mathcal{C}_1 \neq \mathcal{C}_2$, if $\mathcal{C}_1$ is a submodel of $\mathcal{C}_2$, then we delete $\mathcal{C}_1$. Let $\mathcal{M}^*_1$ be the resulting model. Clearly, $\mathcal{M}^*_1$ is a p-morphic image of both St_1 and St_2 and, moreover, it is also isomorphic to an open submodel of $Ch_{\mathsf{LTK}_1}(n)$.

Suppose we have already constructed the model $\mathcal{M}^*_i := \langle \mathcal{F}^*_i, V^*_i \rangle$ s.t.:

(i) $\forall w \in W_{\mathcal{F}^*_i}, \quad depth(w) \leq i$

(ii) $\mathcal{M}^*_i$ is a p-morphic image of the open submodel of $\mathcal{M}_2$ generated by the set $\bigcup \mathcal{C}^{\preccurlyeq}$, where $\mathcal{C}$ is an $\mathsf{R}^2_{\preccurlyeq}$-cluster of depth i.

(iii) $\mathcal{M}^*_i$ is isomorphic to some open submodel of $Ch_{\mathsf{LTK}_1}(n)$.

The following procedure will explain how to obtain the model $\mathcal{M}^*_{i+1}$. Let

$\mathcal{C}$ be the $\mathsf{R}^*_{\precnsim}$-deepest $\mathsf{R}^*_{\precnsim}$-cluster in $\mathcal{M}^*_i$. Consider the $\mathsf{R}^2_{\precnsim}$-cluster $\mathcal{C}_{i+1}$ in $\mathcal{M}_2$ of depth $i+1$. If $\mathcal{C}_{i+1}$ is not a submodel of $\mathcal{C}$, then we adjoin $\mathcal{C}_{i+1}$ as the immediate $\mathsf{R}^*_{\precnsim}$-predecessor of $\mathcal{C}$, otherwise we do not add anything. This procedure ends when we reach the $\mathsf{R}^2_{\precnsim}$-deepest $\mathsf{R}^2_{\precnsim}$-cluster $\mathcal{C}$ in $\mathcal{M}_2$. We denote the resulting model by $\mathcal{M}^*$. Clearly, $\mathcal{M}^*$ is a p-morphic image of the original model $\mathcal{M}_1$, therefore it refutes $\mathtt{A}$. Since $\mathcal{M}^*$ is also isomorphic to some open submodel of $Ch_{\mathsf{LTK}_1}(n)$, it follows $Ch(n) \nVdash_V \mathtt{A}$. ∎

3.2 Definability of worlds

We have seen that there is a procedure to build n-characterising models for LTK_1 for any finite n. What we want to show now is that such models enjoy a very interesting property. In fact we can prove that each world in the base set of these kind of models is *definable*. This means that given any world w we are able to construct a formula such that it is true at w and *only* at w. This gives us a powerful tool to achieve our goal of proving decidability for inference rules. In fact, as we shall see in the last section of this chapter, this is a fundamental condition for our main results.

Let us first give a precise definition of definability:

Definition 3.2.1 *Given a model $\mathcal{M} = \langle \mathcal{F}, V\rangle$, a world $w \in W_{\mathcal{F}}$ is definable if and only if there is a formula $\beta(w)$ such that:*

(i) $(\mathcal{F}, w) \Vdash_V \beta(w)$;

(i) $\forall z \in W_{\mathcal{F}}\ ((\mathcal{F}, z) \Vdash_V \beta(w) \quad \Rightarrow \quad (w = z))$.

In other words, we say a world w definable if and only if there is a formula $\beta(w)$ whose valuation is the set containing w as its only member:

$V(\beta(w)) = \{w\}$.

In order to construct our defining formulae, we shall use the abbreviation S_i for $S_i(Ch_{\mathsf{LTK}_1}(n))$. If $depth(w) = 1$, the expression $\mathcal{T}_{\mathbb{S}}(w)$ will denote the $\mathsf{R}_{\preccurlyeq}$-circle of R_{e}-clusters to which w belongs.

Step 1: Defining a world of depth 1. We start by analysing the case $depth(w) = 1$, that is w belongs to some $\mathcal{T}_{\mathbb{S}}(w) \in S_1(Ch_{\mathsf{LTK}_1}(n))$. we shall use the following formulae:

$$\alpha(w) := \bigwedge_{w \in V(p_i)} p_i \quad \wedge \quad \bigwedge_{w \notin V(p_i)} \neg p_i$$

$$\rho_{\mathsf{a}}(w) := \bigwedge_{w \mathsf{R}_{\mathsf{a}} z} \Diamond_{\mathsf{a}} \alpha(z) \quad \wedge \quad \mathsf{K}_{\mathsf{a}} \bigvee_{w \mathsf{R}_{\mathsf{a}} z} \alpha(z)$$

$$\rho_{\mathsf{e}}(w) := \bigwedge_{z \in \mathcal{C}_{\mathsf{R}_{\mathsf{e}}}(w)} \Diamond_{\mathsf{e}} \Big(\alpha(z) \wedge \rho_{\mathsf{a}}(z)\Big) \quad \wedge \quad \mathsf{K}_{\mathsf{e}} \bigvee_{z \in \mathcal{C}_{\mathsf{R}_{\mathsf{e}}}(w)} \Big(\alpha(z) \wedge \rho_{\mathsf{a}}(z)\Big)$$

$$\rho_{\preccurlyeq}(w) := \bigwedge_{z \in \mathcal{T}_{\mathbb{S}}(w)} \Diamond_{\preccurlyeq} \Big(\alpha(z) \wedge \rho_{\mathsf{a}}(z) \wedge \rho_{\mathsf{e}}(z)\Big) \wedge \Box_{\preccurlyeq} \bigvee_{z \in \mathcal{T}_{\mathbb{S}}(w)} \Big(\alpha(z) \wedge \rho_{\mathsf{a}}(z) \wedge \rho_{\mathsf{e}}(z)\Big)$$

$$\rho_{\succcurlyeq}(w) := \bigwedge_{z \in \mathcal{T}_{\mathbb{S}}(w)} \Box_{\preccurlyeq} \Diamond_{\preccurlyeq} \Big(\alpha(z) \wedge \rho_{\mathsf{a}}(z) \wedge \rho_{\mathsf{e}}(z)\Big)$$

We set the formula $\beta(w)$ to be:

$$\beta(w) := \alpha(w) \quad \wedge \quad \rho_{\mathsf{a}}(w) \quad \wedge \quad \rho_{\mathsf{e}}(w) \quad \wedge \quad \rho_{\preccurlyeq}(w) \quad \wedge \quad \rho_{\succcurlyeq}(w) \tag{3.1}$$

The intuition behind the formulae just defined is:

(i) $\rho_{\mathsf{a}}(w)$ specifies the structure of the $\mathsf{R_a}$-cluster generated by w;

(ii) $\rho_{\mathsf{e}}(w)$ describes the $\mathsf{R_e}$-cluster generated by w;

(iii) $\rho_{\preccurlyeq}(w)$ indicates all the $\mathsf{R}_{\preccurlyeq}$-accessible worlds from w and it also specifies that they are the only ones $\mathsf{R}_{\preccurlyeq}$-seen by w;

(iv) $\rho_{\succcurlyeq}(w)$, finally, says that the $\mathsf{R}_{\preccurlyeq}$-maximal time-cluster that is $\mathsf{R}_{\preccurlyeq}$-accessible from w consists of all the $\mathsf{R_e}$-clusters from $\mathfrak{T}_{\mathbb{S}}(w)$.

Step 2: Defining worlds of depth $i+1$**.** Suppose w is an element of depth $i+1$. The formulae $\alpha(w)$, $\rho_{\mathsf{a}}(w)$ and $\rho_{\mathsf{e}}(w)$ are defined in the same way as the former case. Recall that $w^{<} := \{z \mid w\mathsf{R}_{\preccurlyeq}z \;\&\; \neg(z\mathsf{R}_{\preccurlyeq}w)\}$.

$$\gamma(i) := \bigwedge_{z\in S_i} \neg\beta(z)$$

$$\rho'_{\preccurlyeq}(w) := \bigwedge_{z\in w^{<}} \Diamond_{\preccurlyeq}\beta(z) \quad\wedge \bigwedge_{z\in S_i \& z\notin w^{<}} \neg\Diamond_{\preccurlyeq}\beta(z)$$

$$\delta(w) := \Box_{\preccurlyeq}\Big(\bigvee_{z\in w^{<}} \beta(z) \quad\vee \bigvee_{z\in \mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)} \big(\alpha(z)\wedge\rho_{\mathsf{a}}(z)\wedge\rho_{\mathsf{e}}(z)\wedge\gamma(i)\big)\Big)$$

We can now define $\beta(w)$:

$$\beta(w) := \alpha(w)\wedge\rho_{\mathsf{a}}(w)\wedge\rho_{\mathsf{e}}(w)\wedge\rho'_{\preccurlyeq}(w)\wedge\gamma(i)\wedge\delta(w) \tag{3.2}$$

The formula $\rho'(w)$ says that w $\mathsf{R}_{\preccurlyeq}$-sees a specified set of worlds from S_i, while $\gamma(i)$ avoids the case $w\in S_i$. Finally, $\delta(w)$ says that if a world z is $\mathsf{R}_{\preccurlyeq}$-seen by w, then either it belongs to the set of all the $\mathsf{R}_{\preccurlyeq}$-successors of

w, or it is in the $\mathsf{R}_{\preccurlyeq}$-cluster generated by w.

What we have to show now is that our definition is correct and each world is actually defined by a formula of the type introduced in 3.1 or 3.2.

Lemma 3.2.2 *For any n-characterising model $Ch_{\mathsf{LTK}_1}(n)$, each world w from $W_{Ch(n)}$ is definable.*

PROOF. Clearly the fact that for each world w, $w \Vdash_V \beta(w)$ follows directly from the definition of $\beta(w)$.

Let us show that the second part of Definition 3.2.1 also holds true. We have to check if for any w, z from $Ch_{\mathsf{LTK}_1}(n)$, the assumption $z \Vdash_V \beta(w)$ implies that $(w = z)$. There can be two cases:

CASE 1. Assume w has depth 1 and suppose there is a point z s.t. $z \Vdash_V \beta(w)$.

(i) If $depth(z) = 1$, then the structure of $\beta(w)$ implies that the $\mathsf{R}_{\preccurlyeq}$-open submodels generated by z and w are isomorphic, so they should coincide. Hence, by the structure of $S_1(Ch_{\mathsf{LTK}_1}(n))$, we have $w = z$.

(ii) The case $depth(z) = 2$ is impossible because $\rho_{\succcurlyeq}(w)$ is a conjunct of $\beta(w)$.

(iii) If $depth(z) > 2$, then either $z\mathsf{R}_{\preccurlyeq}w$ or $\neg(z\mathsf{R}_{\preccurlyeq}w)$. The case $z\mathsf{R}_{\preccurlyeq}w$ is impossible for the structure of $S_2(Ch_{\mathsf{LTK}_1}(n))$ (i.e. there should be an R_{e}-cluster $\mathcal{C}$ s.t. $depth(\mathcal{C}) = 2$, $\mathcal{C} \in \mathcal{C}_{\mathsf{R}_{\mathsf{e}}}(z)^{\preccurlyeq}$ and $\mathcal{C} \notin \mathcal{C}_{\mathsf{R}_{\mathsf{e}}}(w)^{\preccurlyeq}$). Since $\rho_{\succcurlyeq}(w)$ is also a conjunct of $\beta(w)$, the case $\neg(z\mathsf{R}_{\preccurlyeq}w)$ is impossible as well.

CASE 2. Assume w has depth $i+1$ and suppose there is a point z s.t. $z \Vdash_V \beta(w)$. By the structure of the conjunct $\gamma(i)$ of $\beta(w)$, we have

$depth(z) > (i+1)$. By the conjunct $\rho'_{\preccurlyeq}$ we have $\forall v \in S_i(w\mathsf{R}_{\preccurlyeq}v \Rightarrow z\mathsf{R}_{\preccurlyeq}v)$. We can have two cases:

(i) If $depth(z) = i+1$, then, by the construction of $Ch_{\mathsf{LTK}_1}(n)$, $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w) = \mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)$ and so $w = z$.

(ii) Suppose $depth(z) > i{+}1$; then either $z\mathsf{R}_{\preccurlyeq}w$ or $\neg(z\mathsf{R}_{\preccurlyeq}w)$. Assume $z\mathsf{R}_{\preccurlyeq}w$; then there are $\mathsf{R}_{\preccurlyeq}$-clusters $\mathcal{C}_1, \ldots, \mathcal{C}_m$ between $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)$ and $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ such that $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z), \mathcal{C}_1, \ldots, \mathcal{C}_m, \mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ is an $\mathsf{R}_{\preccurlyeq}$-chain of $\mathsf{R}_{\preccurlyeq}$-clusters (i.e. $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)\mathsf{R}_{\preccurlyeq}\mathcal{C}_1$, $\mathcal{C}_m\mathsf{R}_{\preccurlyeq}\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ and for each i, j $1 \leq i \leq j \leq m$, $\mathcal{C}_i\mathsf{R}_{\preccurlyeq}\mathcal{C}_j$). By the structure of the conjunct $\delta(w)$, each $\mathsf{R}_{\preccurlyeq}$-cluster from $\mathcal{C}_1, \ldots, \mathcal{C}_m$ is isomorphic to $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ and this is impossible by the construction of $Ch_{\mathsf{LTK}_1}(n)$. Assume $\neg(z\mathsf{R}_{\preccurlyeq}w)$; then there are $\mathsf{R}_{\preccurlyeq}$-clusters $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that $depth(\mathcal{C}_m) = i+1$ and $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z), \mathcal{C}_1, \ldots, \mathcal{C}_m$ is an $\mathsf{R}_{\preccurlyeq}$-chain. Again, by the structure of $\delta(w)$, each $\mathsf{R}_{\preccurlyeq}$-cluster from $\mathcal{C}_1, \ldots, \mathcal{C}_m$ is isomorphic to $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)$ and this is impossible by the construction of $Ch_{\mathsf{LTK}_1}(n)$.

∎

3.3 Decidability for LTK_1 with respect to inference rules

We have now all the tools we need in order to show the main result of this Chapter. Summarising, we know that for any finite n there is a special countable n-characterising model for LTK_1 and in this model all the worlds are defined by a finite and constructible formula. These two important properties enjoyed by our logic shall prove to be of crucial importance for our next result.

As we have anticipated, we want to show that for the case of LTK_1, admissibility of inference rules is a decidable property. Let us start by defining what inference rules are in a more formal fashion:

Definition 3.3.1 (Inference Rule) *An inference rule* $\mathbf{r}$ *is an expression of the form*

$$\mathbf{r} := \frac{\mathtt{A}_1(p_1, \ldots, p_n), \ldots, \mathtt{A}_m(p_1, \ldots, p_n)}{\mathtt{B}(p_1, \ldots, p_n)}$$

where any $\mathtt{A}_i(p_1, \ldots, p_n)$ *and* $\mathtt{B}(p_1, \ldots, p_n)$ *are wff built up from the letters* $p_1, \ldots, p_n$ *(in what follows, we shall sometimes use the expression* $\mathtt{A}_1(p_1, \ldots, p_n), \ldots, \mathtt{A}_m(p_1, \ldots, p_n)/\mathtt{B}(p_1, \ldots, p_n)$*).*

As we said, an inference rule is basically a set of premisses followed by a conclusion. Between the premisses and the conclusion, there might be certain links. For instance, it may be the case that in a class of models the conclusion holds true whenever the premisses also do so. In this case, we would call the rule *valid* or *admissible* for the logic generated by the set of models. This is to say that if the premisses of the rule are *always* theorems of a logic L, the same should be true for its conclusion. Let us analyse this concept more formally.

A substitution σ is a map which assigns a formula to each propositional variable. Given a formula $\mathtt{A}$, $\sigma(\mathtt{A})$ is the result of the application of σ to $\mathtt{A}$.

Definition 3.3.2 *Given a logic* L *and an inference rule* $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$*,* $\mathbf{r}$ *is said to be admissible for* L *if and only if for each substitution* σ*, if* $\sigma(\mathtt{A}_i) \in \mathsf{L}$ *for each* i*, then* $\sigma(\mathtt{B}) \in \mathsf{L}$*.*

Therefore, the greatest class of rules which can be implemented for a given logic, i.e. which are compatible with the set of its valid formulae, is the class of its *admissible rules*: this is the class of all those rules under which the theory itself is closed.

A concept of great importance in this scenario is the one of *definable valuation*:

Definition 3.3.3 (Definable Valuation) *Given a model $\langle \mathcal{F}, V \rangle$, a valuation V' is definable if and only if $\forall p \in \mathsf{Dom}(V')$ there is a formula α_p s.t. $V'(p) = V(\alpha_p)$.*

If we combine an n-characterising model and a definable valuation, we are immediately able to state an important result. In fact, Theorem 3.3.3 in Rybakov [55] follows immediately:

Lemma 3.3.4 *An inference rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_n/\mathtt{B}$ is not admissible for LTK_1 iff there is an n-characterising model $Ch_{\mathsf{LTK}_1}(n) := \langle Ch(n), V \rangle$ and a definable valuation V_2 s.t. $Ch(n) \Vdash_{V_2} \bigwedge_{1 \leq i \leq n} \mathtt{A}_i$ and $Ch(n) \nVdash_{V_2} \mathtt{B}$.*

In order to apply such result to our specific needs, we have to introduce some particular Kripke-frames which are a variant of the *reflexive balloon* previously defined. These are a special kind of 3-modal Kripke-frames. We denote them as LSP-frames, the acronym meaning *Loop String Point*. As their name suggests, these frames contain three different components. We can get an intuitive idea if we imagine a *reflexive balloon* with a *hanging world* before each $\mathsf{R}_{\preccurlyeq}$-cluster plus a single element $\mathsf{R}_{\preccurlyeq}$-maximal cluster. The structure of an LSP-frame is depicted in Figure 3.1.

Definition 3.3.5 (LSP-frames) *Let $\mathcal{F}_{\mathsf{L}}$, $\mathcal{F}_{\mathsf{S}}$ and $\mathcal{F}_{\mathsf{P}}$ be Kripke-frames with the following structure:*

(i) The frame $\mathcal{F}_{\mathsf{L}} = \langle W_{\mathcal{F}_{\mathsf{L}}}, \mathsf{R}^{\mathsf{L}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{L}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{L}}_{\mathsf{a}} \rangle$ (LOOP-component) is as follows: $W_{\mathcal{F}_{\mathsf{L}}}$ is a nonempty set of worlds; $\mathsf{R}^{\mathsf{L}}_{\preccurlyeq} = W_{\mathcal{F}_{\mathsf{L}}} \times W_{\mathcal{F}_{\mathsf{L}}}$; $\mathsf{R}^{\mathsf{L}}_{\mathsf{e}}$ is an equivalence relation on $W_{\mathcal{F}_{\mathsf{L}}}$; $\mathsf{R}^{\mathsf{L}}_{\mathsf{a}}$ is some equivalence relation on $\mathsf{R}^{\mathsf{L}}_{\mathsf{e}}$-clusters;

(ii) Let $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}\rangle$ *be a* finite *$\mathcal{LTK}$-frame (i.e. it is an $\mathcal{LTK}$-frame with a finite base set of worlds. See Definition 2.2.3); let* $\mathcal{C}_1, \dots, \mathcal{C}_i$ *be an enumeration of all the* $\mathsf{R}_{\preccurlyeq}$*-clusters of worlds from* $W_{\mathcal{F}}$*; let* $\mathsf{Dots} := \{w_1, \dots, w_i\}$ *be a set of worlds such that* $\forall w_j, 1 \leq j \leq i(w_j \notin W_{\mathcal{F}})$*. The frame* $\mathcal{F}_{\mathsf{S}} = \langle W_{\mathcal{F}_{\mathsf{S}}}, \mathsf{R}^{\mathsf{S}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{S}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{S}}_{\mathsf{a}}\rangle$ *(STRING-component) has the following structure:* $W_{\mathcal{F}_{\mathsf{S}}} = W_{\mathcal{F}} \cup \mathsf{Dots}$*;* $\mathsf{R}^{\mathsf{S}}_{\preccurlyeq} = \mathsf{R}_{\preccurlyeq} \cup \{\langle w_j, z\rangle \mid w_j \in \mathsf{Dots}\ \&\ z \in \cup\mathcal{C}^{\preccurlyeq}_j\} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$*;* $\mathsf{R}^{\mathsf{S}}_{\mathsf{e}} = \mathsf{R}_{\mathsf{e}} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$*;* $\mathsf{R}^{\mathsf{S}}_{\mathsf{a}} = \mathsf{R}_{\mathsf{a}} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$*.*

(iii) The frame $\mathcal{F}_{\mathsf{P}} := \langle W_{\mathcal{F}_{\mathsf{P}}}, \mathsf{R}^{\mathsf{P}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{P}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{P}}_{\mathsf{a}}\rangle$ *(POINT-component) is such that its base set contains only one world denoted by @,* $W_{\mathcal{F}_{\mathsf{P}}} := \{@\}$*, and all the binary relations on* $W_{\mathcal{F}_{\mathsf{P}}}$ *are universal.*

An LSP-frame (loop-string-point frame) is a tuple $\mathcal{F}_{\mathsf{lsp}} = \langle W_{\mathsf{lsp}}, \mathsf{R}^{\mathsf{lsp}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{lsp}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{lsp}}_{\mathsf{a}}\rangle$ *where* $W_{\mathcal{F}_{\mathsf{lsp}}} = W_{\mathcal{F}_{\mathsf{L}}} \cup W_{\mathcal{F}_{\mathsf{S}}} \cup W_{\mathcal{F}_{\mathsf{P}}}$*;* $\mathsf{R}^{\mathsf{lsp}}_{\preccurlyeq} = \mathsf{R}^{\mathsf{L}}_{\preccurlyeq} \cup \mathsf{R}^{\mathsf{S}}_{\preccurlyeq} \cup \mathsf{R}^{\mathsf{P}}_{\preccurlyeq} \cup \{\langle w, z\rangle \mid w \in W_{\mathcal{F}_{\mathsf{S}}}\ \&\ z \in W_{\mathcal{F}_{\mathsf{L}}}\}$*;* $\mathsf{R}^{\mathsf{lsp}}_{\mathsf{e}} = \mathsf{R}^{\mathsf{L}}_{\mathsf{e}} \cup \mathsf{R}^{\mathsf{S}}_{\mathsf{e}} \cup \mathsf{R}^{\mathsf{P}}_{\mathsf{e}}$*;* $\mathsf{R}^{\mathsf{lsp}}_{\mathsf{a}} = \mathsf{R}^{\mathsf{L}}_{\mathsf{a}} \cup \mathsf{R}^{\mathsf{S}}_{\mathsf{a}} \cup \mathsf{R}^{\mathsf{P}}_{\mathsf{a}}$ *(See Figure 3.1).*

As we have anticipated, LSP-frames play a central role in our proof, in fact we can prove that any inference rule which is not admissible for LTK_1 has a finite and countable counter-model based on a frame of this kind. This is to say that the class of admissible rules for LTK_1 is the class of all the inferences which are valid in all the finite LSP-frames. Since the proof of this result is quite long and technical, we have decided to break it into two theorems corresponding to the two directions of the equivalence.

Theorem 3.3.6 *If an inference rule*
$\mathbf{r} := \mathtt{A}_1, \dots, \mathtt{A}_m/\mathtt{B}$ *is not admissible for* LTK_1*, then there is a finite LSP-frame* $\mathcal{F}_{\mathsf{lsp}}$*, whose size is computable from* $\|Var(\mathbf{r})\|$ *(where* $Var(\mathbf{r})$ *is the set of all the variables occurring in* $\mathbf{r}$*), and a model* $\mathcal{M}_{\mathsf{lsp}} = \langle \mathcal{F}_{\mathsf{lsp}}, V\rangle$ *s.t.* $\mathcal{F}_{\mathsf{lsp}} \Vdash_V \bigwedge_{1\leq i\leq n} \mathtt{A}_i$ *and* $\mathcal{F}_{\mathsf{lsp}} \nVdash_V \mathtt{B}$*.*

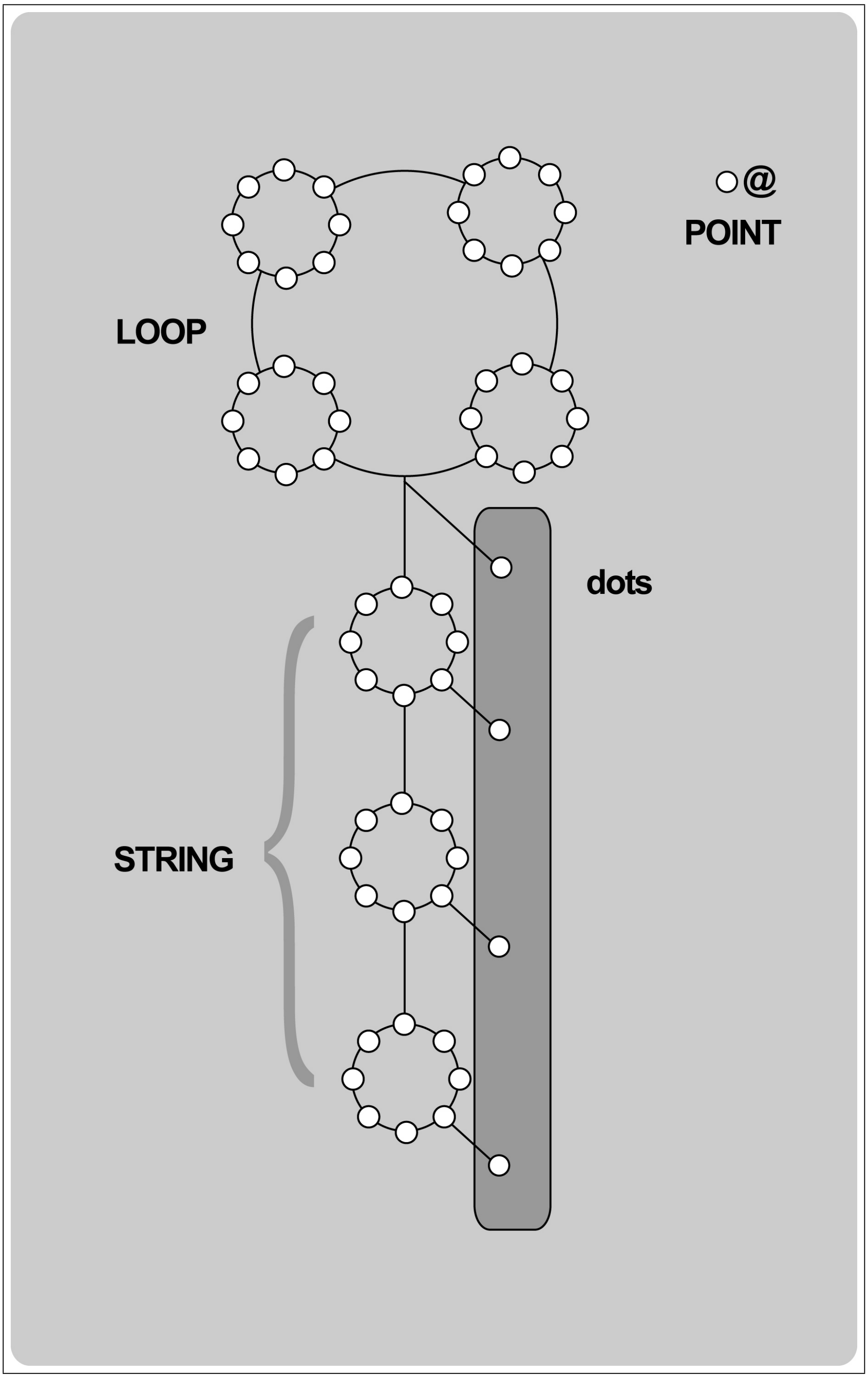

Figure 3.1: Scheme of the structure of an LSP-frame.

Proof. Let us suppose that an inference rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$ is not admissible for LTK_1 and let $p_1, \ldots, p_k$ be all the letters occurring in $\mathbf{r}$. Hence there are formulae $\gamma_1, \ldots, \gamma_j$, $1 \leq j \leq k$, s.t. $\bigwedge_{1 \leq i \leq m} \mathtt{A}_i(\gamma_1, \ldots, \gamma_j) \in \mathsf{LTK}_1$ and $\mathtt{B}(\gamma_1, \ldots, \gamma_j) \notin \mathsf{LTK}_1$. Let $Prop(\gamma)$ be the set of all the propositional letters occurring in $\gamma_1, \ldots, \gamma_j$.

By Lemma 3.3.4 there is an $n+1$-characterising model $Ch_{\mathsf{LTK}_1}(n+1) := \langle Ch(n+1), V\rangle$ and a new definable valuation V_2 with $\mathsf{Dom}(V_2) := Prop(\gamma) \cup \{p_{n+1}\}$, where $p_{n+1} \notin Prop(\gamma)$, s. t. $Ch(n+1) \Vdash_{V_2} \bigwedge_{1 \leq i \leq m} \mathtt{A}_i(\gamma_1, \ldots, \gamma_j)$ and $Ch(n+1) \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)$.

Take a world $w \in W_{Ch(n+1)}$ such that:

(i) $(Ch(n+1), w) \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)$

(ii) $\forall v \in w^{\preccurlyeq}(v \notin V_2(p_{n+1}))$

(iii) $\forall v \in W_{Ch(n+1)}(((Ch(n+1), v) \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)\ \&\ v^{\preccurlyeq} \cap V_2(p_{n+1}) = \emptyset) \Rightarrow \|w^{\preccurlyeq}\| \leq \|v^{\preccurlyeq}\|)$ (i.e. $w^{\preccurlyeq}$ is the smallest set of the kind $v^{\preccurlyeq}$ containing a world refuting $\mathtt{B}$ and such that none of its elements belongs to $V_2(p_{n+1})$).

It can be easily noticed that, since the propositional letter p_{n+1} does not occur in any γ_i, such a world w exists in $Ch_{\mathsf{LTK}_1}(n+1)$.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_i$ be an enumeration of all the $\mathsf{R}_{\preccurlyeq}$-clusters of worlds from $w^{\preccurlyeq}$. Now we take and fix, for each $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}_j$ a world w_j such that:

(i) w_j is an immediate $\mathsf{R}_{\preccurlyeq}$-predecessor of $\mathcal{C}_j$, i.e. $w_j \mathsf{R}_{\preccurlyeq} \mathcal{C}_j$, $\neg(\mathcal{C}_j \mathsf{R}_{\preccurlyeq} w_j)$ and if there is a world z s.t. $w_j \mathsf{R}_{\preccurlyeq} z$, $z \mathsf{R}_{\preccurlyeq} \mathcal{C}_j$ and $\neg(\mathcal{C}_j \mathsf{R}_{\preccurlyeq} z)$, then $z = w_j$.

(ii) for any k, $1 \leq k \leq i$, if $dp(\mathcal{C}_k) > dp(\mathcal{C}_j)$, then $\neg(w_j \mathsf{R}_{\preccurlyeq} \mathcal{C}_k)$

(iii) $w_j \in V_2(p_{n+1})$

The existence of such a world for each $\mathsf{R}_{\preccurlyeq}$-cluster is guaranteed by the construction of $Ch_{\mathsf{LTK}_1}(n+1)$. In fact, since $w_j \in V_2(p_{n+1})$ while none of the worlds from any $\mathcal{C}_j$ belongs to $V_2(p_{n+1})$, we have that for any j, $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w_j)$

is not a submodel of $\mathcal{C}_j$. Let $\mathsf{Dots} = \{w_1, \ldots, w_i\}$ be the set of those worlds w_j. Take and fix a world $@ \in W_{Ch(n+1)}$ such that:

(i) $@ \notin w^{\preccurlyeq} \cup \mathsf{Dots}$

(ii) $@^{\preccurlyeq} = \{@\}$

Let $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}}} := \langle \mathcal{F}_{\mathsf{lsp}}, V_2 \rangle$ be an open submodel of $Ch_{\mathsf{LTK}_1}(n+1)$ where $W_{\mathcal{F}_{\mathsf{lsp}}} := w^{\preccurlyeq} \cup \mathsf{Dots} \cup \{@\}$. Since $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}}}$ is a generated submodel of $Ch_{\mathsf{LTK}_1}(n+1)$, we have $\mathcal{F}_{\mathsf{lsp}} \Vdash_{V_2} \bigwedge_{1 \leq i \leq m} \mathtt{A}_i(\gamma_1, \ldots, \gamma_j)$ and $\mathcal{F}_{\mathsf{lsp}} \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)$. Moreover, by Definition 3.3.5, $\mathcal{F}_{\mathsf{lsp}}$ is an LSP-frame. Though $\mathcal{F}_{\mathsf{lsp}}$ is finite, the number of worlds from its base set is not known. To reduce such number, we apply the technique used in the proof of Theorem 2.3.2 in a slightly different way. Consider the STRING-component $\mathcal{F}_{\mathsf{S}}$ of $\mathcal{F}_{\mathsf{lsp}}$ (cf. Definition 3.3.5, item (ii)). For each $\mathtt{D} \in Sub(\mathtt{B})$, we consider the $\mathsf{R}_{\preccurlyeq}$-maximal world $v \in W_{\mathcal{F}_{\mathsf{S}}}$ such that $(\mathcal{F}_{\mathsf{lsp}}, v) \Vdash_{V_2} \mathtt{D}$. We can have two cases: either $v \in \mathsf{Dots}$ and hence $v = w_j$ for some j, or $v \in \mathcal{C}_j$ for some j. In both cases, by $\mathcal{C}_{\mathtt{D}}$ we denote the set $\bigcup \mathcal{C}_j \cup \{w_j\}$. Likewise, by $\mathcal{C}_{\neg\mathtt{D}}$ we denote the set $\bigcup \mathcal{C}_j \cup \{w_j\}$ such that there is a world $v \in \bigcup \mathcal{C}_j \cup \{w_j\}$ which is the $\mathsf{R}_{\preccurlyeq}$-maximal world refuting $\mathtt{D}$, i.e. $(\mathcal{F}_{\mathsf{lsp}}, v) \nVdash_{V_2} \mathtt{D}$. Then we define a subframe $\mathcal{F}_{\mathsf{lsp}'} := \langle W_{\mathcal{F}_{\mathsf{lsp}'}}, \mathsf{R}^{\mathsf{lsp}'}_{\preccurlyeq}, \mathsf{R}^{\mathsf{lsp}'}_{\mathsf{e}}, \mathsf{R}^{\mathsf{lsp}'}_{\mathsf{a}} \rangle$ where:

$$W_{\mathcal{F}_{\mathsf{lsp}'}} := \bigcup_{\mathtt{D} \in Sub(\mathtt{A})} \mathcal{C}_{\mathtt{D}} \cup \bigcup_{\mathtt{D} \in Sub(\mathtt{A})} \mathcal{C}_{\neg\mathtt{D}} \cup \bigcup \mathcal{C}_1 \cup w_1 \cup \{@\}$$

(recall that $\mathcal{C}_1$ is the $\mathsf{R}_{\preccurlyeq}$-maximal cluster defined in the enumeration $\mathcal{C}_1, \ldots, \mathcal{C}_i$ and w_1 is its *dot*-world).

Let $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}'}} := \langle \mathcal{F}_{\mathsf{lsp}'}, V_3 \rangle$ be a model s.t. $V_3 = V_2 \upharpoonright W_{\mathcal{F}_{\mathsf{lsp}'}}$. It is easy to verify that $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}'}}$ refutes $\mathbf{r}$ and $\mathcal{F}_{\mathsf{lsp}'}$ is an LSP-frame. Moreover, the number of worlds from $W_{\mathcal{F}_{\mathsf{lsp}'}}$ is finite and computably bounded by the size of $Var(\mathbf{r})$ (cf. item (iii) page 75).

$\blacksquare$

Theorem 3.3.7 *For any inference rule* $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$, *if there is a finite* LSP*-frame* $\mathcal{F}_{\mathsf{lsp}}$, *whose size is computable from* $\|Var(\mathbf{r})\|$ *(where* $Var(\mathbf{r})$ *is the set of all the variables occurring in* $\mathbf{r}$*), and a model* $\mathcal{M}_{\mathsf{lsp}} = \langle \mathcal{F}_{\mathsf{lsp}}, V\rangle$ *s.t.* $\mathcal{F}_{\mathsf{lsp}} \Vdash_V \bigwedge_{1\leq i\leq n} \mathtt{A}_i$ *and* $\mathcal{F}_{\mathsf{lsp}} \nVdash_V \mathtt{B}$, *then* $\mathbf{r}$ *is not admissible for* LTK_1.

Proof. Suppose that we have an inference rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$, an LSP-frame $\mathcal{F}_{\mathsf{lsp}}$ and a model $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}}} := \langle \mathcal{F}_{\mathsf{lsp}}, S\rangle$ such that $\mathcal{F}_{\mathsf{lsp}} \Vdash_S \bigwedge_{1\leq i\leq n} \mathtt{A}_i$ and $\mathcal{F}_{\mathsf{lsp}} \nVdash_S \mathtt{B}$. Let $Prop(W_{\mathcal{F}_{\mathsf{lsp}}}) := \{p_w \mid w \in W_{\mathcal{F}_{\mathsf{lsp}}}\}$ and $\mathsf{VAR} := Prop(W_{\mathcal{F}_{\mathsf{lsp}}}) \cup Var(\mathbf{r})$. We define a new valuation S_2 for $\mathcal{F}_{\mathsf{lsp}}$ in the following way:

(i) $\mathsf{Dom}(S_2) = \mathsf{VAR}$

(ii) $\forall p_w \in Prop(W_{\mathcal{F}_{\mathsf{lsp}}})\big(S_2(p_w) = \{w\}\big)$

(iii) $\forall x \in Var(\mathbf{r})\big(S_2(x) = S(x)\big)$

Clearly the new model $\langle \mathcal{F}_{\mathsf{lsp}}, S_2\rangle$ still refutes $\mathtt{B}$, but not any $\mathtt{A}_i$. We construct, following the procedure explained in Section 3.1, the model $Ch_{\mathsf{LTK}_1}(n) := \langle Ch(n), V\rangle$, where $n = \|\mathsf{VAR}\|$. It is easy to see that the model $\langle \mathcal{F}_{\mathsf{lsp}}, S_2\rangle$ formerly defined is (isomorphic to) an open submodel of $Ch_{\mathsf{LTK}_1}(n)$. we shall construct, now, a new definable valuation V_2 for $Ch(n)$ refuting $\mathbf{r}$. The basic idea is finding a way to extend the valuation S_2 from $\mathcal{F}_{\mathsf{lsp}}$ to the whole frame $Ch(n)$. Recall that by Lemma 3.2.2 we know that each world from the base set of $Ch_{\mathsf{LTK}_1}(n)$ is definable (recall that for any world w, by $\beta(w)$ we denote that particular formula defining w).

Let @ be the name of that world from $W_{\mathcal{F}_{\mathsf{lsp}}}$ such that:

(i) $@^{\preccurlyeq} = \{@\}$

(ii) $\forall w \in W_{\mathcal{F}_{\mathsf{lsp}}}\ \big((w\mathsf{R}_{\mathsf{e}}@ \ or\ w\mathsf{R}_{\preccurlyeq}@ \ or\ w\mathsf{R}_{\mathsf{a}}@) \ \Rightarrow\ w = @\big)$ (See Figure 3.2.)

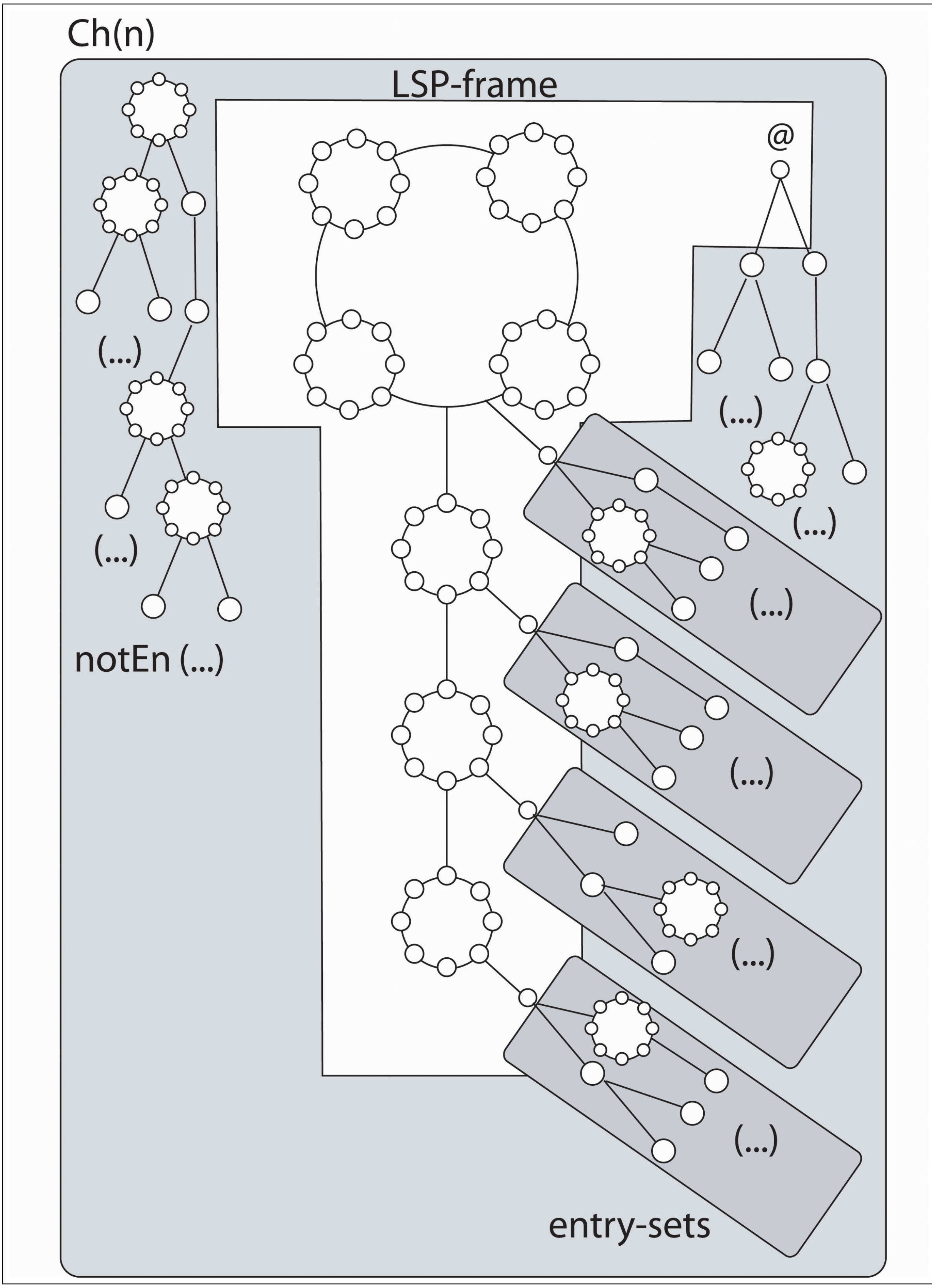

Figure 3.2: Scheme of the structure of $Ch(n)$ and the sets Entry and $\mathsf{notEntry}$.

We define the set of all those worlds from $W_{Ch(n)}$ that are not $\mathsf{R}_{\preccurlyeq}$-related to any point from $[W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}]$ (see Figure 3.2), i.e. we set
$\mathsf{notEntry} := \{v \mid v \nVdash_V \Diamond_{\preccurlyeq}\beta(w),\ \forall w \in [W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}]\}$
Let Dots be a subset of $W_{\mathcal{F}_{\mathsf{lsp}}}$ as defined in Definition 3.3.5, i.e.:
$\mathsf{Dots} = \{w_j \mid w_j \in [W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}]\ \&\ \forall z \in W_{\mathcal{F}_{\mathsf{lsp}}}\ (z\mathsf{R}_{\preccurlyeq} w_j\ \Rightarrow\ w_j = z)\}$
Take and fix, for each $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}_j$ of worlds from $[W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}] - \mathsf{Dots}$ a representative world z_j belonging to $\mathcal{C}_j$. Let Rep be the set containing all those representative elements.
For each representative world z_j from Rep, we shall define, now, an *entry-set* (see Figure 3.2). It contains all those worlds v from $W_{Ch(n)} - W_{\mathcal{F}_{\mathsf{lsp}}}$ which are $\mathsf{R}_{\preccurlyeq}$-predecessors of z_j and such that z_j is the $\mathsf{R}_{\preccurlyeq}$-deepest world belonging to $[W_{\mathcal{F}_{\mathsf{lsp}}} - \mathsf{Dots}]$ which is $\mathsf{R}_{\preccurlyeq}$-accesible from v: $\forall z_j \in \mathsf{Rep}$

$\mathsf{Entry}(z_j) := \{w \mid w \notin W_{\mathcal{F}_{\mathsf{lsp}}}\ \&\ w \Vdash_V \Diamond_{\preccurlyeq}\beta(z_j)\ \&\ \forall v \in [W_{\mathcal{F}_{\mathsf{lsp}}} - \mathsf{Dots}]\big((v\mathsf{R}_{\preccurlyeq} z_j\ \&\ \neg(z_j\mathsf{R}_{\preccurlyeq} v)) \Rightarrow w \nVdash_V \Diamond_{\preccurlyeq}\beta(v)\big)\}$ (See Figure 3.2.)
For each representative world z_j from Rep, we define a formula $\phi(z_j)$ that is true only at those worlds belonging to $\mathsf{Entry}(z_j)$. $\forall z_j \in \mathsf{Rep}$:

$$\phi(z_j) := \bigwedge_{v \in W_{\mathcal{F}_{\mathsf{lsp}}}} \neg\beta(v) \wedge \Diamond_{\preccurlyeq}\beta(z_j) \wedge \bigwedge_{v \in W_{\mathcal{F}_{\mathsf{lsp}}} \& v\mathsf{R}_{\preccurlyeq} z_j \& \neg(z_j \mathsf{R}_{\preccurlyeq} v)} \neg\Diamond_{\preccurlyeq}\beta(v)$$

It can be easily verified that, given a world v, it belongs to $\mathsf{Entry}(z_j)$, for some $z_j \in \mathsf{Rep}$, if and only if $\phi(z_j)$ is true at v under V. Recall that for any $z_j \in \mathsf{Rep}$, by w_j we denote the world from Dots such that z_j is one of its immediate $\mathsf{R}_{\preccurlyeq}$-successors.

To define the valuation V_2, let $\mathsf{Dom}(V_2) := \mathsf{VAR}$;
$\forall p \in \mathsf{VAR}$, $V_2(p) :=$

$$\bigcup_{v \in W_{\mathcal{F}_{\mathsf{lsp}}} \& v \in V(p)} V(\beta(v)) \cup \bigcup_{v \in \mathsf{notEntry} \& @ \in V(p)} V(\beta(v)) \cup \bigcup_{z_j \in \mathsf{Rep} \& z_j \in V(p)} V(\phi(z_j))$$

Obviously the valuation V_2 is definable, in fact, for each $p \in \mathsf{VAR}$, there is a formula α_p such that $V_2(p) = V(\alpha_p)$, namely, $\forall p \in \mathsf{VAR}$:

$$\alpha_p := \bigvee_{v \in W_{\mathcal{F}_{\mathsf{lsp}}} \& v \in V(p)} \beta(v) \vee \bigvee_{v \in \mathsf{notEntry} \& @ \in V(p)} \beta(v) \vee \bigvee_{z_j \in \mathsf{Rep} \& z_j \in V(p)} \phi(z_j)$$

Next step is to show that the inference rule $\mathbf{r}$ is not valid in the new model $\langle Ch(n), V_2 \rangle$. It is sufficient to show that the following claim holds: for any formula $\mathtt{A}$ on $\mathcal{L}^{\mathsf{LTK}}$ containing only letters from VAR

$$\mathcal{F}_{\mathsf{lsp}} \Vdash_{S_2} \mathtt{A} \;\Leftrightarrow\; Ch(n) \Vdash_{V_2} \mathtt{A}.$$

Notice that the three statements below follow immediately by the definition of V_2:

(i) $\forall w \in W_{\mathcal{F}_{\mathsf{lsp}}}$ $(Ch(n), w) \Vdash_{V_2} \mathtt{A} \;\Leftrightarrow\; (\mathcal{F}_{\mathsf{lsp}}, w) \Vdash_{S_2} \mathtt{A}$ (the model $\langle \mathcal{F}_{\mathsf{lsp}}, S_2 \rangle$ being isomorphic to $\langle \mathcal{F}_{\mathsf{lsp}}, V_2 \rangle$ which is an open submodel of $\langle Ch(n), V_2 \rangle$);

(ii) $\forall z_i \in \mathsf{Rep}$, $\forall v \in \mathsf{Entry}(z_i)$ $(Ch(n), v) \Vdash_{V_2} \mathtt{A} \Leftrightarrow (\mathcal{F}_{\mathsf{lsp}}, w_i) \Vdash_{S_2} \mathtt{A}$;

(iii) $\forall v \in \mathsf{notEntry}$ $(Ch(n), v) \Vdash_{V_2} \mathtt{A} \;\Leftrightarrow\; (\mathcal{F}_{\mathsf{lsp}}, @) \Vdash_{S_2} \mathtt{A}$.

Since $W_{Ch(n)} = W_{\mathsf{lsp}} \cup \mathsf{notEntry} \cup \bigcup_{z_j \in \mathsf{Rep}} \mathsf{Entry}(z_j)$, the model $\langle Ch(n), V_2 \rangle$ refutes $\mathbf{r}$. ∎

Corollary 3.3.8 *An inference rule* $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$ *is admissible for* LTK_1

if and only if for any LSP*-frame* $\mathcal{F}$, $\mathbf{r}$ *is valid in* $\mathcal{F}$, *i.e.* $\mathcal{F} \Vdash \mathtt{A}_1, \ldots, \mathtt{A}_m$ *implies* $\mathcal{F} \Vdash \mathtt{B}$.

Corollary 3.3.9 (C. [4]) *The logic* LTK_1 *is decidable with respect to inference rules.*

Chapter 4

The Axiomatic System $\mathcal{AS}_{\mathsf{LTK}}$

This Chapter is entirely devoted to the research of an axiomatic system generating only those theorems belonging to the logic LTK. The work done in Chapter 2 and Chapter 3 is mostly semantical. Summarising we have introduced our logic semantically as the set of all the formulae which are valid in a particular class of frames and proved it to be decidable with respect both to its theorems and to its admissible inference rules. This means that given *any* formula and *any* inference rule we are now able to check whether they belong to LTK or else they are admissible to LTK_1 respectively. What we still lack is an automatic method to generate valid formulae. What we need is a machine which can produce only those formulae which are theorems of LTK. This implies the switch from the study of semantical aspects to syntactical ones. And this is exactly the task we aim at fulfilling in the following pages.

It is clear how helpful it could be having an axiomatic system available for our purposes. However, several difficulties arise whenever one is to prove an axiomatic system to be sound and complete with respect to a class of multi-modal frames. According to Bennett *et al.* [2] and Kurucz [44], if there is no interaction between modalities, a transfer of properties (such as *finite model property*, *decidability*, etc.) from the component simple modal logics to the newly generated multi-modal system does apply. However, as soon as such interaction takes place it is not straightforward anymore to prove that the combined system is conservative with respect to the properties of its components. In some cases the opposite may apply. Let us consider, for instance, our specific case. As it will be clear later in this chapter, in order to axiomatise the behaviour of the modal operator $\Box_{\preccurlyeq}$ we use an $\mathsf{S4.3}$ modal system, known to be sound and complete with respect to the class of linear frames; regarding our epistemic modalities, we use distinct $\mathsf{S5}$ systems, complete with respect to the the class of frames in which the relation is an equivalence over worlds (c.f. [3]). These two well known results do immediately transfer if we do not have any interaction between modal operators, but it is uncertain whether it holds true in case such interaction happens. In these circumstances, a specific investigation is needed and this may turn out not to be trivial. Nevertheless, despite such difficulties, interaction between modalities is necessary to exploit the power of multi-modal languages.

Scheme and Methodology. This Chapter aims at providing the multi-modal logic LTK (formerly introduced in Chapter 2) with a finite, sound and complete axiomatisation with interacting modalities.

In Section 4.1 we introduce a set of axiom schemata and inference rules. The axioms involved are the classic axioms for the calculi $\mathsf{S4.3}$ (temporal operator), $\mathsf{S5}$ (epistemic modalities) and a few axioms governing the interaction between modalities. The most peculiar axiom in the latter set is our multi-modal version of the famous one introduced by Dummett and Lemmon [16]. After providing few basic syntactical definitions, we proceed by providing the reader with a justification and an interpretation of the axioms chosen.

In Section 4.2 we prove our system to be sound with respect to the class of $\mathcal{LTK}$-frames, meaning that the theorems generated by the system are valid in every frame of the class. We carry on this relatively easy proof by by induction on the length of any deduction in the system.

Section 4.3 is devoted to show our completeness results. Due to its length, this section is divided in four subsections:

4.3.1 We define and construct an n-canonical model for $\mathsf{LTK}_{\mathsf{ax}}$ in which all the theorems of LTK built up from letters $p_1, \dots, p_n$ hold true, whereas a formula $\mathsf{B} \notin \mathsf{LTK}_{\mathsf{ax}}$ is false. The frame of such model is not an $\mathcal{LTK}$-frame, the required type, but it is, nevertheless, our starting point.

4.3.2 Here we apply the truth preserving operation of taking a generated submodel of the canonical model formerly defined. The resulting model is linear with respect to the temporal relation $\mathsf{R}_{\preccurlyeq}$, although not yet of the sought kind.

4.3.3 We use the technique of *filtration* to reduce the number of worlds in our linear model. This is a technique developed by Segerberg [65]. Further we prove Lemma 4.3.16, which is the core of the whole work (cf. Goldblatt [26]) and makes a substancial use of our multi-modal version of the axiom presented by Dummett and Lemmon [16].

4.3.4 After having proved Lemma 4.3.16, it is possible to apply a variant of the well known technique of *unraveling*. This will generate the intended frame, namely a finite $\mathcal{LTK}$-frame, which enables us to show our result of completeness: the logic $\mathsf{LTK_{ax}}$ is characterised by the class of $\mathcal{LTK}$-frames and thus it coincides with LTK.

Finally, Section 4.4 presents a generalised case of our logic and axiomatic system. The idea is that if the multi-modal Dummett Axiom is removed from our system, the logic generated is a general version of LTK, useful to describe situations in which more than one environment is possible at each moment in the time line.

As usual, all the results concerning the logic LTK can be transfered to the systems $\mathsf{LTK_1}$, $\mathsf{LTK^-}$ and $\mathsf{LTK^{--}}$ too.

4.1 Axioms and Rules of $\mathcal{AS}_{\mathsf{LTK}}$ (Schemata)

The number of axiom schemata we present is not fixed and depends on the number of agents operating in the system one is to use. In particular although we have a fixed number of axioms regulating the behaviour of the temporal operator $\Box_{\preccurlyeq}$, we have as many groups of axioms for the operators K_i as many the agents operating in the system are. The same holds for the axioms regulating the interactions between modalities.

Axioms (Schemata)

Axioms of CPC (classical propositional calculus)

$K_{\Box_{\preccurlyeq}}$: $\Box_{\preccurlyeq}(\mathtt{A} \to \mathtt{B}) \to (\Box_{\preccurlyeq}\mathtt{A} \to \Box_{\preccurlyeq}\mathtt{B})$

$T_{\Box_{\preccurlyeq}}$: $\Box_{\preccurlyeq}\mathtt{A} \to \mathtt{A}$

$4_{\Box_{\preccurlyeq}}$: $\Box_{\preccurlyeq}\mathtt{A} \to \Box_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{A}$

$3_{\Box_{\preccurlyeq}}$: $\Box_{\preccurlyeq}(\mathtt{A} \wedge \Box_{\preccurlyeq}\mathtt{A} \to \mathtt{B}) \vee \Box_{\preccurlyeq}(\mathtt{B} \wedge \Box_{\preccurlyeq}\mathtt{B} \to \mathtt{A})$

K_{K_ξ}: $\mathsf{K}_\xi(\mathtt{A} \to \mathtt{B}) \to (\mathsf{K}_\xi\mathtt{A} \to \mathsf{K}_\xi\mathtt{B})$, $\quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$

T_{K_ξ}: $\mathsf{K}_\xi\mathtt{A} \to \mathtt{A}$, $\quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$

4_{K_ξ}: $\mathsf{K}_\xi\mathtt{A} \to \mathsf{K}_\xi\mathsf{K}_\xi\mathtt{A}$, $\quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$

5_{K_ξ}: $\neg\mathsf{K}_\xi\mathtt{A} \to \mathsf{K}_\xi\neg\mathsf{K}_\xi\mathtt{A}$, $\quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$

M.1: $\Box_{\preccurlyeq}\mathtt{A} \to \mathsf{K}_{\mathsf{e}}\mathtt{A}$

M.2: $\mathsf{K}_{\mathsf{e}}\mathtt{A} \to \mathsf{K}_i\mathtt{A}$, $\quad 1 \leq i \leq k$

$Dum_{\Box_{\preccurlyeq}}$: $\Box_{\preccurlyeq}(\Box_{\preccurlyeq}(\mathsf{K}_{\mathsf{e}}\mathtt{A} \to \Box_{\preccurlyeq}\mathtt{A}) \to \mathsf{K}_{\mathsf{e}}\mathtt{A}) \to (\Diamond_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{A} \to \Box_{\preccurlyeq}\mathtt{A})$

Inference Rules of $\mathcal{AS}_{\mathsf{LTK}}$:

$$MP : \frac{\mathtt{A}, \quad \mathtt{A} \to \mathtt{B}}{\mathtt{B}} \qquad Nec : \frac{\mathtt{A}}{\Box_{\preccurlyeq}\mathtt{A}}$$

It is easy to notice that we can derive a necessitation rule for the modalities $\mathsf{K}_{\mathsf{e}}, \mathsf{K}_1, \ldots, \mathsf{K}_k$ by means of the axioms M.1 - M.2 and the rule Nec.

Definition 4.1.1 (Derivation, Deduction, Theoremhood) *A* **derivation** *of a formula α from the premises $\beta_1, \ldots, \beta_j$, in symbols $\beta_1, \ldots, \beta_j \vdash_{\mathcal{AS}} \alpha$ in an axiomatic system $\mathcal{AS}$ is a finite sequence of formulae $\alpha_1, \ldots, \alpha_n, \alpha$ s.t. each α_i is either a premise, or an instance of an axiom schema from $\mathcal{AS}$ or it has been obtained from a sequence of formulae $\alpha_{k_1}, \ldots, \alpha_{k_m}$ occurring before α_i via application of an inference rule from $\mathcal{AS}$.*
A **deduction** *in $\mathcal{AS}$ is a derivation with the empty set of premises.*

*A formula α is a **theorem** in $\mathcal{AS}$, denoted by $\vdash_{\mathcal{AS}} \alpha$, if there is a deduction of α in $\mathcal{AS}$.*

Definition 4.1.2 $\mathsf{LTK}_{\mathsf{ax}} := \{\mathtt{A} \in Fma(\mathcal{L}^{\mathsf{LTK}}) \mid \ \vdash_{\mathcal{AS}_{\mathsf{LTK}}} \mathtt{A}\}$.
$\mathsf{LTK}^{-}_{\mathsf{ax}} := \mathsf{LTK}_{\mathsf{ax}} \cap Fma(\mathcal{L}^{-}_{\mathsf{LTK}})$.

The Meaning of the Axioms. Our axioms for the time modality $\Box_{\preccurlyeq}$ give rise to an $\mathsf{S4.3}$ modal system, known to be sound and complete with respect to the class of linear orders (see [3]). Formerly we stated that each agent operating in the system is provided with a a certain knowledge background. In order to give a simple account of it, we associate each agent to an $\mathsf{S5}$-modal system. The assumptions we make are the usual ones:

- Agent i knows $\mathtt{A}$ whenever the fact $\mathtt{A}$ is provided by *all* the resources she/he can access. Thus agents may *know* only those facts which are provided by all the sources they have access to;

- Positive Introspection: if someone knows something, she/he is also aware of it;

- Negative Introspection: if someone ignores something, she/he is aware of it.

Moreover, we assume each agent to be logicallly omniscient (knowing both all the tautologies and all the consequences implicit in her/his knowledge base).

The same assumptions appear to be more natural when it comes to model the behaviour of the environment. In our system, the axioms involving the environment modality play a central role. In the interaction between different modalities, the operator $\mathsf{K_e}$ works like a bridge connecting the others, which otherwise would not interact at all. The axioms M.1 and M.2 state that if something is always true toward the future, then it is also true at the current moment/environment (M.1) and hence each agent knows it (M.2).

More specifically, Axiom M.1 aims at achieving property PM.1, whereas Axiom M.2 is responsable for PM.2. Axiom $Dum_{\Box_{\preccurlyeq}}$ entails the property PM.3 in a less evident and straightforward way. However, this is probably the most interesting one, for it is the one regulating the peculiar relation linking $\Box_{\preccurlyeq}$ to $\mathsf{K_e}$. Indeed, as it is made clear by Lemma 4.2.1, axiom $Dum_{\Box_{\preccurlyeq}}$ achieves two things:

(i) making temporal and environmental clusters coincide;

(ii) ensuring a discrete order of temporal clusters.

4.2 Soundness

The first thing to verify is to check whether our axiomatic system produces formulae which are actually true in every $\mathcal{LTK}$-frame. Any axiomatic system enjoying such property with respect to a class of frames is said to be *sound* with respect to the specified class of frames. Thus we start by proving that the system $\mathsf{LTK_{ax}}$ is a sound system with respect to the class of $\mathcal{LTK}$-frames:

Theorem 4.2.1 (Soundness) $\forall \mathtt{A} \in Fma(\mathcal{L}^{\mathsf{LTK}})\ \ (\mathtt{A} \in \mathsf{LTK_{ax}} \Rightarrow \mathtt{A} \in \mathsf{LTK})$

Proof. (by induction on the length of the deduction $\mathcal{D} = \mathcal{D}_1, \ldots, \mathcal{D}_j$ of a theorem $\mathtt{A} \in \mathsf{LTK_{ax}}$). Suppose $j = 1$, then $\mathtt{A}$ is an axiom from $\mathcal{AS}_{\mathsf{LTK}}$. We provide a proof only for the axioms M.1 , M.2 and $Dum_{\Box_{\preccurlyeq}}$. (a) Suppose there are an $\mathcal{LTK}$-frame $\mathcal{F}$, a valuation V for $\mathcal{F}$, and a world $v \in W_{\mathcal{F}}$ such that $(\mathcal{F}, v) \nVdash_V \Box_{\preccurlyeq}\mathtt{A} \to \mathsf{K_e}\mathtt{A}$. Then $(\mathcal{F}, v) \Vdash_V \Box_{\preccurlyeq}\mathtt{A}$ and $(\mathcal{F}, v) \nVdash_V \mathsf{K_e}\mathtt{A}$. Hence for each world $z \in \{t \mid v\mathsf{R}_{\preccurlyeq}t\}$, $z \Vdash_V \mathtt{A}$ but there is a world $u \in \{t \mid v\mathsf{R_e}t\}$ such that $u \nVdash \mathtt{A}$. Since by definition $\{t \mid v\mathsf{R}_{\preccurlyeq}t\} \supseteq \{t \mid v\mathsf{R_e}t\}$, this leads to a contradiction. Using a similar argument, it can be easily seen that Axiom M.2 is valid too.

(b) Suppose that Axiom $Dum_{\Box_\preccurlyeq}$ is not valid. Then there are an $\mathcal{LTK}$-frame $\mathcal{F}$, a valuation V for $\mathcal{F}$, and a world $v \in W_\mathcal{F}$ such that $(\mathcal{F}, v) \nVdash_V \Box_\preccurlyeq(\Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \to \mathsf{K_e}\mathtt{A}) \to (\Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A} \to \Box_\preccurlyeq\mathtt{A})$, and hence:

$$(\mathcal{F}, v) \Vdash_V \Box_\preccurlyeq(\Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \to \mathsf{K_e}\mathtt{A}) \tag{4.1}$$

and

$$(\mathcal{F}, v) \nVdash_V (\Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \tag{4.2}$$

Condition (4.1) implies that $\forall z \in W_\mathcal{F}\ (v\mathsf{R}_\preccurlyeq z \ \Rightarrow\ (\mathcal{F}, z) \Vdash_V \Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \to \mathsf{K_e}\mathtt{A})$. This means that for each $\mathsf{R}_\preccurlyeq$-successor z of v, at least one of the following conditions should hold:

(4.1.1): $(\mathcal{F}, z) \nVdash_V \Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A})$, then there is a world t such that $z\mathsf{R}_\preccurlyeq t$ & $(\mathcal{F}, t) \Vdash_V \mathsf{K_e}\mathtt{A}$ & $(\mathcal{F}, t) \nVdash_V \Box_\preccurlyeq\mathtt{A}$;

(4.1.2): $(\mathcal{F}, z) \Vdash_V \mathsf{K_e}\mathtt{A}$.

Let us analise condition (4.2): if $(\mathcal{F}, v) \nVdash_V (\Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A} \to \Box_\preccurlyeq\mathtt{A})$, we have that both of the following conditions (4.2.1) and (4.2.2) must hold:

(4.2.1): $(\mathcal{F}, v) \Vdash_V \Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A}$;

(4.2.2): $(\mathcal{F}, v) \nVdash_V \Box_\preccurlyeq\mathtt{A}$;

From condition (4.2.1) and (4.2.2) it follows that there is a world $\mathsf{R}_\preccurlyeq$-accessible from v in which $\mathtt{A}$ is not true, while there is another point $\mathsf{R}_\preccurlyeq$-accessible from v starting from which $\mathtt{A}$ holds true everywhere toward the future. Since each $\mathcal{LTK}$-frame is a linear and discrete order with respect to $\mathsf{R}_\preccurlyeq$-clusters, there is a world v_2 such that $v\mathsf{R}_\preccurlyeq v_2$ and $(\mathcal{F}, v_2) \nVdash_V \mathtt{A}$ and for each world z_2 such that $v_2\mathsf{R}_\preccurlyeq z_2$ & $\neg(z_2\mathsf{R}_\preccurlyeq v_2)$, $(\mathcal{F}, z_2) \Vdash \mathtt{A}$, and hence

$(\mathcal{F}, z_2) \Vdash \Box_{\preccurlyeq}\mathtt{A}$. Trivially, condition (4.1.2) does not hold at v_2, for $\mathsf{R_e}$ is reflexive.

Then condition (4.1.1) should hold. This implies that there is a world t_2 such that $v_2\mathsf{R}_{\preccurlyeq}t_2$, $(\mathcal{F}, t_2) \Vdash_V \mathsf{K_e}\mathtt{A}$ and $(\mathcal{F}, t_2) \nVdash_V \Box_{\preccurlyeq}\mathtt{A}$. Hence $t_2\mathsf{R}_{\preccurlyeq}v_2$ (for, by the way we chose v_2, if $v_2\mathsf{R}_{\preccurlyeq}t_2$ and $\neg(t_2\mathsf{R}_{\preccurlyeq}v_2)$, we should have $(\mathcal{F}, t_2) \Vdash_V \Box_{\preccurlyeq}\mathtt{A}$). Moreover $\neg(t_2\mathsf{R_e}v_2)$. However, this is in contradiction with Definition 2.2.3, for in $\mathcal{LTK}$-frames $\mathsf{R}_{\preccurlyeq}$-clusters and $\mathsf{R_e}$-clusters should coincide.

Thus, if $lg(\mathcal{D}) = n + 1$, it can be easily shown that each inference rule preserves validity. ∎

4.3 Semantic Completeness

We now know that any formula generated by $\mathsf{LTK_{ax}}$ is valid in the class of $\mathcal{LTK}$-frames. Our goal is thus to show that if a formula $\mathtt{B}$ is valid in such class, then it ought to be generated by the axiomatic system as well. An achievement of this kind would tell us that the axiomatic system we have described generates *all* and *only* the theorems of LTK: in two words, the system is *sound and complete*. In order to achieve this task, we use many well known techniques such as canonical models, generated subframes and filtration.

4.3.1 Canonical Models

It is a well known result that any consistent normal k-modal logic L has a model, called its *canonical model*, which is *characterising* for L in the sense that:

Definition 4.3.1 *A model $\mathcal{M} := \langle W, \mathsf{R}_1, \ldots, \mathsf{R}_k, V\rangle$ is characterising for a logic L on a lanagauge $\mathcal{L}$ if for any formula $\mathtt{A} \in \mathcal{L}$ $(\mathtt{A} \in \mathsf{L} \Leftrightarrow \forall w \in W(w \Vdash_V$*

$\mathtt{A}))$.

Let us briefly sketch few standard definitions and results.

Definition 4.3.2 *Given an axiomatic system $\mathcal{AS}$ on a language $\mathcal{L}$, a set $\Delta \subset Fma(\mathcal{L})$ is:*

(i) $\mathcal{AS}$-consistent iff $\Delta \nvdash_{\mathcal{AS}} \bot$;

(ii) $\mathcal{L}$-complete iff $\forall \mathtt{A} \in Fma(\mathcal{L})$ $\mathtt{A} \in \Delta$ or $\neg\mathtt{A} \in \Delta$;

(iii) $\mathcal{AS}$-maximal iff Δ is $\mathcal{AS}$-consistent and $\mathcal{L}$-complete.

Definition 4.3.3 *Let L be a consistent normal k-modal logic on a language $\mathcal{L}$ containing the modal operators $\Box_1, \ldots, \Box_k$. An n-**canonical model** $\mathcal{M}^{\mathsf{c}}_n = \langle W^{\mathsf{c}}_n, \mathsf{R}^{\mathsf{c}}_1, \ldots, \mathsf{R}^{\mathsf{c}}_k, V^{\mathsf{c}}_n \rangle$ for L is such that:*

(i) W^{c}_n is the set of all the possible L-maximal sets w.r.t. those formulae built up from the propositional letters $p_1, \ldots, p_n$;

(ii) $\forall v, z \in W^{\mathsf{c}}_n, \quad v\mathsf{R}^{\mathsf{c}}_i z \iff \{\mathtt{A} \mid \Box_i\mathtt{A} \in v\} \subseteq z$, $1 \leq i \leq k$;

(iii) $V^{\mathsf{c}}_n(p_i) = \{v \in W^{\mathsf{c}}_n \mid p_i \in v\}, \quad 1 \leq i \leq n$.

Lemma 4.3.4 *Let L be a consistent normal k-modal logic and let $\mathcal{M}^{\mathsf{c}}_n = \langle W^{\mathsf{c}}_n, \mathsf{R}^{\mathsf{c}}_1, \ldots, \mathsf{R}^{\mathsf{c}}_k, V^{\mathsf{c}}_n \rangle$ be an n-canonical model for L. Then $\forall v \in W^{\mathsf{c}}_n$, $\forall \mathtt{A}(p_1, \ldots, p_n) \in Fma(\mathcal{L})$ $(\Box_i\mathtt{A} \in v \iff \forall z \in W^{\mathsf{c}}_n(v\mathsf{R}^{\mathsf{c}}_i z \Rightarrow \mathtt{A} \in z))$.*

Lemma 4.3.5 (Truth Lemma) *Let L be a consistent normal k-modal logic and let $\mathcal{M}^{\mathsf{c}}_n = \langle W^{\mathsf{c}}_n, \mathsf{R}^{\mathsf{c}}_1, \ldots, \mathsf{R}^{\mathsf{c}}_k, V^{\mathsf{c}}_n \rangle$ be an n-canonical model for L. Then $\forall v \in W^{\mathsf{c}}_n$, $\forall \mathtt{A}(p_1, \ldots, p_n) \in Fma(\mathcal{L})$*

$$(\mathcal{F}^{\mathsf{c}}_n, v) \Vdash_{V^{\mathsf{c}}_n} \mathtt{A} \iff \mathtt{A} \in v$$

where $\mathcal{F}^{\mathsf{c}}_n$ denotes the n-canonical frame on which $\mathcal{M}^{\mathsf{c}}_n$ is built.

Take and fix for the rest of this Chapter a formula $\mathtt{B}(p_1, \ldots, p_n) \notin \mathsf{LTK}_{\mathsf{ax}}$. Hence the set $\{\neg \mathtt{B}\}$ is $\mathcal{AS}_{\mathsf{LTK}}$-consistent and it follows that there exists an $\mathcal{AS}_{\mathsf{LTK}}$-maximal set w w.r.t. all the formulae built up from $p_1, \ldots, p_n$ such that $\mathtt{B} \notin w$. Therefore there is an n-canonical model $\mathcal{M}^{\mathsf{c}} := \langle \mathcal{F}^{\mathsf{c}}_n, V^{\mathsf{c}}_n \rangle$ for $\mathsf{LTK}_{\mathsf{ax}}$ (where $\mathcal{F}^{\mathsf{c}}_n = \langle W^{\mathsf{c}}_n, \mathsf{R}^{\mathsf{c}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{c}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{c}}_1, \ldots, \mathsf{R}^{\mathsf{c}}_k \rangle$) such that $w \in W^{\mathsf{c}}_n$ and, by Lemma 4.3.5, $(\mathcal{F}^{\mathsf{c}}_n, w) \nVdash_{V^{\mathsf{c}}_n} \mathtt{B}$. Although this model shows some interesting properties, it is not built on an $\mathcal{LTK}$-frame.

However, the binary relations in the n-canonical frame have the following properties:

(i) $\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}$, $\mathsf{R}^{\mathsf{c}}_i$ are reflexive, symmetric and transitive.

(ii) $\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}$ is reflexive, transitive and weakly connected.

(iii) $\forall v, z \in W^{\mathsf{c}}_n \ \ (v\mathsf{R}^{\mathsf{c}}_i z \Rightarrow v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}} z)$.

(iv) $\forall v, z \in W^{\mathsf{c}}_n \ \ (v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}} z \Rightarrow (v\mathsf{R}^{\mathsf{c}}_{\preccurlyeq} z \ \& \ z\mathsf{R}^{\mathsf{c}}_{\preccurlyeq} v))$. Notice that the opposite direction does not hold.

In the following sections, we shall apply several truth preserving operation in order to transform the canonical model we defined into a model based on an $\mathcal{LTK}$-frame.

4.3.2 Generated Subframes and Models

The first truth operation we apply to our canonical model is taking a *generated submodel*. The idea is to consider the world w which we have formerly isolated as one world in which $\mathtt{B}$ is false and reduce the canonical model to only those worlds which are related to w by means of any relation. The resulting model would therefore be *rooted* in the sense that there would be a *first* set of worlds, namely the $\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}$-cluster to which the world w belongs.

Let us define the operation of taking subframes and submodels more formally:

Definition 4.3.6 (Subframe) *An n-modal K-frame $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_1, \ldots, \mathsf{R}_n \rangle$ is a subframe of an m-modal K-frame $\mathcal{S} = \langle W_{\mathcal{S}}, \mathsf{S}_1, \ldots, \mathsf{S}_m \rangle$ if $n = m$, $W_{\mathcal{F}} \subseteq W_{\mathcal{S}}$ and each R_i is the restriction of S_i to $W_{\mathcal{F}}$, i.e. $\mathsf{R}_i = \mathsf{S}_i \restriction W_{\mathcal{F}}$.*

Definition 4.3.7 (Generated subframes and models) *An n-modal K-frame $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_1, \ldots, \mathsf{R}_n \rangle$ is a* generated *subframe of an m-modal K-frame $\mathcal{S} = \langle W_{\mathcal{S}}, \mathsf{S}_1, \ldots, \mathsf{S}_m \rangle$ (notation $\mathcal{F} \sqsubseteq \mathcal{S}$) if $\mathcal{F}$ is a subframe of $\mathcal{S}$ and $\forall v \in W_{\mathcal{F}} \quad \forall z \in W_{\mathcal{S}}$ if there is a relation S_j such that $v\mathsf{S}_j z$ in $\mathcal{S}$, then $z \in W_{\mathcal{F}}$. A model $\langle \mathcal{F}, V \rangle$ is a generated submodel of $\langle \mathcal{S}, S \rangle$ if $\mathcal{F} \sqsubseteq \mathcal{S}$ and V is the restriction of S to $W_{\mathcal{F}}$ (i.e. $V = S \restriction W_{\mathcal{F}}$).*

Lemma 4.3.8 (Generated subframes) *If $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is a generated submodel of $\mathcal{M}_2 = \langle \mathcal{F}_2, V_2 \rangle$, then $\forall v \in W_{\mathcal{F}}$, $(\mathcal{F}, v) \Vdash_V \mathtt{A} \iff (\mathcal{F}_2, v) \Vdash_{V_2} \mathtt{A}$.*

Definition 4.3.9 *Let $\mathcal{F}$ be a Kripke-frame $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{S}_1, \ldots, \mathsf{S}_k \rangle$, w a world w in $W_{\mathcal{F}}$, and S_i a partial order on $W_{\mathcal{F}}$. Then the set $w^{\mathsf{S}_i \leq}$ is the set of all the worlds S_i-accessible from w, i.e. $w^{\mathsf{S}_i \leq} := \{z \mid w\mathsf{S}_i z\}$. Likewise the set $w^{\mathsf{S}_i <}$ is defined as the set of all the* strict *successors of w, i.e. $w^{\mathsf{S}_i <} := \{z \mid w\mathsf{S}_i z \ \& \ \neg(z\mathsf{S}_i w)\}$.*
Given a S_i-cluster $\mathcal{C}$, the set of all the S_i-clusters it has access to is defined as $\mathcal{C}^{\mathsf{S}_i \leq} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{S}_i\mathcal{C}_j\}$. Likewise $\mathcal{C}^{\mathsf{S}_i <} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{S}_i\mathcal{C}_j \ \& \ \neg(\mathcal{C}_j\mathsf{S}_i\mathcal{C})\}$ is the set of all those S_i-clusters which are strictly above *with respect to $\mathcal{C}$.*
The sets $w^{\mathsf{R}_{\preccurlyeq}\leq}$ and $\mathcal{C}^{\mathsf{R}_{\preccurlyeq}\leq}$ shall henceforth be referred to as $w^{\preccurlyeq}$ and $\mathcal{C}^{\preccurlyeq}$ respectively.

Consider the generated submodel of $\langle \mathcal{F}^{\mathsf{c}}_n, V^{\mathsf{c}}_n \rangle$ generated by $w^{\preccurlyeq}$ (recall that w is that world refuting $\mathtt{B}$ in the n-canonical model) and denote it by

$\langle \mathcal{F}_{w^{\preccurlyeq}}, V_n^{\mathsf{c}} \rangle$. Hence $(\mathcal{F}_{w^{\preccurlyeq}}, w) \nVdash_{V_n^{\mathsf{c}}} \mathtt{B}$ which entails $\mathcal{F}_{w^{\preccurlyeq}} \nVdash \mathtt{B}$. As some formerly observed facts (items iii and iv in Section 4.3.1) make clear, the base set of $w^{\preccurlyeq}$ contains also all those worlds which are both $\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}$- and $\mathsf{R}^{\mathsf{c}}_{i}$-related to w. Being then a real generated submodel of the canonical one, all the general results concerning generated submodels do apply.

As the following lemma will clarify, the generated submodel $\langle \mathcal{F}_{w^{\preccurlyeq}}, V_n^{\mathsf{c}} \rangle$ shows some interesting properties shared with $\mathcal{LTK}$-frames:

Lemma 4.3.10 *$\mathcal{F}_{w^{\preccurlyeq}}$ has the following properties:*

(i) The relations $\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{c}}_{1}, \ldots, \mathsf{R}^{\mathsf{c}}_{k}$ are equivalence relations;

(ii) The relation $\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}$ is reflexive, transitive and connected;

(iii) $\forall v, z \in W_F\ v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}z \Rightarrow (v\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}z\ \&\ z\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}v)$;

(iv) $\forall v, z \in W_F\ v\mathsf{R}^{\mathsf{c}}_{i}z \Rightarrow v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}z$;

Proof. (iii) Suppose $\neg(v\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}z)$. Then there is a formula $\Box_{\preccurlyeq}\mathtt{D} \in v$ s.t. $\mathtt{D} \notin z$. By Axiom M.1 it follows $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in v$ and hence $\neg(v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}z)$. The same case arises if we assume $\neg(z\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}v)$. (iv) Suppose $\neg(v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}z)$. Then there is a formula $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in v$ s.t. $\mathtt{D} \notin z$. By Axiom M.2 it follows $\mathsf{K}_{i}\mathtt{D} \in v$ for each i and hence $\neg(v\mathsf{R}^{\mathsf{c}}_{i}z)$ ■

4.3.3 Filtration

A good way to achieve the required property of discreteness (i.e. given any two disctint worlds in a model, there could be only a finite amount of *moments* between them) is to make a filtration of the base set of the model in order to have it finite. This technique can be dated back to the work by Scott and further developed by Segerberg [65]. Although it is a standard technique, it requires a careful and appropriate selection of the *filtration set*. The well known results concerning this method are recalled below and they would allow us to show the following:

Lemma 4.3.11 (Filtration Lemma) $\forall \mathtt{D} \in \Gamma\ \forall v \in W_n^{\mathsf{c}}\ (v \Vdash_{V_n^{\mathsf{c}}} \mathtt{D} \Leftrightarrow [v] \Vdash_{V^\Gamma} \mathtt{D})$

We start by defining the filtration set Γ as the union of several sets:

- $\Gamma_0 := Sub(\mathtt{B})$, where $Sub(\mathtt{B})$ is the set of all the *subformulae* of $\mathtt{B}$

- $\Gamma_1 := Sub\{Dum_{\Box_{\preccurlyeq}}(\mathtt{D}) \mid \mathtt{D} \in \Gamma_0\}$ (where the notation $Dum_{\Box_{\preccurlyeq}}(\mathtt{D})$ is intended to denote the instance of the axiom $Dum_{\Box_{\preccurlyeq}}$ by the formula $\mathtt{D}$)

- $\Gamma_2 := \{\mathsf{K_e}\Box_{\preccurlyeq}\mathtt{D} \mid \Box_{\preccurlyeq}\mathtt{D} \in \Gamma_0 \cup \Gamma_1\}$

- $\Gamma_3 := \{\mathsf{K}_i\mathsf{K_e}\mathtt{D} \mid \mathsf{K_e}\mathtt{D} \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2\}$ for $1 \leq i \leq k$

Let the filtration set Γ be the union of the formerly defined sets:

$$\Gamma := \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

Then we define a new relation between worlds as:

$$\forall v, z \in W_n^{\mathsf{c}}\ \ (v \sim z \Leftrightarrow \forall \mathtt{D} \in \Gamma(v \Vdash_{V_n^{\mathsf{c}}} \mathtt{D} \Leftrightarrow z \Vdash_{V_n^{\mathsf{c}}} \mathtt{D}))$$

and we generate equivalence classes with respect to the relation $\sim$:

$$[v] := \{z \mid v \sim z\}$$

The Γ-filtered model $\mathcal{M}^\Gamma$ is defined as $\langle \mathcal{F}^\Gamma, V^\Gamma\rangle$ where $\mathcal{F}^\Gamma = \langle W^\Gamma, \mathsf{R}^\Gamma_{\preccurlyeq}, \mathsf{R}^\Gamma_{\mathsf{e}}, \mathsf{R}^\Gamma_1, \ldots, \mathsf{R}^\Gamma_k\rangle$ and:

(i) $W^\Gamma := \{[v] \mid v \in W_n^{\mathsf{c}}\}$

(ii) $\mathsf{R}^\Gamma_{\mathsf{e}}$ and each R^Γ_i are standard $\mathsf{S5}$ filtration relations, i.e.:
$[v]\mathsf{R}^\Gamma_\xi[z] \Leftrightarrow \forall \mathsf{K}_\xi\mathtt{D} \in \Gamma((\mathcal{F}_n^{\mathsf{c}}, v) \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_\xi\mathtt{D} \Leftrightarrow (\mathcal{F}_n^{\mathsf{c}}, z) \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_\xi\mathtt{D})$ for $\xi \in \{\mathsf{e}, 1, \ldots, k\}$

(iii) $\mathsf{R}^{\Gamma}_{\preccurlyeq}$ is a standard $\mathsf{S4.3}$ filtration relation, namely:
$[v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z] \;\Leftrightarrow\; \forall \Box_{\preccurlyeq}\mathtt{D} \in \Gamma((\mathcal{F}^{\mathsf{c}}_{n}, v) \Vdash_{V^{\mathsf{c}}_{n}} \Box_{\preccurlyeq}\mathtt{D} \;\Rightarrow\; (\mathcal{F}^{\mathsf{c}}_{n}, z) \Vdash_{V^{\mathsf{c}}_{n}} \Box_{\preccurlyeq}\mathtt{D})$

(iv) $\forall p_i \in \{p_1, \dots, p_n\}\; V^{\Gamma}(p_i) := \{[v] \mid v \in V^{\mathsf{c}}_{n}(p_i)\}$

It remains only to show that such model satisfies the two filtration conditions for each binary relation $\mathsf{R}^{\Gamma}_{\xi}$:

F1 $v\mathsf{R}^{\mathsf{c}}_{\xi}z \;\Rightarrow\; [v]\mathsf{R}^{\Gamma}_{\xi}[z]$, where $\xi \in \{\preccurlyeq, \mathtt{e}, 1 \dots k\}$;

F2.1 $[v]\mathsf{R}^{\Gamma}_{\xi}[z] \;\Rightarrow\; \forall \mathsf{K}_{\xi}\mathtt{D} \in \Gamma(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\xi}\mathtt{D} \;\Rightarrow\; z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D})$, for $\xi \in \{\mathtt{e}, 1 \dots k\}$;

F2.2 $[v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z] \;\Rightarrow\; \forall \Box_{\preccurlyeq}\mathtt{D} \in \Gamma(v \Vdash_{V^{\mathsf{c}}_{n}} \Box_{\preccurlyeq}\mathtt{D} \;\Rightarrow\; z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D})$

Lemma 4.3.12 *The following properties hold true in the model $\mathcal{M}^{\Gamma}$:*

(i) The relation $\mathsf{R}^{\Gamma}_{\mathsf{e}}$ satisfies $F1$ and $F2.1$.

(ii) Each relation R^{Γ}_{i} satisfies $F1$ and $F2.1$.

(iii) The relation $\mathsf{R}^{\Gamma}_{\preccurlyeq}$ satisfies $F1$ and $F2.2$.

Proof. (i). F1. Suppose there are two worlds v and z such that $v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}z$. Then $v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \;\Rightarrow\; z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D}$. Since $v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \to \mathsf{K}_{\mathsf{e}}\mathsf{K}_{\mathsf{e}}\mathtt{D}$ we have $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \;\Rightarrow\; (v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathsf{K}_{\mathsf{e}}\mathtt{D}) \;\Rightarrow\; (z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Suppose $(z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Then $(z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathsf{K}_{\mathsf{e}}\mathtt{D})$. Since $\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}$ is symmetric by Lemma 4.3.10, we have $z\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}v$ and hence $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Therefore, by our definition of $\mathsf{R}^{\Gamma}_{\mathsf{e}}$, it follows $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$.
F2.1. Suppose $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z]$. Then $\forall \mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma \;\; (v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \;\Leftrightarrow\; z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Since $z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \to \mathtt{D}$ we have $z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D}$.
The proof of cases (ii) and (iii) is similar. ∎

Hence by the standard results concerning filtrations, we can state the following:

Lemma 4.3.13 $\forall \mathtt{D} \in \Gamma\ \forall v \in W_n^{\mathsf{c}}\ ((\mathcal{F}_n^{\mathsf{c}}, v) \Vdash_{V_n^{\mathsf{c}}} \mathtt{D} \ \Leftrightarrow\ (\mathcal{F}^{\Gamma}, [v]) \Vdash_{V^{\Gamma}} \mathtt{D})$.

Corollary 4.3.14 $(\mathcal{F}^{\Gamma}, [w]) \nVdash_{V^{\Gamma}} \mathtt{B}$.

Once more again, we can show that our current Γ-filtered model $\mathcal{M}^{\Gamma}$ is conservative with respect to the properties stated by Lemma 4.3.10.

Lemma 4.3.15 *In the model $\mathcal{M}^{\Gamma}$ the following holds:*

(i) $\mathsf{R}_{\mathsf{e}}^{\Gamma}$ *and each* R_i^{Γ} *are reflexive, symmetric and transitive.*

(ii) $\mathsf{R}_{\preccurlyeq}^{\Gamma}$ *is reflexive, transitive and connected.*

(iii) $\forall [v], [z] \in W^{\Gamma}\ \ ([v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[z] \ \Rightarrow\ ([v]\mathsf{R}_{\preccurlyeq}^{\Gamma}[z]\ \&\ [z]\mathsf{R}_{\preccurlyeq}^{\Gamma}[v]))$.

(iv) $\forall [v], [z] \in W^{\Gamma}\ \ ([v]\mathsf{R}_i^{\Gamma}[z] \ \Rightarrow\ [v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[z])$.

Proof. (i) Trivially $(v \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \ \Leftrightarrow\ (v \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Hence $[v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[v]$ and $\mathsf{R}_{\mathsf{e}}^{\Gamma}$ is reflexive.

Suppose $[v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[z]$ and $[z]\mathsf{R}_{\mathsf{e}}^{\Gamma}[u]$. Hence $(v \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \ \Leftrightarrow\ (z \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$ and $(z \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \ \Leftrightarrow\ (u \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$, which entails $(v \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \ \Leftrightarrow\ (u \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Hence $[v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[u]$.

Suppose $[v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[z]$. Then $(v \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \ \Leftrightarrow\ (z \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$ and hence $(z \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \ \Leftrightarrow\ (v \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$ which means $[z]\mathsf{R}_{\mathsf{e}}^{\Gamma}[v]$.

(ii) For the properties of reflexivity and transitivity see the previous case. Since by F1 $(v\mathsf{R}_{\preccurlyeq}^{\mathsf{c}}z)$ implies $([v]\mathsf{R}_{\preccurlyeq}^{\Gamma}[z])$ and $\mathsf{R}_{\preccurlyeq}^{\mathsf{c}}$ is connected (see Lemma 4.3.15), it follows that $\mathsf{R}_{\preccurlyeq}^{\Gamma}$ is connected.

(iii) Suppose $[v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[z]$ and either $\neg([v]\mathsf{R}_{\preccurlyeq}^{\Gamma}[z])$ or $\neg([z]\mathsf{R}_{\preccurlyeq}^{\Gamma}[v])$. If $\neg([v]\mathsf{R}_{\preccurlyeq}^{\Gamma}[z])$, then there is a formula $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma$ such that $v \Vdash_{V_n^{\mathsf{c}}} \Box_{\preccurlyeq}\mathtt{D}$ and $z \nVdash_{V_n^{\mathsf{c}}} \Box_{\preccurlyeq}\mathtt{D}$. By Axiom $4_{\Box_{\preccurlyeq}}$ it follows $v \Vdash_{V_n^{\mathsf{c}}} \Box_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{D}$, hence by Axiom M.1 we have

$v \Vdash_{V_n^c} \mathsf{K_e}\Box_{\preccurlyeq}\mathtt{D}$. But $\mathsf{K_e}\Box_{\preccurlyeq}\mathtt{D} \in \Gamma$ by construction[1], therefore since $[v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[z]$ we have $z \Vdash_{V_n^c} \mathsf{K_e}\Box_{\preccurlyeq}\mathtt{D}$ and, by reflexivity, $z \Vdash_{V_n^c} \Box_{\preccurlyeq}\mathtt{D}$, which is a contradiction. We reach a similar contradiction if we assume $\neg([z]\mathsf{R}_{\preccurlyeq}^{\Gamma}[v])$.
(iv) Suppose $[v]\mathsf{R}_i^{\Gamma}[z]$ for some i and $\neg([v]\mathsf{R}_{\mathsf{e}}^{\Gamma}[z])$. Then there is a formula $\mathsf{K_e}\mathtt{D} \in \Gamma$ s.t. $v \Vdash_{V_n^c} \mathsf{K_e}\mathtt{D}$ and $z \nVdash_{V_n^c} \mathsf{K_e}\mathtt{D}$. Again, by the axioms $4_{\mathsf{K_e}}$ and M.2 we obtain $v \Vdash_{V_n^c} \mathsf{K}_i\mathsf{K_e}\mathtt{D}$. The formula $\mathsf{K}_i\mathsf{K_e}\mathtt{D}$ belongs to Γ by construction[2], therefore, given $[v]\mathsf{R}_i^{\Gamma}[z]$, we have $z \Vdash_{V_n^c} \mathsf{K}_i\mathsf{K_e}\mathtt{D}$ and, by reflexivity, $z \Vdash_{V_n^c} \mathsf{K_e}\mathtt{D}$, which gives rise to a contradiction.

■

Properties of filtered relations. From Lemma 4.3.15 follows that the new binary relations possess certain properties, namely all the knowledge modalities are reflexive, symmetric and transitive, whereas the time relation is connected as well as reflexive and transitive.
Both properties PM.1 and PM.2 hold true:

$$\forall v, z \in W^{\Gamma} \quad (v\mathsf{R}_{\mathsf{e}}^{\Gamma}z \;\Rightarrow\; (v\mathsf{R}_{\preccurlyeq}^{\Gamma}z \;\&\; z\mathsf{R}_{\preccurlyeq}^{\Gamma}v))$$

which means that two information points (worlds) are simultaneous whenever they are from the same environment. But this is something we were able to state even in the previous stages of our construction. The main achievement is that now we have a very important property. In fact, since the base set of the filtered frame is finite, trivially the time relation $\mathsf{R}_{\preccurlyeq}^{\Gamma}$ gives rise to a discrete linear order of temporal ($\mathsf{R}_{\preccurlyeq}^{\Gamma}$-) clusters, which in this context means that given any two distinct worlds, there is only a finite number

[1] Indeed if $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma$, then there are only two possibilities: either $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma_0$ or $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma_1$ and in both cases $\mathsf{K_e}\Box_{\preccurlyeq}\mathtt{D} \in \Gamma_2$ and hence it belongs to Γ as well.
[2] IIn fact if $\mathsf{K_e}\mathtt{D} \in \Gamma$, then either $\mathsf{K_e}\mathtt{D} \in \Gamma_0$ or $\mathsf{K_e}\mathtt{D} \in \Gamma_1$ or else $\mathsf{K_e}\mathtt{D} \in \Gamma_2$; hence $\mathsf{K}_i\mathsf{K_e}\mathtt{D} \in \Gamma_3$ and it belongs to Γ as well.

of moments between them.

However we do not have the property PM.3 yet:

$$\forall v, z \in W^{\Gamma} \quad ((v\mathsf{R}^{\Gamma}_{\preccurlyeq} z \;\&\; z\mathsf{R}^{\Gamma}_{\preccurlyeq} v) \;\Rightarrow\; v\mathsf{R}^{\Gamma}_{\mathsf{e}} z)$$

In other words in this frame we may have $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper clusters of $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters, which in our intended interpretation means that it can be the case that two points, though at the same moment of the flow of time could belong to different environments (see Figure 4.1). Unfortunately, this is not the case for $\mathcal{LTK}$-frames, therefore another transformation seems to be necessary to prove our axiomatic system to be complete with respect to these structures. To achieve this goal, we will construct another frame. The idea is to unravel each $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster, without using the standard technique of *bulldozing*, which would give rise to an infinite, but not discrete (with respect to temporal clusters) frame (cf. Blackburn *et al.* [3], pages 220–222 and Segerberg [65]). In other words we will define a well ordering on $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters inside each $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster, in order to construct a new frame. Such frame will be obtained by substituting each $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster with the finite ordered line formerly defined. The only troubling case is for formulae of the form $\Box_{\preccurlyeq}\mathtt{D}$ from $Sub(\mathtt{B})$. There could be, for example, a world v in an $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster such that it falsifies $\Box_{\preccurlyeq}\mathtt{D}$ and another world, say z which is the only point $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-accessible from v falsifying $\mathtt{D}$. If such world z belonged to the same $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster as v and it were not $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-accessible from v in the new unravelled frame, then we would have a lack of truth values for formulae from $Sub(\mathtt{B})$. However this situation is made impossible by the subsequent lemma, stating that whenever a world v in an $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster falsifies $\Box_{\preccurlyeq}\mathtt{D}$, $\mathtt{D}$ is falsified either in the same environment-cluster to which v belongs, or in another world z which is *strictly above* with respect to v, i.e. $v\mathsf{R}^{\Gamma}_{\preccurlyeq}z$ & $\neg(z\mathsf{R}^{\Gamma}_{\preccurlyeq}v)$.

Lemma 4.3.16 $\forall\Box_{\preccurlyeq}\mathtt{D} \in Sub(\mathtt{B})\forall v \in W^{\Gamma}$ *if* $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{D}$ *and* v *is not final, then there is a world* $z \in W^{\Gamma}$ *s.t.* $v\mathsf{R}^{\Gamma}_{\preccurlyeq}z$*,* $z \nVdash_{V^{\Gamma}} \mathtt{D}$ *and either* $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$ *or* $\neg(z\mathsf{R}^{\Gamma}_{\preccurlyeq}v)$*.*

PROOF. Suppose there are a formula $\Box_{\preccurlyeq}\mathtt{D} \in Sub(\mathtt{B})$ and a non final world v such that $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{D}$. There are only two possible cases.

Case 1. $v \Vdash_{V^{\Gamma}} \Diamond_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{D}$. Hence, since the instance of Axiom $Dum_{\Box_{\preccurlyeq}}$ with respect to the formula $\mathtt{D}$ is true in the model $\mathcal{M}^{\Gamma}$ (for the way we defined Γ, and in particular Γ_1), we have $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}(\Box_{\preccurlyeq}(\mathsf{K_e}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}) \to \mathsf{K_e}\mathtt{D})$, and then $v \Vdash_{V^{\Gamma}} \Diamond_{\preccurlyeq}(\Box_{\preccurlyeq}(\mathsf{K_e}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}) \wedge \Diamond_{\mathsf{e}}\neg\mathtt{D})$.
Therefore there exists a world z such that:

$$(v\mathsf{R}^{\Gamma}_{\preccurlyeq}z) \;\;\&\;\; (z \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}(\mathsf{K_e}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}) \wedge \Diamond_{\mathsf{e}}\neg\mathtt{D}) \tag{4.3}$$

Let us suppose by contradiction that

(i) The formula $\mathtt{D}$ is true in each world from the $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-cluster of v, i.e. $\forall t \in W^{\Gamma}\ (v\mathsf{R}^{\Gamma}_{\mathsf{e}}t \;\Rightarrow\; t \Vdash_{V^{\Gamma}} \mathtt{D})$.

(ii) There is not any world *strictly above* w.r.t. v in which $\mathtt{D}$ is false, i.e. $\forall t \in W^{\Gamma}\ \ (v\mathsf{R}^{\Gamma}_{\preccurlyeq}t$ & $\neg(t\mathsf{R}^{\Gamma}_{\preccurlyeq}v) \;\Rightarrow\; t \Vdash_{V^{\Gamma}} \mathtt{D})$.

From (b) and (4.3) follows that $z\mathsf{R}^{\Gamma}_{\preccurlyeq}v$. By (4.3) we also have $z \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}(\mathsf{K_e}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D})$, hence $v \Vdash_{V^{\Gamma}} \mathsf{K_e}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}$. But from (a) follows that $v \Vdash_{V^{\Gamma}} \mathsf{K_e}\mathtt{D}$, therefore $v \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{D}$, which is a contradiction.

Case 2. $v \nVdash_{V^{\Gamma}} \Diamond_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{D}$. Hence $v \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}\Diamond_{\preccurlyeq}\neg\mathtt{D}$. This implies that $\mathtt{D}$

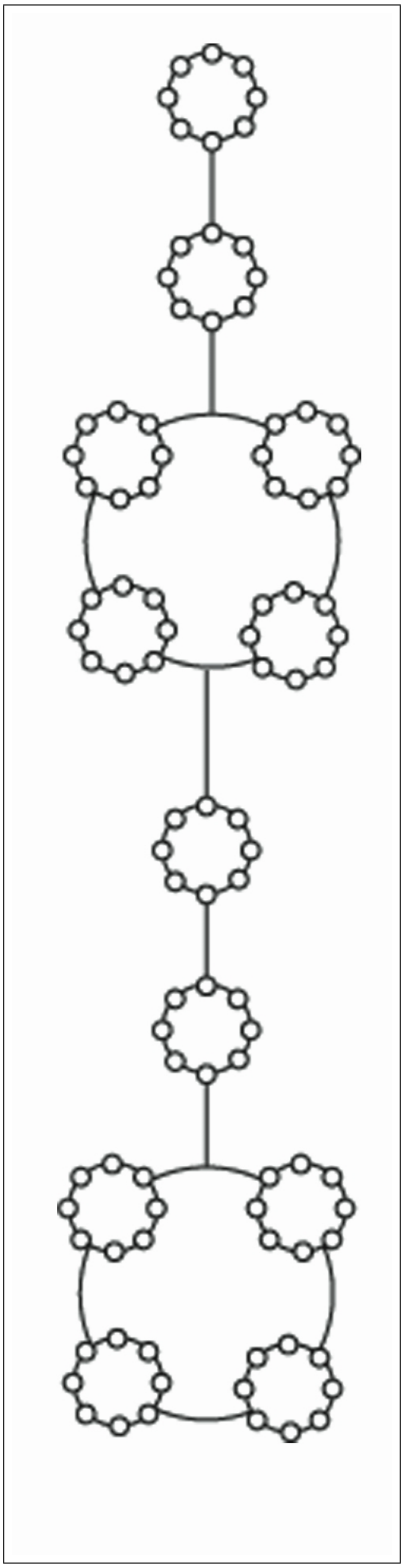

Figure 4.1: Scheme of the structure of the frame $\mathcal{F}^{\Gamma}$: a finite *generalised* reflexive LTK-balloon.

is false at least in some world from the final $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster and such world is *strictly above* w.r.t. v, which is non final by assumption. ∎

4.3.4 Completeness

We are now approaching the final step of our proof. Consider the frame $\mathcal{F}^{\Gamma}$.

(i) Fix a well ordering $\mathcal{C}_1, \ldots, \mathcal{C}_f$ among $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-clusters such that $i < l$ if and only if $(\mathcal{C}_i \mathsf{R}^{\Gamma}_{\preccurlyeq} \mathcal{C}_m) \& \neg(\mathcal{C}_m \mathsf{R}^{\Gamma}_{\preccurlyeq} \mathcal{C}_i)$.

(ii) Fix some well ordering among $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters inside any $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster, so that each $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-cluster would be taken once and only once.

Hence each single world from the base set of $\mathcal{F}^{\Gamma}$ would be displayed as $\langle v_j, i\rangle$, meaning that v belongs to the j-th $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-cluster inside the i-th $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster. Given that the number of $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-clusters is f, we stipulate that the index f denotes that a world $\langle v_j, \mathsf{f}\rangle$ belongs to the *final* $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster $\mathcal{C}_f$.

(iii) Define a new frame $\mathcal{S} = \langle W_{\mathcal{S}}, \mathsf{S}_{\preccurlyeq}, \mathsf{S}_{\mathsf{e}}, \mathsf{S}_1, \ldots, \mathsf{S}_k\rangle$ in the following way:

- $W_{\mathcal{S}} = \bigcup_{v \in W^{\Gamma}} \langle v_j, i\rangle$;

- $\langle v_j, i\rangle \mathsf{S}_{\xi} \langle z_m, l\rangle \iff v \mathsf{R}^{\Gamma}_{\xi} z$, for $1 \leq \xi \leq k$;

- $\langle v_j, i\rangle \mathsf{S}_{\mathsf{e}} \langle z_m, l\rangle \iff i = l \;\&\; j = m$;

- $\langle v_j, i\rangle \mathsf{S}_{\preccurlyeq} \langle z_m, l\rangle \iff (j \leq m \;\&\; i = l)$ or $i < l$ or $l = \mathsf{f}$, i.e. z is $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final;

(iv) Let $\mathcal{M}_{\mathcal{S}} = \langle \mathcal{S}, V^{\mathcal{S}}\rangle$ be a model such that $\forall p \in \{p_1, \ldots, p_n\}$ $V^{\mathcal{S}}(p) = \{\langle v_j, i\rangle \mid v \in V^{\Gamma}(p)\}$. Then clearly the following holds:

Lemma 4.3.17 $\forall \mathtt{D} \in Sub(\mathtt{B}) \forall v \in W^{\Gamma} \quad ((\mathcal{F}^{\Gamma}, v) \Vdash_{V^{\Gamma}} \mathtt{D} \quad \Leftrightarrow \quad (\mathcal{S}, \langle v_j, i \rangle) \Vdash_{V^{\mathcal{S}}} \mathtt{D})$.

PROOF. (i) (By induction on the length of $\mathtt{D}$). Trivially, if $lg(\mathtt{D}) = 1$, $\mathtt{D}$ has the form p, $v \in V^{\Gamma}(p)$ and hence $\langle v_j, i \rangle \in V^{\mathcal{S}}(p)$. Therefore $\langle v_j, i \rangle \Vdash_{V^{\mathcal{S}}} p$.

Suppose $\mathtt{D}$ has the form $\mathsf{K_e}\mathtt{A}$. Then $v \Vdash_{V^{\Gamma}} \mathsf{K_e}\mathtt{A}$ if and only if $\forall z \in W^{\Gamma}$ $(v\mathsf{R}^{\Gamma}_{\mathsf{e}}z \Rightarrow z \Vdash_{V^{\Gamma}} \mathtt{A})$. By inductive hypothesis (IH) for any z such that $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$ we have $\langle z_m, l \rangle \Vdash_{V^{\mathcal{S}}} \mathtt{A}$. Since $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$ implies that both v and z belong to the same $\mathsf{R}^{\Gamma}_{\preccurlyeq}$- and $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters, it follows $i = l$ and $j = m$, and hence $\langle v_j, i \rangle \mathsf{S_e} \langle z_m, l \rangle$, which means $\langle v_j, i \rangle \Vdash_{V^{\mathcal{S}}} \mathsf{K_e}\mathtt{A}$.

Suppose $\langle v_j, i \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$. Then there is a world $\langle z_m, l \rangle$ such that $\langle v_j, i \rangle \mathsf{S}_{\preccurlyeq} \langle z_m, l \rangle$ and $\langle z_m, l \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$. This means that either $i < l$, or $(i = l \;\&\; j \leq m)$ or else $l = \mathsf{f}$. Each of these cases implies $v\mathsf{R}^{\Gamma}_{\preccurlyeq}z$. By IH we have $z \nVdash_{V^{\Gamma}} \mathtt{A}$ and therefore $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{A}$.

(ii) Assume $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{A}$. Suppose v is not $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final. Then, by lemma 4.3.16, there is a world z such that $v\mathsf{R}^{\Gamma}_{\preccurlyeq}z$, $z \nVdash_{V^{\Gamma}} \mathtt{A}$ and either $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$ or $\neg(z\mathsf{R}^{\Gamma}_{\preccurlyeq}v)$. If $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$, it follows that both v and z have the same indices for the $\mathsf{R}^{\Gamma}_{\preccurlyeq}$- and $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters they belong to, i.e. they are displayed as $\langle v_i, j \rangle$ and $\langle z_i, j \rangle$. Hence $\langle v_i, j \rangle \mathsf{S}_{\preccurlyeq} \langle z_i, j \rangle$. By IH $\langle z_i, j \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$ and therefore $\langle v_i, j \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$. Else if $\neg(z\mathsf{R}^{\Gamma}_{\preccurlyeq}v)$, given that the worlds v and z are displayed as $\langle v_j, i \rangle$ and $\langle z_m, l \rangle$, it follows that $i < l$, and hence $\langle v_j, i \rangle \mathsf{S}_{\preccurlyeq} \langle z_m, l \rangle$. Again, by IH we have $\langle z_m, l \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$ and therefore $\langle v_j, i \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$. Finally suppose v is $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final. Then there is a world z which is $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final as well and it is such that $v\mathsf{R}^{\Gamma}_{\preccurlyeq}z$ and $z \nVdash_{V^{\Gamma}} \mathtt{A}$. Since z is displayed as $\langle z_m, \mathsf{f} \rangle$, it follows that $\langle v_j, \mathsf{f} \rangle \mathsf{S}_{\preccurlyeq} \langle z_m, \mathsf{f} \rangle$. By IH $\langle z_m, \mathsf{f} \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$ and therefore $\langle v_j, \mathsf{f} \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$.

■

Corollary 4.3.18 $\mathcal{S} \not\Vdash_{V^{\mathcal{S}}} \mathtt{B}$

The frame $\mathcal{S}$ has the structure depicted in Figure 4.2. This frame has the structure of a reflexive LTK-balloon.

Whenever the $\mathsf{S}_{\preccurlyeq}$-final cluster of our model is an $\mathsf{S}_{\preccurlyeq}$-proper cluster of S_{e}-clusters (i.e. not *simple*), the frame we have obtained is not an $\mathcal{LTK}$-frame. We recall that these frames have no $\mathsf{S}_{\preccurlyeq}$-proper clusters of S_{e}-clusters inside, and hence our final construction cannot be considered as a member of such class. However, this is not a problem, for it follows from [4] and [9] that these structures are nothing but *p-morphic* images of $\mathcal{LTK}$-frames.

Nevertheless, for the sake of completeness, we present an explicit way to turn our formerly obtained frame to an $\mathcal{LTK}$-frame. This will be done by applying a variant of the well known *bulldozing* technique developed by Segerberg in [65]. We aim at substituting the final $\mathsf{S}_{\preccurlyeq}$-proper cluster with an infinite, linear and discrete sequence of S_{e}-clusters in a truth preserving way.

We proceed as follows:

(i) Define for each world $\langle v_j, i\rangle$ a new collection of worlds as follows, where $\mathcal{C}_i$ denotes the $\mathsf{S}_{\preccurlyeq}$-cluster generated by $\langle v_j, i\rangle$:

$$\langle v_j, i\rangle^+ := \begin{cases} \{\langle v_j, i, \emptyset\rangle\} & \text{if } \mathcal{C}_i \text{ is simple w.r.t. } \mathsf{S}_{\mathsf{e}}\text{-clusters} \\ \{\langle v_j, i, h\rangle \mid h = 1, 2, \ldots\} & \text{otherwise} \end{cases}$$

Notice that if if $\mathcal{C}_i$ is simple w.r.t. S_{e}-clusters, it contains at most 1 S_{e}-cluster and hence it is not $\mathsf{S}^{\preccurlyeq}$-final.

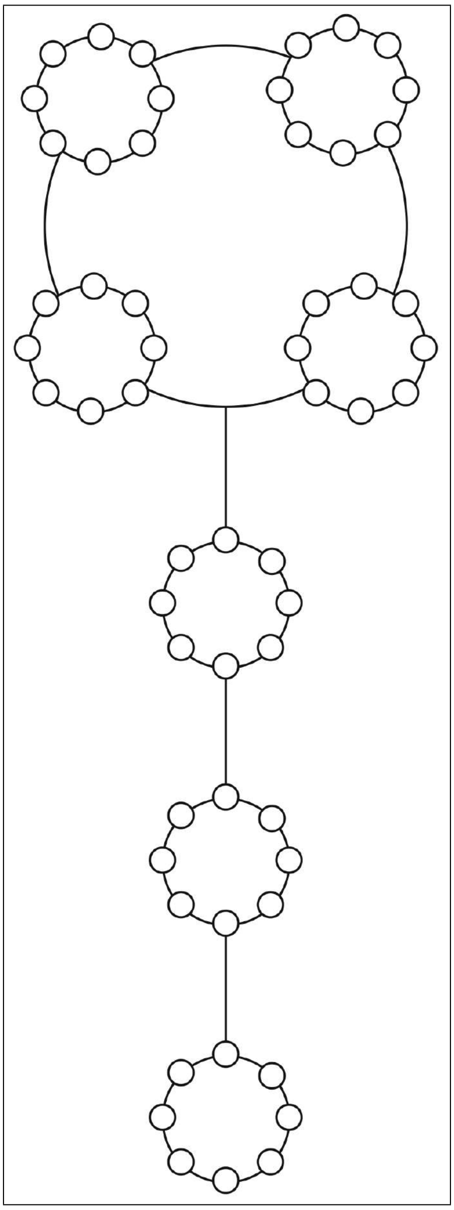

Figure 4.2: Scheme of the structure of $\mathcal{S}$: a case of reflexive LTK-balloon.

(ii) Define a new frame $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$ as:

- $W_{\mathcal{F}} := \bigcup_{\langle v_j, i \rangle \in W_{\mathsf{S}}} \cup \langle v_j, i \rangle^+$

- $\langle v_j, i, h \rangle \mathsf{R}_{\xi} \langle z_m, l \rangle \iff (j = m) \;\&\; (i = l) \;\&\; (h = o) \;\&\; \langle v_j, i \rangle \mathsf{S}_{\xi} \langle z_m, l \rangle$, for $1 \leq \xi \leq k$

- $\langle v_j, i, h \rangle \mathsf{R}_{\mathsf{e}} \langle z_m, l, o \rangle \iff (j = m) \;\&\; (i = l) \;\&\; (h = o)$

- $$\langle v_j, i, h \rangle \mathsf{R}_{\preccurlyeq} \langle z_m, l, o \rangle \iff \begin{cases} (j \leq m) \;\&\; (i = l) \;\&\; (h = o) & \text{or} \\ (h, o \neq \emptyset) \;\&\; (h < o) & \text{or} \\ (h = o = \emptyset) \;\&\; (i < l) & \text{or} \\ (h = \emptyset) \;\&\; (o \neq \emptyset) & \end{cases}$$

(iii) Define a new model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ with the following valuation V:

$\forall p \in \{p_1, \ldots, p_n\}\ V(p) := \{\langle v_j, i, h \rangle \mid \langle v_j, i \rangle \in V^{\mathsf{S}}(p)\}$.

This operation is clearly truth preserving, and therefore we can state the following:

Theorem 4.3.19 *$\forall \mathtt{B} \in Fma(\mathcal{L}^{\mathsf{LTK}})$ if $\mathtt{B} \notin \mathsf{LTK}_{\mathsf{ax}}$ then there is an $\mathcal{LTK}$-frame $\mathcal{F}$ such that $\mathcal{F} \nVdash \mathtt{B}$.*

Corollary 4.3.20 (Soundness and Completeness, C. and Rybakov [11])

(i) $\mathsf{LTK}_{\mathsf{ax}} = \mathsf{LTK}$

(ii) $\mathsf{LTK}^{-}_{\mathsf{ax}} = \mathsf{LTK}^{-}$

(iii) $\mathsf{LTK}^{--}_{\mathsf{ax}} = \mathsf{LTK}^{--}$

Corollary 4.3.21 $\mathsf{LTK}^{1}_{\mathsf{ax}}$ *(the version of* $\mathsf{LTK}_{\mathsf{ax}}$ *with only one agent operating in the system) has the effective finite model property and it is decidable w.r.t. admissible inference rules.*

4.4 A General Case: GLTK

So far we have presented a semantic framework for reasoning about time and knowledge which can be useful whenever the flow of time is considered as linear and discrete and only one situation (environment) is possible at each moment. However, we might be interested in generalising such an approach and presenting a system based on more general theoretical assumptions. A *generalised* $\mathcal{LTK}$-frame can be thought as a structure which is identical to an $\mathcal{LTK}$-frame except for the fact that it allows distinct environment-clusters to be concurrent (see Figure 4.3)[3]. This aspect may result of use whenever we aim at reasoning about simultaneous alternatives to a given state of affairs without assuming the time as branching.

The logic GLTK generated by this class of generalised frames can be easily proven to be decidable with respect to its theorems.

To prove this claim it is sufficient merely to modify our previous proof in the following way:

(i) Let $\mathcal{AS}_{\mathsf{GLTK}}$ be an axiomatic system obtained by deleting Axiom $Dum_{\Box_{\preccurlyeq}}$ from $\mathcal{AS}_{\mathsf{LTK}}$ and let $\mathsf{GLTK}_{\mathsf{ax}}$ be the set of theorems generated by

[3] Following the terminology previously used, a *generalized* $\mathcal{LTK}$-frame can be understood as an $\mathcal{LTK}$-frame lacking the property PM.3.

$\mathcal{AS}_{\mathsf{GLTK}}$.

(ii) Trivially, delete part (b) from Theorem 4.2.1;

(iii) Change the filtration set Γ to $\Gamma^- := \Gamma_0 \cup \Gamma_2^- \cup \Gamma_3^-$ where:

- $\Gamma_0 := Sub(\mathtt{B})$
- $\Gamma_2^- := \{\mathsf{K_e}\Box_{\preccurlyeq}\mathtt{D} \mid \Box_{\preccurlyeq}\mathtt{D} \in \Gamma_0\}$
- $\Gamma_3^- := \{\mathsf{K}_i\mathsf{K_e}\mathtt{D} \mid \mathsf{K_e}\mathtt{D} \in \Gamma_0 \cup \Gamma_2^-\}$ for $1 \leq i \leq k$

At the end of the process of filtration, we obtain a model based on a finite *generalised* $\mathcal{LTK}$-frame.

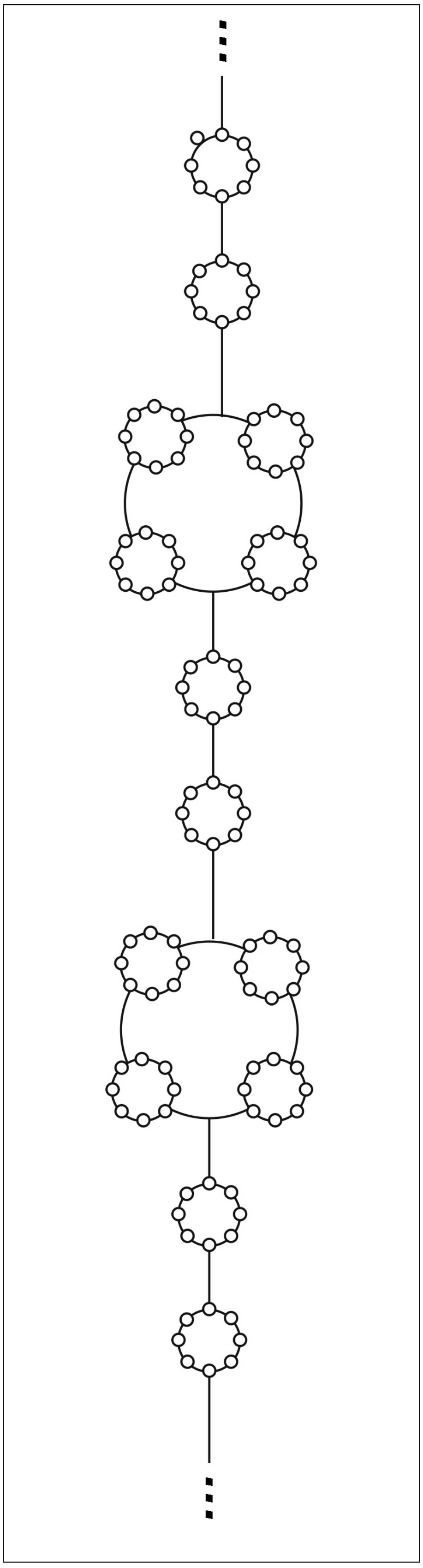

Figure 4.3: Scheme of the structure of a generalized $\mathcal{LTK}$-frame.

Chapter 5

Rules in LTK_1: Structural Incompleteness

In this last Chapter we shall analyse both syntactical and semantic aspects of inference rules. From the results presented in the previous chapters, it follows that we are now able to recognise both theorems and admissible inference rules for the logic LTK_1. In this Chapter we shall describe our last result, namely that the logic LTK_1 is not *structurally complete*. Intuitively this means that there are rules which are not syntactically derivable, but these same rules are, nevertheless, admissible. Thus, there are rules which are valid in LTK_1 even though it is not possible to derive them in its axiomatic system. Moreover, there is an infinite number of these rules. We shall define an infinite class of admissible and not derivable rules. This result is important in order to shorten derivations in $\mathsf{LTK}^1_{\mathsf{ax}}$. In fact since all these rules are valid, they may be applied in syntactical derivations without altering the set of theorems of LTK_1.

In the previous chapters, we used only Kripke (or Relational) Semantics. As we have pointed out in Chapter 2, however, this is not the only semantic tool developed for modal logics. Historically it is Algebraic Semantics, in fact, the first semantic framework which has been introduced. The algebraic analysis of modal logics provides many well known results which can be also applied to our specific case. We shall therefore introduce algebraic semantics tools in order to translate the results we have obtained in this new framework. Moreover we shall clarify all the links which occur between Algebraic and Relational semantics. This shall put our results on a wider framework.

In this Chapter we shall use *all* the results described in the previous chapters, which are briefly summarised below:

- Chapter 2. The logic LTK has the *effective finite model property* (cf. Theorem 2.3.2). This means that both LTK and LTK_1 are decidable with respect to their theorems. In fact, given any formula in the language of these two logics, if this formula is not a theorem, then this formula is not valid in some finite model whose size is computable. As we shall see further in this Chapter, this result shall be used in order to check whether some rules are derivable in $\mathsf{LTK}^1_{\mathsf{ax}}$.

- Chapter 3. We designed an algorithm to recognise admissible inference rules. In particular for any inference rule which is not admissible for LTK_1 there is a finite and computable counter-model (Theorem 3.3.6). We shall use this result in order to check whether some of the rules we shall introduce in this Chapter are admissible or not.

- Chapter 4. We provided a sound and complete axiomatic system for both LTK and LTK_1 (Theorem 4.3.19). This result shall be always used in this Chapter, as we shall describe the links occurring between syntax and semantics concerning inference rules.

Summary and Methodology. This Chapter is divided in three sections.

Section 5.1 Derivable and Admissible rules. Here we formally define the notion of derivability related to inference rules. A rule is derivable inside an axiomatic system whenever one can derive its conclusion assuming its premisses. For any logic, derivability implies admissibility, whereas the converse does not always apply. We shall define an infinite class of admissible though not derivable rules for our logic LTK_1. Thus in our case the set of derivable rules is a proper subset of the one of admissible rules. This implies that LTK_1 is *structurally incomplete*. In order to show this result we shall first define recursively an infinite class of rules. Then, in Lemma 5.1.4, we show that all the rules just introduced are admissible for LTK_1. We show this by using the results provided in Chapter 3, Theorem 3.3.6. Further, we define a specific reflexive LTK_1-balloon. We use this model to prove all the lemmas which follow. In particular, in Lemma 5.1.8 we show that none of the rules introduced is derivable on $\mathsf{LTK}^1_{\mathsf{ax}}$. From this Lemma and Lemma 5.1.4 we can conclude that LTK_1 is not structurally complete.

Section 5.2 Algebraic Semantics. Since Algebraic Semantics is historically the first type of semantics which has been developed to deal with modal logics, this Section is devoted to the analysis of the tools and results it provides.

- We shall first touch upon the basic concepts of algebraic semantics for propositional multi-modal logics by providing the basics of algebraic semantics, as the concepts of algebra, matrixes, valuations *et c.*.

- We turn then our attention to the definition of the so called *Tarski-Lindenbaum algebras*, special algebraic constructions which play a role analogous to canonical models in Kripke Semantics. In order to do so, we start by giving definitions and defining truth preserving operations on algebras, i.e. homomorphisms, isomorphisms, generated subalgebras and the relations occurring between such constructs.

- Here we introduce the *Stone's Representation theorems* which link Kripke-frames to algebraic structures and *vice versa*. In order to do so the definitions of lattices, filters and ultrafilters plus some additional well known results are needed.

- Finally we introduce some well known results which link inference rules to quasi-identities and we apply these results to our case.

Section 5.3 Further work. We shall describe here the piece of research we are currently working on. We start to investigate the problem of finding a finite basis for admissible inference rules. In fact, we aim at finding a set of rules to *axiomatise* all the inference rules admissible for LTK_1, i.e. the smallest set of rules starting from which one can derive all the admissible rules for LTK_1. This topic, as we shall see, is rather problematic and it is currently an open research field. We introduce the reader to the problems related to such investigation and we show our attempts to solve these problems.

5.1 Derivable and Admissible Rules

When facing an axiomatic system, one is usually interested in varying either its axioms or its inference rules while keeping the set of generated theorems consistent. The problem of how to vary the set of inference rules, however, does not admit a general and straightforward solution. Nevertheless having a larger number of applicable inference rules is extremely useful in order to shorten derivations.

In Chapter 3 we have designed an algorithm which recognises admissible rules for the logic LTK_1. In this Chapter we would rather turn our attention to some syntactic aspects related to the study of inference rules. As we have already seen in Chapter 3, the class of admissible rules is the set of all those inferences under which a logic is closed, i.e. the set of all those rules which can be implemented for a given logic without altering its set of theorems. Thus the notion of admissible rules of a logic is invariant in the sense that it does not depend on the choice of the axiomatic system one uses. This notion is extremely general and comprehensive and should not be confused with the syntactical concept of *derivable* rules. From an intuitive point of view, a rule is derivable in an axiomatic system whenever there is a derivation of its conclusions given its premisses as assumptions. This means that a rule is derivable in the axiomatic system $\mathcal{AS}$ of a logic L if:

(i) there is a derivation of its conclusion assuming its premisses

(ii) the derivation is carried out using only axioms and rules from the axiomatic system itself. More formally this is:

Definition 5.1.1 (Derivability) *Given a logic* L *generated by the axiomatic system* $\mathcal{AS}$*, an inference rule* $\mathbf{r} = \mathtt{A}_1, \ldots, \mathtt{A}_n/\mathtt{B}$ *is derivable in* $\mathcal{AS}$ *(in symbols* $\vdash_{\mathcal{AS}} \mathbf{r}$*) if* $\mathtt{A}_1 \wedge \ldots \wedge \mathtt{A}_n \vdash_{\mathcal{AS}} \mathtt{B}$.

The collection of all the rules derivable in a logic depends on the choice of the axiomatic system one uses. In fact the set of derivable rules of a logic changes according to which axiomatic system one choses in order to axiomatise the logic itself. Clearly each derivable rule is also admissible by definition. On the other hand the set of admissible rules for a logic contains *all* the rules under which the logic itself is closed. The set of derivable rules of a logic is, therefore, a subset of the class of all the rules admitted by the logic itself. Hence derivability implies admissibility, whereas the converse might not apply.

In some systems the set of derivable rules and the one of admissible rules coincide. We call these logics *structurally complete* as they are, in some sense, self contained. If a logic is structurally complete, then any admissible rule can be derived in its axiomatic system. More formally this is:

Definition 5.1.2 *A logic* L *generated by an axiomatic system* $\mathcal{AS}$ *is structurally complete if and only if for any inference rule* $\mathbf{r}$ *which is admissible for* L*,* $\vdash_{\mathcal{AS}} \mathbf{r}$*.*

For many logics, however, the set of derivable rules is a *proper* subset of the class of admissible rules. These logics admit rules which are not syntactically derivable. For instance Harrop (see [32]) discovered that the Intuitionistic Propositional Calculus IPC admits rules which are not derivable in the system itself: this system is not structurally complete. This extremely interesting result led a great deal of research in the area.

In this Chapter we shall give an answer to the question whether LTK_1 is structurally complete.

As we have anticipated, many well known logics admit rules which are not derivable in their axiomatic systems. We shall show that the system LTK_1 is no exception and that it actually admits an infinite number of rules which are not derivable in its axiomatic system. Before turning our attention to the analysis of these rules, however, we would provide some technical details we shall use in what follows. The following Lemma shows that if a rule has the specific structure described below, than such a rule is admissible for LTK_1.

Lemma 5.1.3 *Any inference rule with a structure as* $\Diamond_{\preccurlyeq} x \rightarrow (\Diamond_{\mathsf{e}} y \wedge \Diamond_{\mathsf{e}} \neg y)/\neg x$ *is admissible for* LTK_1*.*

PROOF. Suppose by contradiction that a rule $\mathbf{r} := \Diamond_{\preccurlyeq} x \to (\Diamond_{\mathsf{e}} y \wedge \Diamond_{\mathsf{e}} \neg y)/\neg x$ is not admissible for LTK_1. Then by Theorem 3.3.6 there is a finite LSP-frame $\mathcal{F}$ (cf. Definition 3.3.5), a valuation V for $\mathcal{F}$ and a world v such that $\mathcal{F} \Vdash_V \Diamond_{\preccurlyeq}\mathtt{A} \to (\Diamond_{\mathsf{e}}\mathtt{B} \wedge \Diamond_{\mathsf{e}}\neg\mathtt{B})$ and $(\mathcal{F}, v) \nVdash \neg\mathtt{A}$, i.e. $(\mathcal{F}, v) \Vdash \mathtt{A}$. Since the *dot-world* d associated to v is $\mathsf{R}_{\preccurlyeq}$-related to v, it follows that $(\mathcal{F}, d) \Vdash_V \Diamond_{\preccurlyeq}\mathtt{A}$ and therefore $(\mathcal{F}, d) \Vdash \Diamond_{\mathsf{e}}\mathtt{B} \wedge \Diamond_{\mathsf{e}}\neg\mathtt{B}$. Since, by definition of $\mathcal{F}$, the world d is a single element R_{e}-cluster, we reach a contradiction.

∎

We shall now recursively define an infinite class of rules in the language of LTK_1. For any $n \in \mathbb{N}$, let $\phi(n)$ be the following formula:

1. $\phi_1 := x_1 \wedge \Diamond_{\preccurlyeq}(\Box_{\preccurlyeq}\neg x_1)$
2. $\phi_{n+1} := x_{n+1} \wedge \Diamond_{\preccurlyeq}(\Box_{\preccurlyeq}\neg x_{n+1} \wedge \phi_n)$

Let

$$\mathbf{r}_0 := \frac{\Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}x_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg x_0) \to (\Diamond_{\mathsf{e}}y \wedge \Diamond_{\mathsf{e}}\neg y)}{\neg\Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}x_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg x_0)}$$

$$\mathbf{r}_n := \frac{\Diamond_{\preccurlyeq}\phi_n \to (\Diamond_{\mathsf{e}}y \wedge \Diamond_{\mathsf{e}}\neg y)}{\neg\phi_n}$$

Let $\mathcal{R} := \{\mathbf{r}_0\} \cup \{\mathbf{r}_n \mid n \in \mathbb{N}\}$.

Lemma 5.1.4 *Any inference rule from the set $\mathcal{R}$ is admissible in LTK_1.*

PROOF. According to Theorem 3.3.6 (page 73), an inference rule $\mathbf{r} = \mathtt{A}_1, \ldots, \mathtt{A}_n/\mathtt{B}$ is admissible in LTK_1 if and only if for any model $\mathcal{M}$ based on a finite LSP-frame the following implication holds: if $\mathcal{M} \Vdash \mathtt{A}_1, \ldots, \mathtt{A}_n$ then $\mathcal{M} \Vdash \mathtt{B}$.

(i) Suppose by contradiction that $\mathbf{r}_0$ is not admissible for LTK_1. From Theorem 3.3.6 it follows that there are a finite LSP-frame $\mathcal{F}$, a valuation V for $\mathcal{F}$ and a world v such that $\mathcal{F} \Vdash_V \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0) \to (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$ and $(\mathcal{F}, v) \nVdash \neg\Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0)$. The last condition implies that

$(\mathcal{F}, v) \Vdash \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0)$. In particular in the maximal $\mathsf{R}_{\preccurlyeq}$-cluster (of depth 1) there ought to be two R_{e}-clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1 \Vdash \mathsf{K}_{\mathsf{e}}p_0$ and $\mathcal{C}_2 \Vdash \mathsf{K}_{\mathsf{e}}\neg p_0$. If d is the name of the *dot-world* associated to v (i.e. that single element R_{e}-cluster which is an immediate $\mathsf{R}_{\preccurlyeq}$-predecessor of v), it follows that $(\mathcal{F}, d) \Vdash_V \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0)$ and since $(\mathcal{F}, d) \Vdash_V \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0) \to (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$ we obtain $(\mathcal{F}, d) \Vdash_V (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$, which is contradictory to the fact that d is a single-element R_{e}-cluster.

(ii) Any rule $\mathbf{r}_n \in \mathcal{R}$ has a structure like the one described in Lemma 5.1.3 and it is, therefore, admissible for LTK_1.

∎

We shall now construct a specific LTK_1-balloon we shall use in the following lemmas. Let $\mathcal{F}$ be any finite LTK_1-balloon such that:

(i) $dp(\mathcal{F}) = 2n + 1$ for some $n \geq 1$;

(ii) $\forall w \in |\mathcal{F}|$ if $dp(w)$ is EVEN, then $\mathcal{C}_{\mathsf{e}}(w) := \{w\}$;

(iii) $\forall w \in |\mathcal{F}|$ if $dp(w)$ is ODD, then $\| \bigcup \mathcal{C}_{\mathsf{e}}(w) \| \geq 2$;

(iv) There are at least 2 R_{e}-clusters of depth 1.

Let $\mathcal{C}^1_1, \ldots, \mathcal{C}^1_m$ be an enumeration of all the R_{e}-cluster of depth 1. For each R_{e}-cluster $\mathcal{C}^1_i$ of depth 1, let $w_i(1,1), \ldots, w_i(1,l)$ be an enumeration of all the worlds it contains.

For any R_{e}-cluster whose depth is i, where $i \geq 2$, define a well ordering of the worlds it contains. Display any world as $w(i,m)$, meaning that $w(i,m)$

is the m-th world of depth i.

Let $t, p_0, p_1, \ldots, p_n$ be some propositional letters.

Define a model on $\mathcal{F}$ as follows:

- $V(t) := \{w(i,m) \mid i$ and m are ODD $\}$
- $V(p_0) := \{w \mid dp(w) = 1 \;\&\; w \in \mathcal{C}_1^1\}$
- $\forall i,\ 1 \leq i \leq n \;\; V(p_i) := \{w \mid dp(w) = 2i+1\}$

The variable occurring in the rule $\mathbf{r}_n$ from $\mathcal{R}$ are exactly $y, x_0, x_1, \ldots, x_n$. Fix a mapping $f : Var(\mathbf{r}_n) \mapsto \{t, p_0, p_1, \ldots, p_n\}$ such that:

- $f(y) = t$
- $f(x_i) = p_i$ for $0 \leq i \leq n$.

Please refer to Figure 5.1 for a graphical representation of this model.

Lemma 5.1.5 *In the model $\langle \mathcal{F}, V\rangle$ as depicted in Figure 5.1 the following holds:*

$$\forall m \forall w \;\; w \Vdash_V \phi_m \;\Leftrightarrow\; dp(w) = 2m+1$$

PROOF. (i) The right direction of the implication is easy. Suppose $w \Vdash_V \phi_m$; it follows that $w \Vdash_V p_m \wedge \Diamond_{\preccurlyeq}(\Box_{\preccurlyeq}\neg p_m \wedge \phi_{m-1})$. This implies that $w \in V(p_m)$ and hence $dp(w) = 2m+1$ by definition of V.

(ii) Suppose $dp(w) = 2m+1$. By induction on m:

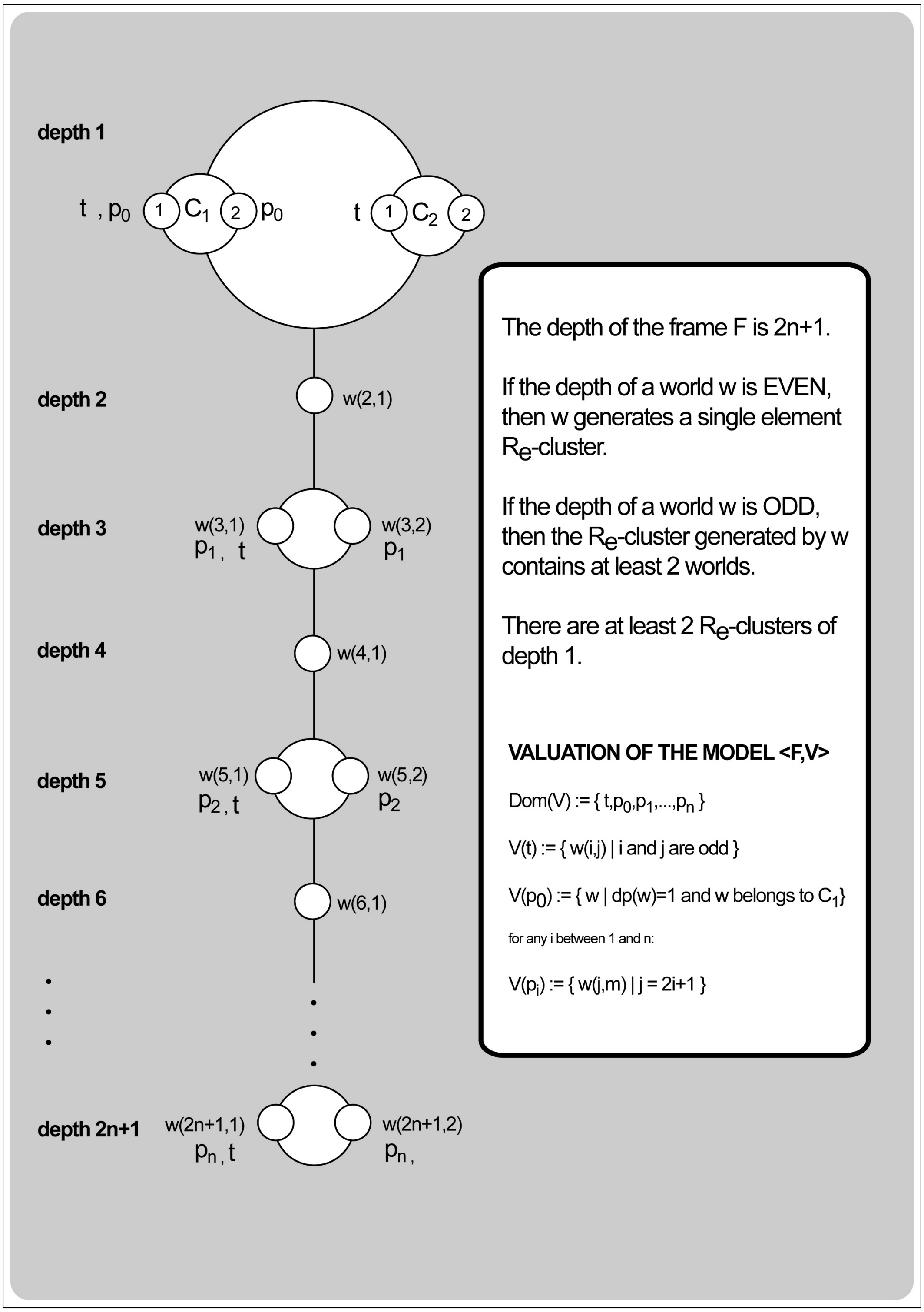

Figure 5.1: Scheme of the model $\langle \mathcal{F}, V\rangle$ based on a LTK$_1$-balloon used in Lemmas 5.1.5, 5.1.6, 5.1.8 and 5.1.7.

(a) $m = 1$. Then $dp(w) = 3$ and $w \Vdash_V p_1$. Moreover, by construction, $w \Vdash_V \Diamond_{\preccurlyeq}\Box_{\preccurlyeq}\neg p_1$ and therefore $w \Vdash_V \phi_1$.

(b) $m = j+1$. Then $dp(w) = 2(j+1)+1 = 2j+3$. By definition of V we get $w \Vdash_V p_{j+1}$ and for any world v such that $w\mathsf{R}_{\preccurlyeq}v$ and $\neg(v\mathsf{R}_{\preccurlyeq}w)$ (i.e. $dp(w) > dp(v)$) we have $v \Vdash_V \Box_{\preccurlyeq}\neg p_{j+1}$. By assumption we have that $dp(w) \geq 5$ and then by IH there is a world z such that $dp(z) = 2j+1$, $dp(z) \geq 3$ and $z \Vdash_V \phi_j$. Since $dp(z) < dp(w)$ (i.e. $w\mathsf{R}_{\preccurlyeq}z$) it follows that $z \Vdash_V \phi_j \wedge \Box_{\preccurlyeq}\neg p_{j+1}$ and hence $w \Vdash_V \Diamond_{\preccurlyeq}(\phi_j \wedge \Box_{\preccurlyeq}\neg p_{j+1})$. Therefore $w \Vdash_V \phi_{j+1}$

∎

Lemma 5.1.6 *In the model $\langle \mathcal{F}, V\rangle$ as depicted in Figure 5.1 the following holds: $\forall w \forall z$ if $dp(w) = 2m+1$ and $dp(z) \geq 2m+1$ then $z \Vdash_V \Diamond_{\preccurlyeq}\phi_m$.*

Lemma 5.1.7 *In the model $\langle \mathcal{F}, V\rangle$ as depicted in Figure 5.1 the following holds: for any subframe $\mathcal{F}_i$ generated by a world w such that $dp(w) = i$ and i is even $\mathcal{F}_i \Vdash I(\sigma(\mathbf{r}_m))$ for any m.*

Proof. By induction on n.

(i) $n = 1$, then the depth of $\mathcal{F}_i$ is 3. Hence the only possible case is the frame $\mathcal{F}_2$ generated by a world w of depth 2. By induction on m.

(a) $m = 1$. Suppose $\mathcal{F}_2 \nVdash I(\sigma(\mathbf{r}_1))$, then there are a valuation S and a world w such that $\mathcal{F}_2 \Vdash_S \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_1 \to (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$ and $(\mathcal{F}_2, w) \nVdash_S \neg\phi_1$. Then $w \Vdash_S \phi_1$ and by Lemma 5.1.5 we have $dp(w) = 3$ which is in contradiction with the assumption that $\mathcal{F}$ has depth 2.

(b) $m = k+1$. Suppose $\mathcal{F}_2 \nVdash I(\sigma(\mathbf{r}_{k+1}))$. Again this implies that there are a valuation S and a world w such that $\mathcal{F}_2 \Vdash_S \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_{k+1} \to (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$ and $(\mathcal{F}_2, w) \nVdash_S \neg\phi_{k+1}$. Then $w \Vdash_S \phi_{k+1}$ and by Lemma 5.1.5 we have

$dp(w) = 2k+3$. Since $2k+3 \geq 5$, this is in contradiction with the assumption that $\mathcal{F}$ has depth 2.

(ii) $n = j+1$. We consider here only the subframe generated by a world w of depth $2j+2$ as the other cases can be easily shown using our inductive hypothesis.

By induction on m.

(a) $m = 1$. Suppose $\mathcal{F}_{2j+2} \nVdash I(\sigma(\mathbf{r}_1))$, then there are a valuation S and a world w such that $\mathcal{F}_{2j+2} \Vdash_S \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_1 \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$ and $(\mathcal{F}_2, w) \nVdash_S \neg\phi_1$. Then $w \Vdash_S \phi_1$ and by Lemma 5.1.5 we have $dp(w) = 3$. Since $\mathcal{F}_{2j+2}$ is deep at least 4, there is a world v of depth 4 wich is, by construction, a single element R_{e}-cluster. Clearly $v \Vdash_S \Diamond_{\preccurlyeq}\phi_1$ and $v \nVdash_S (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$ which is in contradiction with the assumption that $\mathcal{F}_{2j+2} \Vdash_S \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_1 \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$.

(b) $m = k+1$. Suppose $\mathcal{F}_{2j+2} \nVdash I(\sigma(\mathbf{r}_{k+1}))$, then there are a valuation S and a world w such that $\mathcal{F}_{2j+2} \Vdash_S \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_{k+1} \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$ and $(\mathcal{F}_2, w) \nVdash_S \neg\phi_{k+1}$. Then $w \Vdash_S \phi_{k+1}$ and by Lemma 5.1.5 we have $dp(w) = 2k+3$, meaning that w is odd. Since $\mathcal{F}_{2j+2}$ is generated by an even world, this means that there must be a world of depth $2k+4$ which is an immediate $\mathsf{R}_{\preccurlyeq}$-predecessor of w and whose depth is even. Again, consider such v: this world generates, by construction, a single element R_{e}-cluster. Clearly $v \Vdash_S \Diamond_{\preccurlyeq}\phi_{k+1}$ and $v \nVdash_S (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$ which is in contradiction with the assumption that $\mathcal{F}_{2j+2} \Vdash_S \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_{k+1} \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$.

∎

Remark. In the light of Definition 5.1.1, we can derive the following double implication: for any inference rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_n/\mathtt{B}$, for any logic L

generated by the axiomatic system $\mathcal{AS}$,

$$\vdash_{\mathcal{AS}} \mathbf{r} \;\Leftrightarrow\; (\mathtt{A}_1 \wedge \ldots \wedge \mathtt{A}_n) \vdash_{\mathcal{AS}} \mathtt{B}$$

In any normal modal logic, this means that $\mathbf{r}$ is derivable if and only if the implication $\Box(\mathtt{A}_1 \wedge \ldots \wedge \mathtt{A}_n) \to \mathtt{B}$ is a theorem of the logic itself[1]. In the case of LTK_1, this implication would be just $\Box_{\preccurlyeq}(\mathtt{A}_1 \wedge \ldots \wedge \mathtt{A}_n) \to \mathtt{B}$ as the modal operator $\Box_{\preccurlyeq}$ is the *strongest* modality as defined by the axioms[2]. Given a rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_n/\mathtt{B}$, we shall henceforth refer to the implication $\Box_{\preccurlyeq}(\mathtt{A}_1 \wedge \ldots \wedge \mathtt{A}_n) \to \mathtt{B}$ as $I(\mathbf{r})$[3]. The implication $I(\mathbf{r})$ is hence the implication associated to $\mathbf{r}$. This implies that in LTK_1 in order to prove that an inference rule is derivable it is sufficient to show that the implication $I(\mathbf{r})$ is a theorem of LTK_1. As LTK_1 has the *efmp* (see Chapter 2, Theorem 2.3.2), it is enough to show that the assumption that $I(\mathbf{r})$ is falsified by some finite reflexive LTK_1-balloon leads to a contradiction.

Lemma 5.1.8 *In the model $\langle \mathcal{F}, V\rangle$ as depicted in Figure 5.1 the following holds: for any submodel $\langle \mathcal{F}_i, V\rangle$ generated by a world w such that $dp(w) = i$ and i is odd, $(\mathcal{F}_i, w) \not\Vdash_V I(\sigma(\mathbf{r}_m))$ for $m = \frac{i-1}{2}$.*

Proof. By induction on n where $2n + 1$ is the depth of the frame $\mathcal{F}$.

(i) $n = 1$, then $\mathcal{F}$ has depth 3. We can have only two cases: the submodel generated by a world of depth 1 and the one generated by a world of depth 3. For both cases we should show that $\mathcal{F}_i \not\Vdash_V I(\sigma(\mathbf{r}_m))$ for $m = \frac{i-1}{2}$.

[1] cf. Troelstra and Schwichtenberg [68], page 285.

[2] As we have stated in Chapter 4, page 87, necessitation rules for the modal operators $\mathsf{K}_\mathsf{e}, \mathsf{K}_\mathsf{a}$ can be derived by means of the necessitation rule for $\Box_{\preccurlyeq}$ and the axioms $M.1$ and $M.3$.

[3] Notice that in following lemmas we might sometimes use $I(\sigma(\mathbf{r}))$. This means that we take under consideration the implication associated to the inference rule $\mathbf{r}$ under the substitution σ.

(a) $dp(\mathcal{F}_1) = 1$. Clearly we only have to show that $I(\sigma(\mathbf{r}_0)) := \Box_{\preccurlyeq}(\Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0) \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\preccurlyeq}\neg t) \rightarrow \neg\Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0))$ does not hold. Notice that $\langle\mathcal{F}_1, V\rangle$ is the $\mathsf{R}_{\preccurlyeq}$-maximal part of $\langle\mathcal{F}, V\rangle$. Trivially $\mathcal{F}_1 \Vdash_V \Diamond_{\mathsf{e}}t \wedge \Diamond_{\preccurlyeq}\neg t$ by construction, then $\mathcal{F}_1 \Vdash_V \Box_{\preccurlyeq}(\Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0) \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\preccurlyeq}\neg t))$ holds. Moreover $\mathcal{F}_1 \Vdash_V \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0$, then $\mathcal{F}_1 \Vdash_V \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0)$ and hence $\mathcal{F}_1 \nVdash_V \neg\Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}p_0 \wedge \Diamond_{\preccurlyeq}\mathsf{K}_{\mathsf{e}}\neg p_0)$; therefore $\mathcal{F}_1 \nVdash_V I(\sigma(\mathbf{r}_0))$.

(b) $dp(\mathcal{F}_3) = 3$. We should now show that $I(\sigma(\mathbf{r}_1))$ does not hold in the model, where $I(\sigma(\mathbf{r}_1)) := \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_1 \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)) \rightarrow \neg\phi_1$. By Lemma 5.1.5 it follows that for any world w in the base set of $\mathcal{F}_3$, $dp(w) = 3 \Leftrightarrow w \Vdash_V \phi_1$. Consider a world w of depth 3, clearly $w \nVdash_V \neg\phi_1$ and hence, by reflexivity, $w \Vdash_V \Diamond_{\preccurlyeq}\phi_1$. Moreover $dp(w)$ is odd, and then by construction we have $w \Vdash_V \Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t$ and for any world v such that $w\mathsf{R}_{\mathsf{e}}v$ (i.e. w, v have the same depth) we have $v \Vdash_V \Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t$ and hence $v \Vdash_V \Diamond_{\preccurlyeq}\phi_1 \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$. Consider now any world z such that $dp(z) < dp(w)$ (i.e. $w\mathsf{R}_{\preccurlyeq}z$ and $\neg(z\mathsf{R}_{\preccurlyeq}w)$); clearly $z \nVdash_V \Diamond_{\preccurlyeq}\phi_1$ and hence $z \Vdash_V \Diamond_{\preccurlyeq}\phi_1 \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$ holds true. From these observations it follows that $w \Vdash_V \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_1 \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$ which with $w \nVdash_V \neg\phi_1$ implies $w \nVdash_V I(\sigma(\mathbf{r}_1))$.

(ii) $n = j+1$, then the depth of $\mathcal{F}$ is $2(j+1)+1 = 2j+3$. We should show that for any submodel of $\langle\mathcal{F}_{2j+3}, V\rangle$ generated by an odd world w of depth i, $\mathcal{F}_i \nVdash_V I(\sigma(\mathbf{r}_m))$ for $m = \frac{i-1}{2}$. By induction hypothesis we can state that the claim holds for any submodel generated by an odd world from $\langle\mathcal{F}_i, V\rangle$ where $i \leq 2j+1$ i.e. $n \leq j$. Hence we must show only that in the model $\langle\mathcal{F}_{2j+3}, V\rangle$ the implication $I(\sigma(\mathbf{r}_m))$ is not true, where $m = \frac{2j+3-1}{2} = j+1$. Recall that $I(\sigma(\mathbf{r}_{j+1})) := \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_{j+1} \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)) \rightarrow \neg\phi_{j+1}$. Consider a root-world, i.e. a world w such that $dp(w) = 2j+3$; by Lemma 5.1.5 we have $w \not\Vdash_V \neg\phi_{j+1}$. Although by reflexivity we have that $w \Vdash_V \Diamond_{\preccurlyeq}\phi_{j+1}$, by construction it holds $w \Vdash_V \Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t$ and hence $w \Vdash_V \Diamond_{\preccurlyeq}\phi_{j+1} \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$. For any world v such that $w\mathsf{R}_{\mathsf{e}}v$ (i.e. w, v have the same depth) we have $v \Vdash_V \Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t$ and hence $v \Vdash_V \Diamond_{\preccurlyeq}\phi_{j+1} \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$. Consider now any world z such that $dp(z) < dp(w)$ (i.e. $w\mathsf{R}_{\preccurlyeq}z$ and $\neg(z\mathsf{R}_{\preccurlyeq}w)$); clearly $z \not\Vdash_V \Diamond_{\preccurlyeq}\phi_{j+1}$ and hence $z \Vdash_V \Diamond_{\preccurlyeq}\phi_{j+1} \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t)$ holds true. From these observations it follows that $w \Vdash_V \Box_{\preccurlyeq}(\Diamond_{\preccurlyeq}\phi_{j+1} \rightarrow (\Diamond_{\mathsf{e}}t \wedge \Diamond_{\mathsf{e}}\neg t))$ which with $w \not\Vdash_V \neg\phi_{j+1}$ implies $w \not\Vdash_V I(\sigma(\mathbf{r}_{j+1}))$.

∎

The structure depicted in Figure 5.1 is a case of reflexive LTK_1-balloon. In Lemma 5.1.8 we have showed that for any rule $\mathbf{r}_n$ in the class $\mathcal{R}$ there is a finite reflexive LTK_1-balloon in which the antecedent of $I(\mathbf{r}_n)$ is valid whereas the consequent is not. From this observation and the Remark on page 122 it follows that none of the rules in $\mathcal{R}$ is derivable in $\mathsf{LTK}^1_{\mathsf{ax}}$.

Corollary 5.1.9 (Structural Incompleteness) *All the rules from $\mathcal{R}$ are admissible but not derivable in LTK_1, therefore the logic $\mathsf{LTK}^1_{\mathsf{ax}}$ is not structurally complete.*

5.2 Algebraic semantics for LTK_1

The Possible Worlds Framework, or Kripke Semantics, provide intuitive tools and it is widely adopted. Nevertheless, as we highlighted in Chapter **??**, Section 2.2.1, Algebraic Semantics is historically the first type of semantics which has been developed to deal with modal logics (cf. Goldblatt [27]). Understanding algebraic semantics and its links with Kripke Semantics can give a wider perspective on the topic. We have therefore decided to dedicate this Section to the analysis of Algebraic Semantics and to some results which are of use in our research. Moreover, Algebraic structures shall prove themselves of use in order to highlight the semantic counterpart of the notion of derivability. All the standard results and definitions presented in this Section are taken from Blackburn *et al.* [3], Chapter 5, Rybakov [55], Chagrov and Zakharyaschev [72].

5.2.1 Basic Definitions

In order to define Algebraic Semantics for LTK_1, we need some basic definitions and Lemmas.

Definition 5.2.1 (Algebra) *An* ***algebra*** *is a structure of the form* $\mathfrak{A} = \langle A, f_1, \ldots, f_n\rangle$ *where* A *is a set, called* base set *or* universe*, and* $f_1, \ldots, f_n$ *are operations on* A*.*
An algebra is finite if its universe contains a finite number of elements.

Given a propositional language $\mathfrak{L}$, an $\mathfrak{L}$-algebra is an algebra $\mathfrak{A} = \langle A, f_1, \ldots, f_n\rangle$ such that the operations $f_1, \ldots, f_n$ correspond to the logical connectives in $\mathfrak{L}$.

Given an $\mathfrak{L}$-algebra $\mathfrak{A} = \langle A, f_1, \ldots, f_n\rangle$, a **valuation (assignment)** is a mapping $V: \ Fma(\mathfrak{L}) \mapsto A$ such that for each propositional letter $p \in P$, $V(p) \in A$ and, given a formula $\mathtt{A}(p_1, \ldots, p_n)$, $V(\mathtt{A}(p_1, \ldots, p_n)) = \mathtt{A}(V(p_1), \ldots, V(p_n))$.

Given an $\mathfrak{L}$-algebra $\mathfrak{A} = \langle A, f_1, \ldots, f_n\rangle$ and a non-empty subset Δ of A, the pair $\langle \mathfrak{A}, \Delta\rangle$ is called an **$\mathfrak{L}$-matrix** and Δ is the set of distinguished elements. Less formally, the universe A is the set of all the possible truth values, whereas Δ is the set of all the *true*, or designated, truth values.

Given an $\mathfrak{L}$-matrix $\langle \mathfrak{A}, \Delta \rangle$, a formula $\mathtt{A} \in Fma(\mathfrak{L})$ is:
true in $\langle \mathfrak{A}, \Delta \rangle$ under the valuation V, in symbols $\langle \mathfrak{A}, \Delta \rangle \models_V \mathtt{A}(p_1, \ldots, p_n)$, iff $V(\mathtt{A}(p_1, \ldots, p_n)) \in \Delta$;
valid in $\langle \mathfrak{A}, \Delta \rangle$, in symbols $\langle \mathfrak{A}, \Delta \rangle \models \mathtt{A}(p_1, \ldots, p_n)$ iff for each valuation V, $\langle \mathfrak{A}, \Delta \rangle \models_V \mathtt{A}(p_1, \ldots, p_n)$.

In what follows, we shall always be dealing with algebras with only one designated truth value, i.e. the element $\top$, and we will denote an $\mathfrak{L}$-matrix simply as $\mathfrak{A} = \langle A, f_1, \ldots, f_n, \top \rangle$. We shall also use the abbreviation $\mathfrak{A} \models_V \mathtt{A}$ instead of $\langle \mathfrak{A}, \top \rangle \models_V \mathtt{A}$. Clearly for any formula $\mathtt{A}$, $\mathfrak{A} \models_V \mathtt{A}$ if and only if $\mathtt{A} = \top$ holds under V.

Definition 5.2.2 *Given a logic* L *on a language* $\mathfrak{L}$*, an* $\mathfrak{L}$*-matrix, or just an* $\mathfrak{L}$*-algebra* $\mathfrak{A}$ *for simplicity, is* ***characterising*** *for* L *if and only if for any formula* $\mathtt{A}$*,* $\mathtt{A} \in \mathsf{L}$ *iff* $\mathfrak{A} \models \mathtt{A}$*.*

Definition 5.2.3 (Variety) *Given a logic* L *on the language* $\mathfrak{L}$*, the* ***variety generated by*** L *is the set:* $Var(\mathsf{L}) := \{\mathfrak{A} \mid \forall \mathtt{A} \in \mathsf{L} \quad \mathfrak{A} \models \mathtt{A} = \top \ \ \& \ \ \forall \mathtt{A} \forall \mathtt{B} \in Fma(\mathfrak{L})(\mathsf{L} \vdash \mathtt{A} \leftrightarrow \mathtt{B} \Rightarrow \mathfrak{A} \models \mathtt{A} = \mathtt{B})\}$*.*

Definition 5.2.4 *Given an inference rule* $\mathbf{r} = \mathtt{A}_1(p_j), \ldots, \mathtt{A}_n(p_j)/\mathtt{B}(p_j)$ *in the language of a logic* L*:*

- $\mathbf{r}$ *is valid in an algebra* $\mathfrak{A} \in Var(\mathsf{L})$*, in symbols* $\mathfrak{A} \models \mathbf{r}$*, if and only if for every valuation* V *of variables from* $\mathbf{r}$ *in* $\mathfrak{A}$*,* $\mathfrak{A} \models V(\mathtt{B}) = \top$ *provided that* $\forall i \ \ \mathfrak{A} \models V(\mathtt{A}_i) = \top$*;*

- $\mathbf{r}$ *is a semantic corollary of (or it follows semanticly from) a set of inference rules* $\mathcal{R}$*,* $\mathcal{R} \models_{\mathsf{L}} \mathbf{r}$*, if and only if for any algebra* $\mathfrak{A} \in Var(\mathsf{L})$ *if*

$\forall \mathbf{r}_i \in \mathcal{R} \;\; \mathfrak{A} \vDash_{\mathsf{L}} \mathbf{r}_i$ *then* $\mathfrak{A} \vDash_{\mathsf{L}} \mathbf{r}$.

The following result clarifys the link occurring between the concept of being derivable and the one of following semanticly from a set of inference rules: the two notions actually coincide.

Theorem 5.2.5 *Let $\mathcal{R}$ be a collection of inference rules in the language of a logic L. Then for any rule $\mathbf{r}$, $\mathcal{R} \vdash_{\mathsf{L}} \mathbf{r}$ if and only if $\mathcal{R} \vDash_{\mathsf{L}} \mathbf{r}$.*

5.2.2 The Tarski-Lindenbaum Construction

In this section we shall introduce special algebraic sructures which play a key role when dealing with admissible inference rules. These systems are called *free algebras* and we shall define them starting from the so called Tarski-Lindenbaum matrixes.

Let L be a propositional logic in the language $\mathcal{L}$. The Tarski-Lindenbaum matrix for L is given by: $\mathfrak{M}(\mathsf{L}) := \langle Fma(\mathcal{L}), Con(\mathcal{L}), \mathsf{L}\rangle$, where $Con(\mathcal{L})$ is simply the set of all the logical connectives from $\mathcal{L}$. Clearly the Tarski-Lindenbaum matrix is characterising for L. Since this kind of semantics is too general, we shall define another characteristic matrix for L with only one designated element. Let $\mathfrak{F}(\mathsf{L}) := \langle [Fma(\mathcal{L})], Con(\mathcal{L}), \top\rangle$ be an $\mathcal{L}$-matrix where:

(i) $[\mathtt{A}] := \{\mathtt{B} \in Fma(\mathcal{L}) \mid \mathtt{A} \leftrightarrow \mathtt{B} \in \mathsf{L}\}$

(ii) $[Fma(\mathcal{L})] := \{[\mathtt{A}] \mid \mathtt{A} \in Fma(\mathcal{L})\}$

(iii) for any n-ary connective $\odot$ from $Con(\mathcal{L})$ $(\odot([\mathtt{A}]_1, \ldots, [\mathtt{A}]_n) = [\odot(\mathtt{A}_1, \ldots, \mathtt{A}_n)])$

(iv) $\top := [\top]$ (Recall that $\top := \bot \to \bot$).

This definition is correct due to the equivalent replacement theorem.

Theorem 5.2.6 *For any algebraic logic L, the algebra $\mathfrak{F}(\mathsf{L})$ with the single designated element $\top$ is characterising for L.*

Two algebras $\mathfrak{A} = \langle A, f_1, \ldots, f_n\rangle$ and $\mathfrak{B} = \langle B, f'_1, \ldots, f'_m\rangle$ are **similar** if $n = m$ and the operations f_i and f'_i are of the same arity. In what follows, when dealing with similar algebras, we shall often denote the operations in both algebras with the same symbols.

Definition 5.2.7 (Homomorphisms, Isomorphisms, Embeddings) *Let $\mathfrak{A} = \langle A, f_1, \ldots, f_n\rangle$ and $\mathfrak{B} = \langle B, f_1, \ldots, f_n\rangle$ be two similar $\mathfrak{L}$-algebras and let h be a mapping $h : A \mapsto B$ into. Then h is a homomorphism if and only if the following condition holds: for any n-ary operation f from $\mathfrak{A}$, for any n-tuple of elements $a_1, \ldots, a_n$ from A*

$$h(f(a_1, \ldots, a_n)) = f(h(a_1), \ldots h(a_n))$$

*If the homomorphism h is **surjective (onto)**, then h is an **embedding**;*

*If h is **injective (one-to-one)**, then h is an **isomorphism** and $\mathfrak{B}$ is an isomorphic image of $\mathfrak{A}$;*

*if h is a **bijection (isomorphism onto)**, then $\mathfrak{A}$ and $\mathfrak{B}$ are **isomorphic**, in symbols $\mathfrak{A} \cong \mathfrak{B}$.*

Definition 5.2.8 (subalgebra) *Let $\mathfrak{A} = \langle A, f_1, \ldots, f_n\rangle$ and $\mathfrak{B} = \langle B, f_1, \ldots, f_n\rangle$ be two similar $\mathfrak{L}$-algebras. We say that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, in symbols $\mathfrak{A} \sqsubseteq \mathfrak{B}$, if the following conditions hold true:*

(i) $A \subseteq B$

(ii) For any n-ary operation f from $\mathfrak{A}$ and for any $n{+}1$-tuple $a_1, \ldots, a_n, b$ of elements from A, $\mathfrak{A} \models f(a_1, \ldots, a_n) = b \quad \Leftrightarrow \quad \mathfrak{B} \models f(a_1, \ldots, a_n) = b$

Definition 5.2.9 (Generated subalgebras) *Given an algebra $\mathfrak{A} = \langle A, I\rangle$, the sub-algebra generated by B, $\mathfrak{A}[B]$ is the smallest subalgebra of $\mathfrak{A}$ containing B. The set B is the set of generators of $\mathfrak{A}[B]$.*

Theorem 5.2.10 *Given two algebras* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$*, if* $\mathfrak{A}_1 \sqsubseteq \mathfrak{A}_2$ *then* $\forall \mathtt{A}$ $(\mathfrak{A}_2 \models \mathtt{A} \Rightarrow \mathfrak{A}_1 \models \mathtt{A})$.

Definition 5.2.11 (Free Algebras) *Given a class of algebras* $\mathcal{K}$*,* $\mathfrak{F}_d$ *is a free algebra in* $\mathcal{K}$ *if and only if* $\mathfrak{F}_d \in \mathcal{K}$ *and* $\mathfrak{F}_d = \mathfrak{A}[B]$ *for some* B *such that for any* $\mathfrak{A}_i \in \mathcal{K}$*, for any mapping* $f: B \mapsto |\mathfrak{A}_i|$*,* f *can be extended to an homomorphism* $|\mathfrak{F}_d| \mapsto |\mathfrak{A}_i|$ *into.*
If $\mathfrak{A}[B]$ *is a free algebra with* $\|B\| = d$*, then* $\mathfrak{A}[B]$ *is of rank* d*.*

Theorem 5.2.12 *Given an algebraic logic* L*, the algebra* $\mathfrak{F}(\mathsf{L})$ *is a free algebra of countable rank from the variety* $Var(\mathsf{L})$ *with the set* $\{[p_1], [p_2], \ldots\}$ *as the set of generators, where each* p_i *belongs to the set of propositional letters.*

The free algebra of infinite countable rank ω for an algebraic logic L will henceforth be referred to as $\mathfrak{F}_\omega(\mathsf{L})$.

5.2.3 Algebras with operators, Filters and Ultrafilters

The algebraic constructions known as boolean algebras play a central role in providing multi-modal propositional logics with suitable algebraic semantics. More specifically, we shall use boolean algebras with operators. In order to define these systems we need to introduce lattices and distributive lattices:

Definition 5.2.13 (Lattice) *A lattice is an algebra* $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ *where* $\wedge$ *(meet) and* $\vee$ *(join) are two binary operations satisfying the following conditions. For each element* a, b, c *from* A*:*

(i) $a \wedge a = a$; $a \vee a = a$ *(idempotency)*

(ii) $a \wedge b = b \wedge a$; $a \vee b = b \vee a$ *(commutativity)*

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *(associativity)*

(iv) $a \wedge (a \vee b) = a$; $a \vee (a \wedge b) = a$ *(absorption)*

A lattice is said distributive if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ holds for each triple of elements.

Definition 5.2.14 (Boolean Algebras) *A boolean algebra is an algebra* $\mathfrak{A} = \langle A, \wedge, \vee, \neg, \bot, \top \rangle$ *where:*

(i) $\langle A, \wedge, \vee \rangle$ *is a distributive lattice;*

(ii) $\bot$ *and* $\top$ *are nullary operations on* A*, i.e.* $\bot, \top \in A$*;*

(iii) $\forall a \in A \quad \bot \vee a = a$ *and* $\top \wedge a = a$*;*

(iv) $\neg$ *is a unary operation on* A*;*

(v) $\forall a \in A \quad a \vee \neg a = \top$ *and* $a \wedge \neg a = \bot$

Definition 5.2.15 *An algebra* $\mathfrak{A} = \langle A, \neg, \wedge, \vee, \Box_1, \ldots, \Box_k, \top \rangle$ *is called a k-modal algebra if* $\langle A, \neg, \wedge, \vee, \top \rangle$ *is a Boolean algebra and each* $\Box_i$ *is a unary operation on* A *satisfying the following conditions:*

(i) $\Box_i(a \to b) \to (\Box_i a \to \Box_i b) = \top$*;*

(ii) $\Box_i \top = \top$, $1 \leq i \leq k$.

Definition 5.2.16 *Given a lattice* $\mathfrak{A} = \langle A, \wedge, \vee \rangle$*, a subset* ∇ *of* A *is a filter on* A *provided that:*

(i) $\forall a \in A \; \forall d \in \nabla \; (a \wedge d = d \Rightarrow a \in \nabla)$

(ii) $\forall d_1, d_2 \in \nabla \; (d_1 \wedge d_2 \in \nabla)$

A filter ∇ *is proper if* $\bot \notin \nabla$*. A filter* ∇ *is maximal if it is proper and for any proper filter* ∇_2 *on* A *if* $\nabla \subseteq \nabla_2$ *then* $\nabla = \nabla_2$*.*

An ultrafilter on A *is a proper filter* ∇ *such that for each element* $a \in A$ *either* a *or* $\neg a$ *belongs to* ∇*.*

Lemma 5.2.17 *Given a boolean lattice* $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ *the following holds:*

(i) Each maximal filter is an ultrafilter and vice versa.

(ii) If $\mathfrak{A}$ is either finite or finitely generated, each ultrafilter ∇ has the form $a^{\leq}$, where a is an element of A.

(iii) (Zorn Lemma) If ∇ is a proper filter on A, then ∇ can be extended to an ultrafilter ∇^* on A.

5.2.4 Stone's Theorems

We are now able to introduce the kind of systems whose definition is due to Stone. We shall then state few important results linking Kripke-semantics to algebraic semantics. The main idea is to associate each Kripke-frame with a special *wrapping algebra*. Then it can be easily proved that in this algebra the only true formulae are exactly those which are true in the original Kripke-frame. Likewise one could take any multi-modal algebra and construct its *Stone's representation frame*. Again the same result holds: these two structures share the same set of true formulae. But let us analise these constructions in more detail.

Definition 5.2.18 (Wrapping Algebras) *Given a k-modal Kripke-frame $\mathcal{F} = \langle F, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$, its Stone's wrapping algebra is the k-modal algebra $\mathcal{F}^+ = \langle F^+, \vee, \wedge, \neg, \Box_1, \ldots, \Box_k, \bot, \top \rangle$ where $F^+ = \mathcal{P}(F)$ (the power set of F, i.e. the set of all the subsets of F), $\langle F^+, \vee, \wedge, \neg, \bot, \top \rangle$ is the boolean algebra of all the subsets of F and $\Box_i A := \{v \mid v \in F \;\&\; \forall x \in F(v\mathsf{R}_i x \Rightarrow x \in A)\}$. If $\langle \mathcal{F}, V \rangle$ is a k-modal Kripke-model with $Dom(V) = P$, its wrapping algebra is the algebra $\mathcal{F}^+[\{V(p) \mid p \in P\}]$, i.e. the smallest subsystem of $\mathcal{F}^+$ containing $\{V(p) \mid p \in P\}$.*

Definition 5.2.19 (Stone's Frames) *Given a multi-modal algebra $\mathfrak{A} = \langle A, \wedge, \vee, \neg, \Box_1, \ldots, \Box_k, \top \rangle$ its Stone's representation frame is the k-modal Kripke-frame $\mathfrak{A}^+ = \langle A^+, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$ where $A^+ := \{\nabla \mid \nabla$ is an ultrafilter* *on* $A\}$ *and* $\nabla_1 \mathsf{R}_i \nabla_2$ *iff* $\forall x \in A(\Box_i x \in \nabla_1 \Rightarrow x \in \nabla_2)$.

Moreover its Stone's representation model is given by $\langle \mathfrak{A}^+, V \rangle$ *where* $Dom(V) = A$ *and* $\forall a \in A\ V(a) := \{\nabla \mid a \in \nabla\}$.

Theorem 5.2.20 (Stone's Representation Theorem) *For any* k*-modal algebra* $\mathfrak{A} = \langle A, \wedge, \vee, \neg, \Box_1, \dots, \Box_k, \top \rangle$ *the mapping* $i: \ A \mapsto A^{++}$ *where* $i(a) := \{\nabla \mid a \in \nabla\}$ *is an isomorphism, i.e.* $\mathfrak{A} \sqsubseteq \mathfrak{A}^{++}$. *If* A *is finite, then* $\mathfrak{A} \cong \mathfrak{A}^{++}$ *and for any finite* k*-modal Kripke-frame* $\mathcal{F}$, $\mathcal{F} \cong \mathcal{F}^{++}$.

Notice that given a Kripke-model $\langle \mathcal{F}, V \rangle$, by Definition 2.2.1 we know that V is a mapping which associates to each propositional letter from the set P a *subset* of worlds from the universe of $\mathcal{F}$, i.e. $V : P \mapsto \mathcal{P}(W_{\mathcal{F}})$. Therefore such valuation V is *also* a valuation in $\mathcal{F}^+$. Moreover when dealing with Kripke-frames, we say a formula $\mathtt{A}$ to be true under a valuation V whenever $\mathtt{A}$ is true in each single world from $W_{\mathcal{F}}$. In algebraic terms this means that $V(\mathtt{A}) = F$ and in $\mathcal{F}^+$ we have $F = \top$. This observation plus an easy induction on the length of the formula $\mathtt{A}$ leads to state the following:

Corollary 5.2.21 *Given a Kripke-model* $\langle \mathcal{F}, V \rangle$, *for each formula* $\mathtt{A}$, $\mathcal{F} \Vdash_V \mathtt{A}$ *iff* $\mathcal{F}^+ \vDash_V \mathtt{A} = \top$.

The following well known Lemma concerning the interactions between Kripke-frames and Stone's algebras will be useful in what follows (cf. Rybakov [55], Lemma 2.5.9):

Lemma 5.2.22 *If a frame* $\mathcal{F}_1$ *is a generated subframe of a frame* $\mathcal{F}_2$, *then there is a homomorphism* h *from* $\mathcal{F}_2^+$ *onto* $\mathcal{F}_1^+$ *such that* $\forall A \in \mathcal{P}(|\mathcal{F}_2|)$, $h(A) := A \cap |\mathcal{F}_1|$.

5.2.5 Quasi-identities and Inference Rules

We have already provided the necessary tools in order to analyse the concept of derivability from an algebraic perspective. In particular we shall now define special algebraic structures, called *free algebras*.In this Section we shall clarify the link occurring between free algebras and inference rules. In order to do so, we need some preliminar definitions.

Definition 5.2.23 (Identities, Quasi-identities)

- *A quasi-identity is an expression of the form* $f_1 = g_1 \wedge \ldots \wedge f_n = g_n \Rightarrow f = g$*;*

- *An identity is a quasi-identity whose set of premisses is empty, i.e.* $f = g$*;*

- $Fma_Q(\mathcal{L})$*,* $Fma_I(\mathcal{L})$ *are the sets of all the quasi-identities, identities on the language* $\mathcal{L}$*.*

- *A variety (quasi-variety) for a set of identities (quasi-identities)* Γ *on* $\mathcal{L}$ *is the set* $Var(\Gamma) = \{\mathfrak{A} \mid \forall \mathtt{A} \in \Gamma,\ \mathfrak{A} \models \mathtt{A}\}$*.*

Definition 5.2.24 *Given a class of algebras* $\mathcal{K}$ *on* $\mathcal{L}$

- *The elementary theory of* $\mathcal{K}$ *is the set* $Th(\mathcal{K}) = \{\mathtt{A} \in Fma(\mathcal{L}) \mid \forall \mathfrak{A} \in \mathcal{K},\ \mathfrak{A} \models \mathtt{A}\}$*.*

- *The equational theory of* $\mathcal{K}$ *is* $Th_I(\mathcal{K}) = \{\mathtt{A} \in Fma_I(\mathcal{L}) \mid \forall \mathfrak{A} \in \mathcal{K},\ \mathfrak{A} \models \mathtt{A}\}$*.*

- *The quasi-equational theory of* $\mathcal{K}$ *is* $Th_Q(\mathcal{K}) = \{\mathtt{A} \in Fma_Q(\mathcal{L}) \mid \forall \mathfrak{A} \in \mathcal{K},\ \mathfrak{A} \models \mathtt{A}\}$*.*

Definition 5.2.25 *Given a set of quasi-identities* Q*, a quasi-identity* q *is a semantic corollary of* Q*, in symbols* $Q \models q$*, if and only if for any algebra*

$\mathfrak{A}$ the following implication holds: $(\mathfrak{A} \models Q \Rightarrow \mathfrak{A} \models q)$.

Given a set of quasi-identities Q closed under semantic corollaries, a set of quasi-identities B is a basis for Q if and only if for any quasi-identity q, $q \in Q \Leftrightarrow B \models q$.

Definition 5.2.26 *Given an inference rule $\mathbf{r} = \mathtt{A}_1(p_j), \dots, \mathtt{A}_n(p_j)/\mathtt{B}(p_j)$ the quasi-identity associated with it is $q(\mathbf{r}) := (\mathtt{A}_1 = \top \,\&\dots\&\, \mathtt{A}_n = \top) \Rightarrow (\mathtt{B} = \top)$.*

Given a quasi-identity $q := f_1 = g_1 \wedge \dots \wedge f_i = g_i \Rightarrow f = g$, the rule associated with it is $\mathbf{r}(q) := f_1 \leftrightarrow g_1 \wedge \dots \wedge f_i \leftrightarrow g_i / f \leftrightarrow g$.

The following important results state the link occurring between admissible rules for a given logic and quasi-identities in free algebras: an inference rules is, in fact, admissible in a logic L exactly when the quasi-identity associated to it is valid in the Tarski-Lindenbaum algebra of the logic itself. Likewise a quasi-identity q is valid in the Tarski-Lindenbaum algebra $\mathfrak{F}_\omega(\mathsf{L})$ if and only if the rule associated to it is admissible for L (Please refer to Rybakkov [55]).

Theorem 5.2.27 *For any inference rule $\mathbf{r}$, any quasi-identity q and any logic L on $\mathfrak{L}$:*

(i) $\mathbf{r} \in Adm(\mathsf{L})$ *iff* $\mathfrak{F}_\omega(\mathsf{L}) \models q(\mathbf{r})$

(ii) $\mathfrak{F}_\omega(\mathsf{L}) \models q$ *iff* $\mathbf{r}(q) \in Adm(\mathsf{L})$.

In the light of this well known Theorem and the results provided in Chapter 3 we can now state the following:

Corollary 5.2.28 *The quasi-equational theory $Th_Q(\mathsf{LTK}_1)$ is decidable.*

5.3 Further work: the research of a finite Basis

In Chapter 3 we designed an algorithm to check whether a given rule is admissible for LTK_1. Nevertheless the set of admissible rules might contain an infinite number of elements and it could be too complicated for a direct description. In other words we already have a tool which tells us, given a rule, if it is admissible or not, but we are still unable to generate the set of all the admissible inference rules for LTK_1. This is the topic on which our interest is currently focused on and this last Section is devoted to the analysis of the methodology we are using in order to give this problem an answer.

Any Hilbert-style axiomatic system usually contains both axiom schemata and inference rules. As we have seen, there are many cases of axiomatic systems which are not structurally complete: these system admit rules which are, nevertheless, not derivable on the system itself. A basis of admissible rules is nothing but a set of rules which enables the derivation of all the admissible rules for a given logic. If we add a basis of rules to an axiomatic system, this would immediately become structurally complete. In fact, a basis of rules is the smallest set containing those rules which are necessary in order to derive *all* the admissible rules for a system.

A formal definition of what we mean by *basis of inference rules* is the following:

Definition 5.3.1 *Given a set of rules $\mathcal{R}$ for a logic generated by a system $\mathcal{AS}$:*

- *a rule $\mathbf{r} = \mathtt{A}_1(p_j), \ldots, \mathtt{A}_n(p_j)/\mathtt{B}(p_j)$ is derivable in $\mathcal{AS}$ from $\mathcal{R}$ (in symbols $\mathcal{R} \vdash_{\mathcal{AS}} \mathbf{r}$) if $\mathbf{r}$ is derivable in $\mathcal{AS} \oplus \mathcal{R}$, i.e. if there is a derivation in $\mathcal{AS}$ of $\mathtt{B}(p_j)$ having $\mathtt{A}_i(p_j)$ as premisses and using the rules from $\mathcal{AS}$ as well as* *the ones from* $\mathcal{R}$*;*

- *a set of rules* $\mathcal{B}$ *is a basis for* $\mathcal{R}$ *provided that* $\mathcal{B} \subseteq \mathcal{R}$ *and each rule* $\mathbf{r}$ *in* $\mathcal{R}$ *is derivable from* $\mathcal{B}$ *in* $\mathcal{AS}$*, i.e.* $\forall \mathbf{r} \in \mathcal{R} \;\; \mathcal{B} \vdash_{\mathcal{AS}} \mathbf{r}$*.*

Definition 5.3.2 (Basis of Inference Rules) *A collection of admissible rules* $\mathcal{B}$ *for a logic* L *is a basis for all the rules admitted by* L *if and only if for every rule* $\mathbf{r}$*,* $\mathbf{r} \in Adm(\mathsf{L})$ *iff* $\mathcal{B} \vdash_{\mathsf{L}} \mathbf{r}$*.*

We shall now analyse the case of our logic LTK_1. As we have anticipated, we shall not present any result here, but we shall only analyse the track we are following in order to find a finite basis of rules.

In Chapter 3 we introduced some special n-characterising models called $Ch_{\mathsf{LTK}_1}(n)$ (see the construction in Section 3.1). In the following Lemma, we shall use the wrapping algebras associated to these models. We consider for each natural number n, the particular wrapping algebra generated by the valuation of the propositional letters $p_1, \ldots, p_n$ in the model $\langle Ch_{\mathsf{LTK}_1}(n), V\rangle$ as defined in Section 3.1 and we prove it to be a free algebra of rank n from the variety of LTK_1.

Theorem 5.3.3 *For each* n*,* $Ch_{\mathsf{LTK}_1}(n)^+[V(p_1), \ldots, V(p_n)]$ *is a free-algebra of rank* n *in* $Var(\mathsf{LTK}_1)$ *generated by* $V(p_1), \ldots, V(p_n)$*.*

Proof. Consider $Ch_{\mathsf{LTK}_1}(n)^+[V(p_1), \ldots, V(p_n)]$ for some n and some $\mathfrak{A} \in Var(\mathsf{LTK}_1)$ and define a mapping $h : \{V(p_1), \ldots, V(p_n)\} \mapsto |\mathfrak{A}|$ such that $h(V(p_i)) = a_i$ for some $a_i \in |\mathfrak{A}|$. We extend such mapping to a homomorphism from $|Ch_{\mathsf{LTK}_1}(n)^+|$ into $\mathfrak{A}$ in the following way: for each multi-modal term t, $h(t(V(p_1), \ldots, V(p_n))) = t(h(V(p_1)), \ldots, h(V(p_n)))$. In order to prove this definition to be correct (i.e. h is indeed a homomorphism) we want

to show that given two elements $a, b \in |Ch_{\mathsf{LTK}_1}(n)^+|$, $h(a) = h(b)$ whenever $a = b$. Suppose that there are two terms t_1 and t_2 denoting the same element in $Ch_{\mathsf{LTK}_1}(n)^+$, $t_1(V(p_1), \ldots, V(p_n)) = t_2(V(p_1), \ldots, V(p_n))$. Since V is a valuation on $Ch_{\mathsf{LTK}_1}(n)^+$ as well as the valuation of the model $Ch_{\mathsf{LTK}_1}(n)$, by definition of valuation, it follows that $V(t_1(p_1, \ldots, p_n)) = V(t_2(p_1, \ldots, p_n))$, thus $Ch_{\mathsf{LTK}_1}(n) \Vdash_V t_1(p_1, \ldots, p_n) \leftrightarrow t_2(p_1, \ldots, p_n)$ holds. Since the model $Ch_{\mathsf{LTK}_1}(n)$ is n-characterising for LTK_1, $t_1(p_1, \ldots, p_n) \leftrightarrow t_2(p_1, \ldots, p_n) \in \mathsf{LTK}_1$. On the other hand, $\mathfrak{A} \in Var(\mathsf{LTK}_1)$, therefore $\mathfrak{A} \vDash t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$. In particular this holds for $x_i = a_i$, hence
$\mathfrak{A} \vDash t_1(h(V(p_1)), \ldots, h(V(p_n))) = t_2(h(V(p_1)), \ldots, h(V(p_n)))$ which by definition of h means $\mathfrak{A} \vDash h(t_1(V(p_1), \ldots, V(p_n))) = h(t_2(V(p_1), \ldots, V(p_n)))$. ∎

Consider now the of Kripke-structures we have introduced in Chapter 3, in Definition 3.3.5. We introduce now some new Kripke-frames which are very similar to LSP-frames. In fact, they are LSP-frames without the Point-component (see Figure 5.2).

Definition 5.3.4 *Let $\mathcal{F}_{\mathsf{L}}$ and $\mathcal{F}_{\mathsf{S}}$ be Kripke-frames with the following structure:*

(i) The frame $\mathcal{F}_{\mathsf{L}} = \langle W_{\mathcal{F}_{\mathsf{L}}}, \mathsf{R}^{\mathsf{L}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{L}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{L}}_{\mathsf{a}} \rangle$ (LOOP-component) is as follows: $W_{\mathcal{F}_{\mathsf{L}}}$ is a nonempty set of worlds; $\mathsf{R}^{\mathsf{L}}_{\preccurlyeq} = W_{\mathcal{F}_{\mathsf{L}}} \times W_{\mathcal{F}_{\mathsf{L}}}$; $\mathsf{R}^{\mathsf{L}}_{\mathsf{e}}$ is an equivalence relation on $W_{\mathcal{F}_{\mathsf{L}}}$; $\mathsf{R}^{\mathsf{L}}_{\mathsf{a}}$ is some equivalence relation on $\mathsf{R}^{\mathsf{L}}_{\mathsf{e}}$-clusters;

(ii) Let $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}} \rangle$ be a finite *$\mathcal{LTK}$-frame (i.e. it is an $\mathcal{LTK}$-frame with a finite base set of worlds. See Definition 2.2.3); let $\mathcal{C}_1, \ldots, \mathcal{C}_i$ be an enumeration of all the $\mathsf{R}^{\mathsf{S}}_{\preccurlyeq}$-clusters of worlds from $W_{\mathcal{F}}$; let $\mathsf{Dots} := \{w_1, \ldots, w_i\}$ be a set of worlds such that $\forall w_j, 1 \leq j \leq i (w_j \notin W_{\mathcal{F}})$. The frame $\mathcal{F}_{\mathsf{S}} = \langle W_{\mathcal{F}_{\mathsf{S}}}, \mathsf{R}^{\mathsf{S}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{S}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{S}}_{\mathsf{a}} \rangle$ (STRING-component) has the following structure: $W_{\mathcal{F}_{\mathsf{S}}} = W_{\mathcal{F}} \cup \mathsf{Dots}$; $\mathsf{R}^{\mathsf{S}}_{\preccurlyeq} = \mathsf{R}_{\preccurlyeq} \cup \{\langle w_j, z \rangle \mid w_j \in \mathsf{Dots} \;\&\; z \in \mathcal{C}_j\} \cup$*

$\{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$; $\mathsf{R}^{\mathsf{S}}_{\mathsf{e}} = \mathsf{R}_{\mathsf{e}} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$; $\mathsf{R}^{\mathsf{S}}_{\mathsf{a}} = \mathsf{R}_{\mathsf{a}} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$.

An LS*-frame (loop-string frame) is a tuple* $\mathcal{F}_{\mathsf{ls}} = \langle W_{\mathsf{ls}}, \mathsf{R}^{\mathsf{ls}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{ls}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{ls}}_{\mathsf{a}}\rangle$ *where* $W_{\mathcal{F}_{\mathsf{ls}}} = W_{\mathcal{F}_{\mathsf{L}}} \cup W_{\mathcal{F}_{\mathsf{S}}}$; $\mathsf{R}^{\mathsf{ls}}_{\preccurlyeq} = \mathsf{R}^{\mathsf{L}}_{\preccurlyeq} \cup \mathsf{R}^{\mathsf{S}}_{\preccurlyeq} \cup \{\langle w, z\rangle \mid w \in W_{\mathcal{F}_{\mathsf{S}}} \;\&\; z \in W_{\mathcal{F}_{\mathsf{L}}}\}$; $\mathsf{R}^{\mathsf{ls}}_{\mathsf{e}} = \mathsf{R}^{\mathsf{L}}_{\mathsf{e}} \cup \mathsf{R}^{\mathsf{S}}_{\mathsf{e}}$; $\mathsf{R}^{\mathsf{ls}}_{\mathsf{a}} = \mathsf{R}^{\mathsf{L}}_{\mathsf{a}} \cup \mathsf{R}^{\mathsf{S}}_{\mathsf{a}}$ *(See Figure 5.2).*

In the following Lemma we consider a family of LS-frames such that each of them is a generated subframe of the frame of $Ch_{\mathsf{LTK_1}}(n)$. Then we prove that the wrapping algebra of their disjoint union is a subalgebra of $Ch_{\mathsf{LTK_1}}(n)^+[V(p_1), \ldots, V(p_n)]$.

Lemma 5.3.5 *For each family of* LS*-frames* $(\mathcal{F}_i)_{i\in I}$ *such that for each* i, $\mathcal{F}_i \sqsubseteq Ch_{\mathsf{LTK_1}}(n)$*, the following holds:* $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+ \sqsubseteq Ch_{\mathsf{LTK_1}}(n)^+[V(p_1), \ldots, V(p_n)]$ *(where @ is a single element* $\mathsf{R}_{\preccurlyeq}$*-cluster disjoint from each* $\mathcal{F}_i$*).*

Proof. Let $\mathcal{C}_{j_1}, \ldots, \mathcal{C}_{j_m}$ be an enumeration of each $\mathsf{R}_{\preccurlyeq}$-cluster from the *linear* part of each $\mathcal{F}_j$ such that $\mathcal{C}_{j_l} \mathsf{R}_{\preccurlyeq} \mathcal{C}_{i_k}$ iff $j = 1$ and $l \leq k$. Let $d_{j_1}, \ldots, d_{j_{m-1}}$ be an enumeration of all the *dots* from $\mathcal{F}_j$ where for each k, $d_{j_k} \mathsf{R}_{\preccurlyeq} \mathcal{C}_{j_k}$, $\neg(\mathcal{C}_{j_k} \mathsf{R}_{\preccurlyeq} d_{j_k})$ and if $l < k$ then $\neg(d_{j_k} \mathsf{R}_{\preccurlyeq} \mathcal{C}_{j_l})$.

(1) By Lemma 3.2 each world v from the base set of the model $Ch_{\mathsf{LTK_1}}(n)$ is definable by a formula $\beta(v)$. We display each world v from any LS-frame $\mathcal{F}_j$ as v_{j_k}, meaning that v belongs to the k-th $\mathsf{R}_{\preccurlyeq}$-cluster of the frame $\mathcal{F}_j$.

For each frame $\mathcal{F}_j$ and each *dot-world* d_{j_k} we define a formula $\gamma(d_{j_k}) := \gamma_1(d_{j_k}) \wedge \gamma_2(d_{j_k}) \wedge \gamma_3(d_{j_k})$ where:

(i) $\gamma_1(d_{j_k}) := \bigwedge_{k<i} \Diamond_{\preccurlyeq} \beta(v_{j_i})$

(ii) $\gamma_2(d_{j_k}) := \bigwedge_{k>i} \neg\Diamond_{\preccurlyeq} \beta(v_{j_i)}$

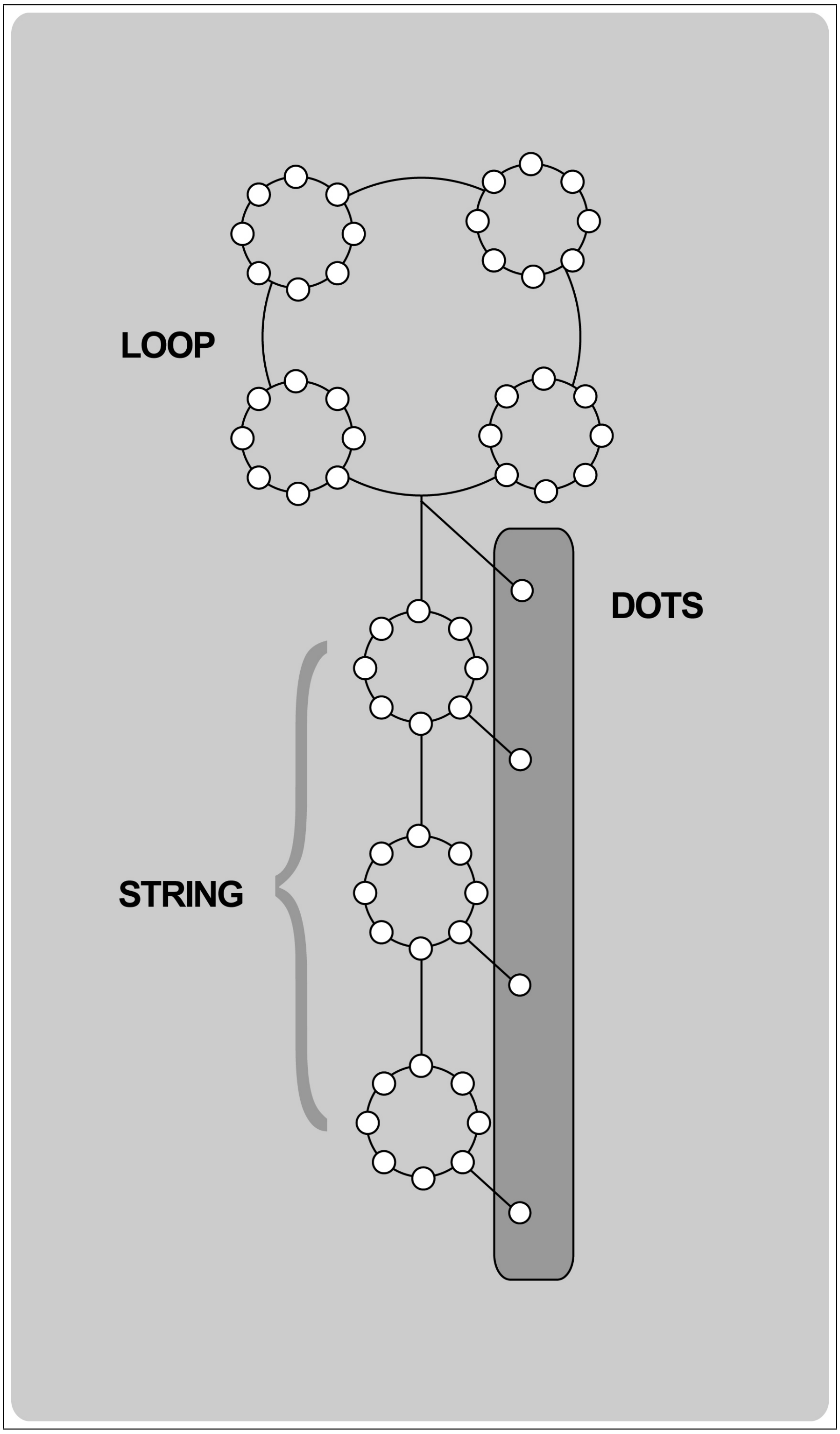

Figure 5.2: Scheme of the structure of an LS-frame.

(iii) $\gamma_3(d_{j_k}) := \bigwedge_{v \in \mathcal{C}_{j_k}} \neg\beta(v)$

Notice that for each world $v \in |Ch_{\mathsf{LTK_1}}(n)|$, $(Ch(n), v) \Vdash_V \gamma(d_{j_k})$ whenever $v\mathsf{R}_{\preccurlyeq}\mathcal{C}_{j_k}$ and $\neg(v\mathsf{R}_{\preccurlyeq}\mathcal{C}_{j_{k-1}})$.

(2) Since $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @) \sqsubset Ch(n)$[4], by Lemma 5.2.22 it follows that there is a homomorphism $Ch_{\mathsf{LTK_1}}(n)^+ \mapsto (\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+$ onto where $h(A) := A \cap |(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+|$.

(3) Let $\mathfrak{B}$ be a generated subalgebra of $Ch_{\mathsf{LTK_1}}(n)^+$ with generators $[\{V(\gamma(d_{i_k})) \mid d_{i_k} \in \mathsf{Dots}_i,\ i \in I\},\ \ \{V(\beta(z)) \mid z \in |\bigsqcup_{i\in I} \mathcal{F}_i| \ \&\ z \notin \mathsf{Dots}_i,\ i \in I\}]$. Clearly the restriction of the homorphism h to $|\mathfrak{B}|$ is into. We shall show that it is also both onto and one to one, i.e. the two algebras are isomorphic.

(4) We start by showing that the homomorphism h is onto. In order to achieve this we show that each singleton $\{z\}$ in $|(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+|$ has a pro-image in $|\mathfrak{B}|$, i.e. for each world $z \in |(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)|$ there is a set of worlds $A \in |\mathfrak{B}|$ such that $h(A) = \{z\}$, which means that $A \cap |(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)| = \{z\}$. Consider any singleton $\{z\}$ in $|(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+|$. The world z must fulfill one of the following requirements:

(i) $z \in \mathcal{F}_j$ for some j and $z \notin \mathsf{Dots}_j$;

(ii) $z = @$;

(iii) $z = d_{j_k}$ for some j and k.

(i) Suppose $z \in \mathcal{F}_j$ for some j and $z \notin \mathsf{Dots}_j$. By definition, $V(\beta(z))$ is a

[4] recall that $Ch(n)$ is the frame on which the model $Ch_{\mathsf{LTK_1}}(n)$ is built

generator of $\mathfrak{B}$ and since $V(\beta(z)) = \{z\}$ it follows that $h(V(\beta(z))) = \{z\}$.

(ii) Suppose $z = @$. Since $(Ch(n), @) \not\Vdash_V \beta(v)$, for all worlds $v \neq @$ and $(Ch(n), @) \not\Vdash_V \gamma(d_{j_k})$ for each $d_{j_k} \in \mathsf{Dots}_j$ and for each $j \in I$, it follows that $@ \in V(\bigwedge_{v \neq @} \neg\beta(v) \wedge \bigwedge_{j \in I} \neg\gamma(d_{j_k}))$. It is easy to verify that the intersection of this set and $|\bigsqcup_{i \in I} \mathcal{F}_i \circ @|$ is $\{@\}$.

(iii) $z = d_{j_k}$ for some j and k. Clearly $(Ch(n), d_{j_k}) \Vdash_V \gamma(d_{j_k})$, but for any world $v \in |\bigsqcup_{i \in I} \mathcal{F}_i \circ @|$ such that $v \neq d_{j_k}$, $(Ch(n), v) \not\Vdash_V \gamma(d_{j_k})$. Therefore $V(\gamma(d_{j_k})) \cap |\bigsqcup_{i \in I} \mathcal{F}_i \circ @| = \{d_{j_k}\}$.

(5) We shall prove now that the homomorphism h is one-to-one, by showing that given any $A \in |\mathfrak{B}|$, if $A \neq \emptyset$, then $h(A) \neq \emptyset$.

Each set $A \in |\mathfrak{B}|$ can be represented as $t(V(\gamma(d^i_{j_k})), V(\beta(u^l)))$ where t is a multi-modal term applied to the generators $V(\gamma(d^i_{j_k})), V(\beta(u^l))$. The set A is non-empty by assumption and therefore, by definition of valuation, it follows that $V(t((\gamma(d^i_{j_k})), \beta(u^l))) \neq \emptyset$. Therefore there is a world $v \in |Ch_{\mathsf{LTK}_1}(n)|$ such that: $(Ch(n), v) \Vdash_V t((\gamma(d^i_{j_k})), \beta(u^l))$. Such world v must satisfy one of the following conditions:

(i) $v \in |\bigsqcup_{i \in I} \mathcal{F}_i \circ @|$;

(ii) $v \notin |\bigsqcup_{i \in I} \mathcal{F}_i \circ @|$ and $\forall z \in |\bigsqcup_{i \in I} \mathcal{F}_i \circ @| \ \ (v\mathsf{R}_{\preccurlyeq} z \ \Rightarrow \ z = @)$;

(iii) $v \notin |\bigsqcup_{i \in I} \mathcal{F}_i \circ @|$ and $\exists z \in |\bigsqcup_{i \in I} \mathcal{F}_i| \ \ v\mathsf{R}_{\preccurlyeq} z$.

(i) If $v \in |\bigsqcup_{i \in I} \mathcal{F}_i \circ @|$ then clearly $h(A) \neq \emptyset$.

(ii) Suppose that $v \notin |\bigsqcup_{i \in I} \mathcal{F}_i \circ @|$ and $\forall z \in |\bigsqcup_{i \in I} \mathcal{F}_i \circ @| \ \ (v\mathsf{R}_{\preccurlyeq} z \ \Rightarrow \ z = @)$.

Notice that $(Ch(n), @) \not\Vdash_V \bigvee_{j\in I}\gamma(d_{j_k})$ and $(Ch(n), @) \not\Vdash_V \bigvee_{z\in|\bigsqcup_{i\in I}\mathcal{F}_i|}\beta(z)$. Moreover $\forall y \quad (@\mathsf{R}_{\preccurlyeq} y \quad \Rightarrow \quad ((Ch(n), y) \not\Vdash_V \bigvee_{j\in I}\gamma(d_{j_k}) \ \& \ (Ch(n), y) \not\Vdash_V \bigvee_{z\in|\bigsqcup_{i\in I}\mathcal{F}_i|}\beta(z)))$. Since this holds true for v as well, it is easy to verify that for each multi-modal term t':
$(Ch(n), @) \Vdash_V t'((\gamma(d^i_{j_k})), \beta(u^l))$ iff $(Ch(n), v) \Vdash_V t'((\gamma(d^i_{j_k})), \beta(u^l))$. Hence $@ \in V(t((\gamma(d^i_{j_k})), \beta(u^l)))$ and $h(A) \neq \emptyset$.

(iii) Assume $v \notin |\bigsqcup_{i\in I}\mathcal{F}_i \circ @|$ and $\exists z \in |\bigsqcup_{i\in I}\mathcal{F}_i| \quad v\mathsf{R}_{\preccurlyeq} z$. Let z be the $\mathsf{R}_{\preccurlyeq}$-deepest world $\mathsf{R}_{\preccurlyeq}$-seen by v; then z is not $\mathsf{R}_{\preccurlyeq}$-final by construction of $Ch_{\mathsf{LTK}_1}(n)$ and it belongs to some $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}_{j_m}$. This cluster is associated to a *dot-world* d_{j_m}. The following statements hold true both for v and d_{j_m}:
$(Ch(n), v) \Vdash_V \gamma(d_{j_m})$;
$(Ch(n), v) \not\Vdash_V \bigvee_{u\in|\bigsqcup_{i\in I}\mathcal{F}_i|}\beta(u)$;

If there is a world y such that either $v\mathsf{R}_{\mathsf{e}}y$ or $v\mathsf{R}_{\mathsf{a}}y$ or else $v\mathsf{R}_{\preccurlyeq}y$ and $y\mathsf{R}_{\preccurlyeq}\mathcal{C}_{j_m}$ but y is not in $\mathcal{C}_{j_m}$, then $(Ch(n), y) \Vdash_V \gamma(d_{j_m})$ and $(Ch(n), y) \not\Vdash_V \bigvee_{u\in|\bigsqcup_{i\in I}\mathcal{F}_i|}\beta(u)$. Moreover starting from $\mathcal{C}_{j_m}$ on, both v and d_{j_m} have exactly the same $\mathsf{R}_{\preccurlyeq}$-successors, therefore for each multi-modal term t':
$(Ch(n), v) \Vdash_V t'((\gamma(d^i_{j_k})), \beta(u^l))$ iff $(Ch(n), d_{j_m}) \Vdash_V t'((\gamma(d^i_{j_k})), \beta(u^l))$. Hence $d_{j_m} \in V(t((\gamma(d^i_{j_k})), \beta(u^l)))$ and $h(A) \neq \emptyset$.

■

Any algebra as the one introduced in Lemma 5.3.6 shows, therefore, an interesting property: it is the subalgebra of $Ch_{\mathsf{LTK}_1}(n)^+[V(p_1), \ldots, V(p_n)]$ for some finite n. Moreover in the following Lemma we prove such algebras to have another interesting property, namely that all those formulae which are valid in *all* the algebras of kind $(\bigsqcup_{i\in I}\mathcal{F}_i \circ @)^+$ are exactly those which are valid in the free algebra of infinite countable rank $\mathfrak{F}_\omega(\mathsf{LTK}_1)$.

Lemma 5.3.6 *For any formula* $\mathtt{A}$, $\mathfrak{F}_\omega(\mathsf{LTK}_1) \vDash \mathtt{A}$ *if and only if for any algebra* $\mathfrak{A}$ *of type* $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+$ *as introduced in Lemma* **??**, $\mathfrak{A} \vDash \mathtt{A}$.

Proof. 1. Suppose there is an algebra of kind $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+$ such that $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+ \nvDash \mathtt{A} = \top$ for some formula $\mathtt{A}$. By Lemma 5.3.5 it follows that for some natural number n, $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+ \sqsubseteq Ch_{\mathsf{LTK}_1}(n)^+[V(p_1), \dots, V(p_n)]$. Thus, by Lemma 5.2.10, we get $Ch_{\mathsf{LTK}_1}(n)^+[V(p_1), \dots, V(p_n)] \nvDash \mathtt{A}$ and since $Ch_{\mathsf{LTK}_1}(n)^+[V(p_1), \dots, V(p_n)] \cong \mathfrak{F}_n(\mathsf{LTK}_1)$, we have $\mathfrak{F}_n(\mathsf{LTK}_1) \nvDash \mathtt{A}$ and therefore $\mathfrak{F}_\omega(\mathsf{LTK}_1) \nvDash \mathtt{A}$.

2. Suppose there is a formula $\mathtt{A}$ such that $\mathfrak{F}_\omega(\mathsf{LTK}_1) \nvDash \mathtt{A} = \top$. Such $\mathtt{A}$ can be represented as:

$$\mathtt{A} := \bigwedge_k f_k = \top \to \bigvee_m \Box_{\preccurlyeq} g_m = \top$$

Since there is a finite d such that
$\mathfrak{F}_d(\mathsf{LTK}_1) \nvDash \mathtt{A} = \top$ and $\mathfrak{F}_d(\mathsf{LTK}_1) \cong Ch_{\mathsf{LTK}_1}(d)^+[V(p_1), \dots, V(p_d)]$, it follows that $Ch(d) \Vdash_V \bigwedge_k f_k$ and there is a world $z \in |Ch(d)|$ such that $(Ch(d), z) \nVdash_V \bigvee_m \Box_{\preccurlyeq} g_m$.

Moreover there is by construction a world $@ \in |Ch(d)|$ that is a single element $\mathsf{R}_{\preccurlyeq}$-maximal cluster and clearly $(Ch(d), @) \Vdash_V \bigwedge_k f_k$.

Take the model $\langle (z^{\preccurlyeq} \circ @), V\rangle$ (where V is an abbreviation for $V \restriction |z^{\preccurlyeq} \circ @|$). Consider any non final $\mathsf{R}_{\preccurlyeq}$-cluster from this model and define a well ordering where $m \leq n$ iff $\mathcal{C}_m \mathsf{R}_{\preccurlyeq} \mathcal{C}_n$.

For each non-final $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}_j$ consider a new world d_j and join such d_j to the model $\langle (z^{\preccurlyeq} \circ @), V\rangle$ as follows: $\forall v \in \mathcal{C}_j^{\preccurlyeq}$ $(d_j \mathsf{R}_{\preccurlyeq} v \ \&\ \neg(v \mathsf{R}_{\preccurlyeq} d_j))$,

$d_j \mathsf{R}_{\preccurlyeq} d_j$, $d_j \mathsf{R}_{\mathsf{e}} d_j$ and $d_j \mathsf{R}_{\mathsf{a}} d_j$.

Extend the valuation V to any new world d_j in an arbitrary way and call the resulting model $\langle \mathcal{F}, V \rangle$.

Notice that for each world d_j, the frame of $d_j^{\preccurlyeq}$ is a frame for $\mathsf{LTK_1}$ and moreover the truth value of formulae of type $\Box_{\preccurlyeq}\mathtt{B}$ at any world v would not be affected by the presence of the new worlds d_j. The conjunction $\bigwedge_k f_k$ belongs to $\mathsf{LTK_1}$ (recall that such formula is true in $\mathfrak{F}_\omega(\mathsf{LTK_1})$), therefore it is also true at each world d_j. Hence $\mathcal{F} \Vdash_V \bigwedge_k f_k$ and $(\mathcal{F}, z) \nVdash_V \bigvee_m \Box_{\preccurlyeq} g_m$. Clearly $\mathcal{F}$ belongs to the class of frames introduced in Lemma **??**, therefore $(\mathcal{F})^+ \not\models \mathtt{A} = \top$.

∎

Lemma 5.3.7 *Let $\mathfrak{A}[a_1, \ldots, a_n]$ be a finitely generated algebra from $Var(\mathsf{LTK_1})$ such that $\|A\| > 1$. Let $q := \Diamond_{\preccurlyeq} x \wedge \Diamond_{\preccurlyeq} \neg x = \top \Rightarrow y = \top$. If $\mathfrak{A} \models q$ then $\mathfrak{A}^+$ has a single element $\mathsf{R}_{\preccurlyeq}$-maximal cluster.*

Proof. Consider the Stone's representation frame of $\mathfrak{A}$, $\mathfrak{A}^+ := \langle A^+, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}} \rangle$ as in Definition 5.2.19 and define a valuation V such that $Dom(V) = \{p_1, \ldots, p_n\}$ and $V(p_i) := \{\nabla \mid a_i \in \nabla\}$. We start by showing that

$$\forall \mathtt{A}(p_1, \ldots, p_n), \forall \nabla \in A^+ \quad \nabla \Vdash_V \mathtt{A}(p_1, \ldots, p_n) \Leftrightarrow \mathtt{A}(a_1, \ldots, a_n) \in \nabla \tag{5.1}$$

In fact suppose (by induction on the length of $\mathtt{A}(p_1, \ldots, p_n)$) that $\mathtt{A}(p_1, \ldots, p_n) = p_i$. Then $\nabla \Vdash_V p_i$ if and only if $\nabla \in V(p_i)$, which means $a_i \in \nabla$.
Supose $\mathtt{A}(p_1, \ldots, p_n) = \Box_{\preccurlyeq}\mathtt{B}(p_1, \ldots, p_n)$. Then $\nabla \Vdash_V \Box_{\preccurlyeq}\mathtt{B}(p_1, \ldots, p_n)$ if and only if $\forall \nabla_2 \in A^+(\nabla \mathsf{R}_{\preccurlyeq} \nabla_2 \Rightarrow \nabla_2 \Vdash_V \mathtt{B}(p_1, \ldots, p_n))$. By Inductive Hypothesis (IH henceforth) we have $\forall \nabla_2 \in A^+(\nabla \mathsf{R}_{\preccurlyeq} \nabla_2 \Rightarrow \mathtt{B}(a_1, \ldots, a_n) \in \nabla_2)$. Recall that the mapping $i : \quad A \mapsto A^{++}$ where $i(a) = \{\nabla \mid a \in \nabla\}$ is

an isomorphism from $\mathfrak{A}$ into $\mathfrak{A}^{++}$, therefore $\mathtt{B}(a_1,\ldots,a_n) \in \nabla_2$ iff $\nabla_2 \in i(\mathtt{B}(a_1,\ldots,a_n))$. Since $i(\mathtt{B}(a_1,\ldots,a_n)) \in A^{++}$, by Definition 5.2.18 we get $\nabla \in \Box_{\preccurlyeq} i(\mathtt{B}(a_1,\ldots,a_n))$, and hence $\nabla \in i(\Box_{\preccurlyeq}\mathtt{B}(a_1,\ldots,a_n))$ which implies $\Box_{\preccurlyeq}\mathtt{B}(a_1,\ldots,a_n) \in \nabla$. Clearly the same holds for the modal operators $\mathsf{K_e}, \mathsf{K_a}$.

Consider any $\mathsf{R}_{\preccurlyeq}$-chain of $\mathsf{R}_{\preccurlyeq}$-clusters $\mathcal{C} := \mathcal{C}_1\mathsf{R}_{\preccurlyeq}\mathcal{C}_2\mathsf{R}_{\preccurlyeq}\ldots$. Since $S4_{\Box_{\preccurlyeq}} \subset \mathsf{LTK}_1$, it follows that $\forall\nabla_1\forall\nabla_2(\nabla_1\mathsf{R}_{\preccurlyeq}\nabla_2 \Rightarrow (\Box_{\preccurlyeq}\mathtt{A} \in \nabla_1 \Rightarrow \Box_{\preccurlyeq}\mathtt{A} \in \nabla_2))$. Let $\Box_{\preccurlyeq}\mathcal{C}_i := \{\Box_{\preccurlyeq}\mathtt{A} \mid \Box_{\preccurlyeq}\mathtt{A} \in \nabla,\ \forall\nabla \in \mathcal{C}_i\}$, then clearly $\Box_{\preccurlyeq}\mathcal{C}_i \subseteq \{\mathtt{A} \mid \mathcal{C}_j \Vdash_V \mathtt{A}\}$ whenever $\mathcal{C}_i\mathsf{R}_{\preccurlyeq}\mathcal{C}_j$. Let $\Box_{\preccurlyeq}\mathcal{C} := \bigcup_{\mathcal{C}_i\in\mathcal{C}} \Box_{\preccurlyeq}\mathcal{C}_i$ be the union of all the $\Box_{\preccurlyeq}\mathcal{C}_i$ such that $\mathcal{C}_i$ is in the chain $\mathcal{C}$. Clearly $\Box_{\preccurlyeq}\mathcal{C}$ is consistent and it is also a subset of the carrier of the algebra $\mathfrak{A}$, therefore $\bigwedge \Box_{\preccurlyeq}\mathcal{C} \neq \emptyset$ in $\mathfrak{A}$. The filter $\Box_{\preccurlyeq}\mathcal{C}^{\leq}$ is proper and hence by Lemma 5.2.17, (iii) it can be extended to an ultrafilter ∇^*, which belongs to the base set of $\mathfrak{A}^+$. Since $\forall\nabla \in \mathcal{C}$ we have that for any $\Box_{\preccurlyeq}\mathtt{A}$ if $\Box_{\preccurlyeq}\mathtt{A} \in \nabla$, then $\Box_{\preccurlyeq}\mathtt{A} \in \nabla^*$, it follows that each single ultrafilter ∇ in $\mathcal{C}$ is $\mathsf{R}_{\preccurlyeq}$-related to $\mathcal{C}(\nabla^*)$ i.e. the $\mathsf{R}_{\preccurlyeq}$-cluster containing ∇^*, which is, therefore, $\mathsf{R}_{\preccurlyeq}$-maximal in $\mathcal{C}$. Hence each $\mathsf{R}_{\preccurlyeq}$-chain in $\mathfrak{A}^+$ has an $\mathsf{R}_{\preccurlyeq}$-maximal cluster.

Since the algebra $\mathfrak{A}$ is finitely generated by the elements $a_1,\ldots,a_n$, there are at most 2^n ultrafilters on $\mathfrak{A}$ and then 2^{2^n} possible $\mathsf{R}_{\preccurlyeq}$-maximal $\mathsf{R}_{\preccurlyeq}$-clusters (i.e. subsets of ultrafilters) in $\mathfrak{A}^+$. Let $\mathcal{C}_1,\ldots,\mathcal{C}_k$ be all the $\mathsf{R}_{\preccurlyeq}$-maximal $\mathsf{R}_{\preccurlyeq}$-clusters in $\mathfrak{A}^+$ and suppose by contradiction the each $\mathcal{C}_i$ contains more than one ultrafilter, i.e. there are no $\mathsf{R}_{\preccurlyeq}$-maximal single-element $\mathsf{R}_{\preccurlyeq}$-clusters.

Take for each cluster $\mathcal{C}_i$ a representative ultrafilter ∇_i from it and let let $at(\nabla_i)$ be that element a such that $\nabla_i = a^{\leq}$ (clearly this elements does not belong to any other member of $\mathcal{C}_i$). Moreover take for each cluster $\mathcal{C}_j \neq \mathcal{C}_i$

an element $\Box_{\preccurlyeq} a(i,j)$ such that $\Box_{\preccurlyeq} a(i,j) \in \bigcup \mathcal{C}_i$ and $\Box_{\preccurlyeq} a(i,j) \notin \bigcup \mathcal{C}_j$[5]. Now consider the element $b := \bigvee_{1 \leq i \leq k}(at(\nabla_i) \wedge \bigwedge_{i \neq j} \Box_{\preccurlyeq} a(i,j))$. Suppose $\Box_{\preccurlyeq} b \neq \bot$ in $\mathfrak{A}$, then there is some ultrafilter $\nabla \in A^+$ such that $\Box_{\preccurlyeq} b \in \nabla$. Moreover such ∇ $\mathsf{R}_{\preccurlyeq}$-sees some $\mathsf{R}_{\preccurlyeq}$-maximal cluster $\mathcal{C}_m$. By assumption the cluster $\mathcal{C}_m$ contains more than one ultrafilter, so there is an ultrafilter $\nabla_l \neq \nabla_m$ such that $\Box_{\preccurlyeq} b \in \nabla_l$. This entails that $b \in \nabla_l$ and hence for some i $(at(\nabla_i) \wedge \bigwedge_{i \leq j} \Box_{\preccurlyeq} a(i,j)) \in \nabla_l$, which means that $i = m$, $at(\nabla_m) \in \nabla_l$ and since by assumption $\nabla_m \neq \nabla_l$ this leads to a contradiction and $\Box_{\preccurlyeq} b = \bot$.

Suppose now that $\Box_{\preccurlyeq} \neg b \neq \bot$. Again there is some ultrafilter $\nabla \in A^+$ such that $\Box_{\preccurlyeq} \neg b \in \nabla$ and ∇ $\mathsf{R}_{\preccurlyeq}$-sees some $\mathsf{R}_{\preccurlyeq}$-maximal cluster $\mathcal{C}_m$. This implies that $\Box_{\preccurlyeq} \neg b \in \nabla_m$ as well as $\neg b \in \nabla_m$. Nevertheless since $at(\nabla_m) \wedge_{m \neq j} a(m,j)$, it follows that $b \in \nabla_m$ and then $\bot \in \nabla_m$ which is a contradiction. Therefore $\Box_{\preccurlyeq} \neg b = \bot$.

From this facts it follows that $\Diamond_{\preccurlyeq} b \wedge \Diamond_{\preccurlyeq} \neg b = \top$ in $\mathfrak{A}$. But $\mathfrak{A} \models \Diamond_{\preccurlyeq} x \wedge \Diamond_{\preccurlyeq} \neg x = \top \Rightarrow y = \top$, hence even in the case $y = \bot$, $\mathfrak{A} \models y = \top$ holds, and this is a contradiction. Therefore the frame $\mathfrak{A}^+$ has at least one single element $\mathsf{R}_{\preccurlyeq}$-maximal $\mathsf{R}_{\preccurlyeq}$-cluster.

∎

Our first hypothesis was to show the following:

Conjecture 5.3.8 *The set of all the quasi-identities valid in the variety generated by* LTK_1, $Th_Q(Var(\mathsf{LTK}_1))$ *has a finite basis* Q^* *which is a basis for all the axioms from* $\mathcal{AS}_{\mathsf{LTK}_1}$ *(i.e. a basis for* $\{\mathtt{A} = \top \mid \mathtt{A}$ *is an axiom of* $\mathsf{LTK}_1\}$*) plus the quasi-identity* $q := \Diamond_{\preccurlyeq} x \wedge \Diamond_{\preccurlyeq} \neg x = \top \Rightarrow y = \top$.

[5]Notice that both these elements do exist for each cluster $\mathcal{C}_i$. In fact any $\mathsf{R}_{\preccurlyeq}$-cluster does not contain duplicate ultrafilters and each ultrafilter ∇ has the form $a^{\leq}$ for some atom a. Moreover if two clusters are distinct from each other and not related there must be at least one element $\Box_{\preccurlyeq} a$ which belongs to the intersection of the first cluster but not to the one of the other.

We have, however, encountered several difficulties while attempting to prove this conjecture. The details of the proof we have attempted to provide follow below.

Recall that the set Q^* is a basis for $Th_Q(\mathfrak{F}_\omega(\mathsf{LTK}_1))$ if and only if $\forall q_i(q_i \in Th_Q(\mathfrak{F}_\omega(\mathsf{LTK}_1)) \Leftrightarrow Q^* \vDash q_i)$. This means that we attempt to prove that for any quasi-identity q_i the following holds:

$$\mathfrak{F}_\omega(\mathsf{LTK}_1) \vDash q_i \Leftrightarrow \forall \mathfrak{A} \in Var(\mathsf{LTK}_1)(\mathfrak{A} \vDash Q^* \Rightarrow \mathfrak{A} \vDash q_i)$$

Clearly for the *left* part of the implication above there are no problems. In fact suppose that $\forall \mathfrak{A} \in Var(\mathsf{LTK}_1)(\mathfrak{A} \vDash Q^* \Rightarrow \mathfrak{A} \vDash q_i)$.
Clearly $\mathfrak{F}_\omega(\mathsf{LTK}_1) \vDash Th_I(\mathfrak{F}_\omega(\mathsf{LTK}_1))$. Consider any frame with the same structure as $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)$ as defined in Lemma ??. Then $((\bigsqcup_{i\in I} \mathcal{F}_i \circ @), @) \nVdash \Diamond_{\preccurlyeq} \mathtt{A} \wedge \Diamond_{\preccurlyeq} \neg \mathtt{A}$ for any formula $\mathtt{A}$ and any valuation. This implies that for any algebra of type $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+$, $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+ \vDash \Diamond_{\preccurlyeq} x \wedge \Diamond_{\preccurlyeq} \neg x \neq \top$ holds for any value assigned to x. Therefore it follows that $(\bigsqcup_{i\in I} \mathcal{F}_i \circ @)^+ \vDash q$. By Lemma 5.3.6, $\mathfrak{F}_\omega(\mathsf{LTK}_1) \vDash q$. This means that $\mathfrak{F}_\omega(\mathsf{LTK}_1) \vDash Q^*$, thus $\mathfrak{F}_\omega(\mathsf{LTK}_1) \vDash q_i$ for any q_i.

Proving that the *right arrow* of the implication above holds true shows, however, several difficulties. We shall sketch the track we have been following below.

Suppose that there is an algebra $\mathfrak{A}$ in the variety of LTK_1 such that $\mathfrak{A} \vDash Q^*$ and $\mathfrak{A} \nvDash q_i$ for some quasi-identity q_i. The quasi-identity q_i can be represented as $\bigwedge_k (f_k(x_1, \dots, x_n) = \top) \Rightarrow g(x_1, \dots, x_n) = \top$.

We should show that $\mathfrak{F}_\omega(\mathsf{LTK}_1) \not\models q_i$. In order to do this, we could proceed as follows:

(i) Find a model based on a finite LTK_1-reflexive balloon (as introduced in Theorem 2.3.2). In this model any formula $f_i(p_1, \dots, p_n)$ from the conjunction $\bigwedge_k (f_k(x_1, \dots, x_n))$ in q_i is true, whereas the formula g is not;

(ii) Join to such model a single world @ so that in the resulting model any formula $f_i(p_1, \dots, p_n)$ from q_i is still true;

(iii) Join a set of *dot-worlds* to our latest model keeping the truth value of any $f_i(p_1, \dots, p_n)$ from q_i true. The resulting model would then be based on an LSP-frame $\mathcal{F}$ in which the formula $\Box_{\preccurlyeq} \bigwedge_i (f_i(p_1, \dots, p_n))$ is true whereas g is not. Therefore in the algebra $\mathcal{F}^+$ the quasi-identity q_i would be false and therefore it would also be falsified by $\mathfrak{F}_\omega(\mathsf{LTK}_1)$.

As it will be clear in the further development of this argument, item (iii) cannot be easily fulfilled. Let us see why it is so.

(i) We start by finding a model $\langle \mathcal{F}, V \rangle$ based on a finite LTK_1-reflexive balloon frame for LTK_1 as introduced in Theorem 2.3.2 such that $\mathcal{F} \Vdash_V \Box_{\preccurlyeq}(\bigwedge_k (f_k(p_1, \dots, p_n)) = \top)$ and $\mathcal{F} \not\Vdash_V g(p_1, \dots, p_n)$ for some n-tuple of propositional letters $p_1, \dots, p_n$.

Clearly the implication $\Box_{\preccurlyeq} \bigwedge f(x_1, \dots, x_n) \to g(x_1, \dots, x_n)$ does not belong to LTK_1. Indeed suppose $\Box_{\preccurlyeq} \bigwedge f(x_1, \dots, x_n) \to g(x_1, \dots, x_n) \in \mathsf{LTK}_1$. In the algebra $\mathfrak{A}$ from $Var(\mathsf{LTK}_1)$ there is by assumption an n-tuple of elements $a_1, \dots, a_n$ such that $\mathfrak{A} \models \bigwedge_k f_k(a_1, \dots, a_n) = \top$ and $\mathfrak{A} \models g(a_1, \dots, a_n) \neq \top$. Since $\mathfrak{A}$ is a modal algebra, $\mathfrak{A} \models \Box_{\preccurlyeq} \top = \top$ and therefore we have $\mathfrak{A} \models \Box_{\preccurlyeq} \bigwedge_k f_k(a_1, \dots, a_n) = \top$ and hence, since the implication $\Box_{\preccurlyeq} \bigwedge_k f_k(x_1, \dots, x_n) \to g(x_1, \dots, x_n)$ belongs to LTK_1 by hypothesis, it follows $\mathfrak{A} \models g(a_1, \dots, a_n) =$

$\top$ which is, clearly, a contradiction.

By Theorem 2.3.2 the logic LTK_1 has the finite model property, therefore there is a finite model $\langle \mathcal{F}, V\rangle$ where $\mathcal{F}$ has the form $z^{\preccurlyeq}$ for some world z such that $(\mathcal{F}, z) \nVdash_V \Box_{\preccurlyeq} \bigwedge_k f_k(p_1,\ldots,p_n) \rightarrow g(p_1,\ldots,p_n)$. Clearly $\mathcal{F} \Vdash_V \Box_{\preccurlyeq} \bigwedge_k f_k(p_1,\ldots,p_n)$.

(ii) Now we want to join to the formerly defined model $\langle \mathcal{F}, V\rangle$ a single world @ so that the resulting disjoint union $\langle \mathcal{F} \circ @, V\rangle$ is such that $\mathcal{F} \circ @ \Vdash_V \Box_{\preccurlyeq} \bigwedge_k f_k(p_1,\ldots,p_n)$ and $(\mathcal{F} \circ @, z) \nVdash_V g(p_1,\ldots,p_n)$.

Consider the algebra $\mathfrak{A}$ from $Var(\mathsf{LTK}_1)$ and suppose it is finitely generated by the elements $a_1,\ldots,a_n$. Take its Stone's representation model $\mathfrak{A}^+ := \langle A^+, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}, V^+\rangle$ where A^+ is the set of all the ultrafilters on $|\mathfrak{A}|$, $Dom(V^+) := \{p_i \mid i \in I\}$ and for each p_i, $V^+(p_i) := \{\nabla \mid a_i \in \nabla\}$.

From Lemma 5.3.7, 5.1, it follows that $\forall \mathtt{A}(p_1,\ldots,p_n)\forall\nabla \in A^+ \quad (\nabla \Vdash_V \mathtt{A}(p_1,\ldots,p_n) \Leftrightarrow \mathtt{A}(a_1,\ldots,a_n) \in \nabla)$. Since $\mathfrak{A} \vDash \bigwedge f(a_1,\ldots,a_n) = \top$, we have $\mathfrak{A}^+ \Vdash_{V^+} \bigwedge_k f_k(p_1,\ldots,p_n)$.

Since by assumption $\mathfrak{A}^+ \vDash q$, by Lemma 5.3.7 it follows that $\mathfrak{A}^+$ has a single element $\mathsf{R}_{\preccurlyeq}$-maximal cluster @ and clearly $@ \Vdash_{V^+} \bigwedge f(p_1,\ldots,p_n)$.

Take the submodel of $\mathfrak{A}^+$ generated by the set $\{@\}$[6] and take the disjoint union model $\langle \mathcal{F}, V\rangle \sqcup \langle @^{\preccurlyeq}, V^+\rangle$ and denote it by $\langle \mathcal{F} \circ @, V\rangle$ for simplicity. Clearly $(\mathcal{F} \circ @) \Vdash_V \bigwedge_k f_k(p_1,\ldots,p_n)$ whereas $(\mathcal{F} \circ @, z) \nVdash_V g(p_1,\ldots,p_n)$.

[6] Notice that the universe of this model is just the set $\{@\}$.

(iii) Since the component $\mathcal{F}$ of the model we defined formerly is not an LS-frame (it is just an LTK_1-reflexive balloon), our first guess was to join to $\mathcal{F}$ as many *dot-worlds* as the number of $\mathsf{R}_{\preccurlyeq}$-clusters in the frame. This means that if in the frame there are $\mathcal{C}_1, \ldots, \mathcal{C}_n$ $\mathsf{R}_{\preccurlyeq}$-cluster, we would add some new single element $\mathsf{R}_{\preccurlyeq}$-clusters $d_1, \ldots, d_n$ assuming that for each i, $1 \leq i \leq n$, d_1 is the immediate $\mathsf{R}_{\preccurlyeq}$-predecessor of $\mathcal{C}_i$ and for each $\mathcal{C}_j$ such that $j < i$, $\neg(d_i \mathsf{R}_{\preccurlyeq} \mathcal{C}_j)$ and $\neg(\mathcal{C}_j \mathsf{R}_{\preccurlyeq} d_i)$. This turns out, however, to be impossible.

Consider in fact any non final $\mathsf{R}_{\preccurlyeq}$-cluster from $\mathcal{F}$ and define a well ordering where $m \leq n$ iff $\mathcal{C}_m \mathsf{R}_{\preccurlyeq} \mathcal{C}_n$. For each non-final $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}_j$ consider a new world d_j and join such d_j to $\mathcal{F} \circ @$ as follows: $\forall v \in \mathcal{C}_j^{\preccurlyeq}$ $(d_j \mathsf{R}_{\preccurlyeq} v \;\&\; \neg(v \mathsf{R}_{\preccurlyeq} d_j))$, $d_j \mathsf{R}_{\preccurlyeq} d_j$, $d_j \mathsf{R}_{\mathsf{e}} d_j$ and $d_j \mathsf{R}_{\mathsf{a}} d_j$.

Extend the valuation V to any new world d_j in the following way: for any propositional letter p_i, $d_j \in V(p_i)$ if and only if $@ \in V(p_i)$.

Denote this model by $\langle \mathcal{F}' \circ @, S \rangle$. The frame on which this model is built on has the same form as the one defined in Lemma 5.3.5. Clearly in the resulting model it is still true that $z \nVdash_S g(p_1, \ldots, p_n)$. It is problematic, however, to show that for any world $v \in |\mathcal{F}' \circ @|$ the statement $v \Vdash_S \bigwedge_k f_k(p_1, \ldots, p_n)$ holds true.

The presence of the newly added *dot-worlds*, however, does indeed affect the truth value of any formula in $\langle \mathcal{F} \circ @, V \rangle$. It is therefore impossible to show the following, i.e. that for any formula f_i from $\bigwedge_k f_k(p_1, \ldots, p_n)$ and for any *dot-world* d_j: $d_j \Vdash_S f_i \quad \Leftrightarrow \quad @ \Vdash_V f_i$.

Our guess is that we need something more than this set of rules to provide a basis, namely some quasi-identities Q which can guarantee the property that given a finitely generated algebra $\mathfrak{A}[a_1, \ldots, a_n]$ from $Var(\mathsf{LTK_1})$ such that $\|A\| > 1$, there is a quasi-identity q such that if $\mathfrak{A} \models q$ then in $\mathfrak{A}^+$ each $\mathsf{R}_{\preccurlyeq}$-cluster has a single element $\mathsf{R}_{\preccurlyeq}$-cluster which is its $\mathsf{R}_{\preccurlyeq}$-immediate predecessor.

Chapter 6

Conclusions

6.1 Summary of the Thesis

In order to have a clear and systematic view of the results provided in the previous chapters, we would like to provide a summary of our research as well as to highlight our contributions to common knowledge.

Chapter 1. **Introduction.** We introduced our research topic and clarified the reasons that led us to work with multi-modal logics. We explained why we decided to adopt multi-modal languages and logics in order to deal with multi-agent reasoning. We surveyed briefly some major contributions in the field and then turned our attention to the problem of inference rules, which are the core of our whole research.

Chapter 2. **A Semantic Definition of** LTK**.** We provided a semantic definition of some multi-modal propositional systems. In particular we introduced the logic LTK. We used a semantic approach. In fact we have started our research by defining a set of Kripke frames. These structures are useful whenever one is to model the behaviour of a set of agents operating on a temporal framework. Thus we defined our logics as the set of all those formulae which are valid in this class of frames. We made a substantial use of the so called *Possible World Semantics* or Kripke-semantics. Further, we showed that the logic LTK has the *effective finite model property* and it is hence decidable with respect to its theorems. This means that for each formula $\mathtt{A}$ which is *not* a theorem of L we can build a model $\mathcal{M}$ whose size is finite *and* computable from the size of $\mathtt{A}$ such that it verifies all theorems of L and falsifies $\mathtt{A}$. Hence the logic LTK is decidable with respect to its theorems. In fact in order to check whether a formula is a theorem it is enough to check only those models which are smaller than a certain finite number n which can be calculated from the size of the formula itself.

Chapter 3. **Admissible Rules in LTK_1: Decidability.** In Chapter 3 we started our semantic analysis of logical consequences. We defined the set of admissible inference rules as the class containing all those rules which can be applied to a given logic without altering its set of theorems. We designed an algorithm which can check, given any inference rule, if this rule is or is not admissible for LTK_1.

Chapter 4. **The Axiomatic System $\mathcal{AS}_{\mathsf{LTK}}$.** In Chapter 4 we provided some axiom schemata and rules which allow the interaction between modalities. As we saw, this is a useful tool in order to deal with the concepts of *learning* and *forgetting*. A language that lacks the power to combine different modalities is, in fact, useless in order to deal with both *learning* and *forgetting* and it cannot handle, therefore, changing knowledge bases. On the other hand, proving that an axiomatic system with combined modalities is sound and complete with respect to a class of frames is neither easy, nor straightforward. Nevertheless we provided a sound and complete axiomatisation with combined modalities.

Chapter 5. **Rules in LTK_1: Structural Incompleteness.** Finally, in Chapter 5 we presented both our latest results and our current research topic. We started by proving that the logic LTK_1 is not *structurally complete*. This means that there are inference rules which are not derivable on the axiomatic system which generates LTK_1. These rules, are, nevertheless, admissible for LTK_1. In this Chapter we define an infinite set of rules with this property. Since all admissible rules can be applied in derivations without altering the set of theorems of a logic, the class of admissible and not derivable rules we present here adds new syntactical tools which can be used in derivations.

Then we provided Algebraic Semantics for LTK_1 and, finally, we introduced the further work and the piece of research we are currently working on. We started to investigate the problem of finding a finite basis for admissible inference rules. We aim at finding a set of rules to *axiomatise* all the inference rules admissible for LTK_1, i.e. the smallest set of rules starting from which one can derive all the admissible rules for LTK_1.

6.2 Contributions

We have introduced some new logical systems which are useful whenever one is to model a situation with several agents operating in a temporal framework. As we have seen in Chapter 1, typical agents may be computer programs running in parallel on some platform or buffers and other devices. Nevertheless, agents may also be seen as players in some strategic game, human beings operating and co-operating in a social environment in order to reach a common goal. Our results can hence provide analytical tools to be used in several fields. Besides Computer Science and Artificial Intelligence, one may apply our logics to the study of social-economic phenomena (e.g. game theory, economical analysis of markets, local and global social interactions).

Besides the interpretation provided and the possible contingent applications, however, we want to highlight the theoretical relevance of our research. We have analysed some logical systems and we have provided some intended models, as the multi-agent framework and the temporal multi-epistemic approach. On the theoretical side, however, we have described some new logics and we have analysed them from a perspective that, as far as we are concerned, has never been considered before. In particular we have built a multi-modal system with combined modalities, LTK_1, which is the result of the fusion and the interaction of three distinct modal systems, namely two $\mathsf{S5}$ systems and one $\mathsf{S4.3}$. We have showed that this logic is:

(i) Decidable with respect to its theorems;

(ii) Decidable with respect to its inference rules;

(iii) Generated by a finite axiomatic system;

(iv) Structurally incomplete.

Thus, these results are available to any researcher willing to model a logical system with the properties we have described. Anyone can chose both the interpretation of the modal operators and the intended models most suitable for his/her purposes.

Moreover, we have analysed our systems from the point of view of inference rules, contributing our decidability results to the field. The study of inference rules applied to multi-modal logics has, in fact, started only recently and there is still much work to be done. Our decidability results are, therefore, a further step towards a systematic and complete analysis of the wide field of inference rules applied to multi-modal propositional logics.

Appendix A

Published Works

(i) Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, semantic approach. *Bulletin of the Section of Logic*, 34(1):13–21, 2005.

(ii) Erica Calardo. Admissible inference rules in the linear logic of knowledge and time LTK. *Logic Journal of the IGPL*, 14(1):15–34, 2006.

(iii) Erica Calardo and Vladimir V. Rybakov. An Axiomatisation for the Multi-modal Logic of Knowledge and Linear Time LTK. *Logic Journal of the IGPL*, 15(3):239–254, 2007.

E.Calardo
V.Rybakov

Combining Time and Knowledge, Semantic Approach

Abstract

The paper investigates a semantic approach for combining knowledge and time. We introduce a multi-modal logic $L(\mathcal{T}_{\mathcal{K}})$ containing modalities for knowledge and time in a semantic way, as the set of all $\mathcal{T}_{\mathcal{K}}$-valid formulae for a class of special frames $\mathcal{T}_{\mathcal{K}}$. The main result of our paper is the theorem stating that $L(\mathcal{T}_{\mathcal{K}})$ is decidable and giving a resolving algorithm. The result is proven by using standard tools: filtration, bulldozing and contracting p-morphisms.

1 Introduction

The paper is devoted to study a semantic approach to model knowledge and time. Study of time and knowledge within framework of modal logic is an active area nowadays (cf. [2, 3, 4, 6] and references therein). Sound and complete axiomatizations for a number of different logics involving modalities for knowledge and time are found in [4]. Our approach, in a sense, is from an opposite site: we generate a logic combining knowledge and time in a semantic manner, via a class of frames which defines such a logic. Our aim is to study the question about decidability. We would like to investigate to which extend a standard technique of modal logic works, we like to construct a deciding algorithm using only standard technique of modal logics without involving heavy technique as automatons or the Rabin theorem.

We model the time as a linear discrete sequence of time states, and the knowledge is represented by a tuple of modal-like operations K_i (imitating knowledge of agents) which operates in time states containing a set of information nodes. We start by introduction of a certain class of multi-modal Kripke frames which have the structure described above and generate the logic $L(\mathcal{T}_{\mathcal{K}})$ as the set of all formulae which are true in these frames. We assume time flow to be linear and discrete and agents operating synchronously: they have access to a sort of *shared clock*[1], each agent knowing what time it is and distinguishing present from future time. The main result of our paper is the theorem stating that $L(\mathcal{T}_{\mathcal{K}})$ is decidable and giving a resolving algorithm.

2 Notation, Definitions

General notation and definitions concerning modal logics which we will use can be found, or instance, in [1, 5]. To study the combination of knowledge and time we will use the language of multi-modal logic. Our language $\mathcal{L}^{\mathcal{T}_{\mathcal{K}}}$ is chosen as follows: the alphabet of $\mathcal{L}^{\mathcal{T}_{\mathcal{K}}}$ contains propositional letters $P := \{p_1, ..., p_n, ...\}$, round brackets $(,)$, standard boolean operations, and the set of modal operations $\{\Box_{\preccurlyeq}, \Box_{\sim}, \{K_i \mid i \in I := \{1, ..., k\}\}$. Well formed formulae (wff) are defined in the standard way, in particular, if A is a wff, than $\Box_{\preccurlyeq}A$, $\Box_{\sim}A$, K_iA, for all $i \in I$, are wff. $Fma(L)$ is the set of all well formed formulae of $\mathcal{L}^{\mathcal{T}_{\mathcal{K}}}$. The informal meaning of the modal operations is as follows. The set $I := \{1, ..., k\}$ indicates k distinct agents. $\Box_{\preccurlyeq}A$ means: the formula A *always will be true*; K_iA: the agent i knows A in the current time state and the current information node; $\Box_{\sim}A$: the wise agent knows A in the current time state and current information node.

Semantics for this language is based on linear and discrete time flow, associating a time point with any natural number n. As semantic tools we will use the following Kripke-Hintikka frames: $\mathcal{T}_{\mathcal{K}} := \langle W_{\mathcal{T}_{\mathcal{K}}}, R_{\preccurlyeq}, R_{\sim}, R_1, ..., R_k \rangle$, where the base set of $\mathcal{T}_{\mathcal{K}}$ is the disjoint union of sets $\mathcal{C}^n$, $W_{\mathcal{T}_{\mathcal{K}}} := \bigcup_{n \in \mathbb{N}} \mathcal{C}^n$,. Binary relations $R_{\preccurlyeq}$, $R_{\sim}$, and $R_1, ..., R_k$ are as follows: $R_{\preccurlyeq}$ is the following linear, reflexive and transitive relation on $W_{\mathcal{T}_{\mathcal{K}}} \times W_{\mathcal{T}_{\mathcal{K}}}$:

$$\forall x, y \in W_{\mathcal{T}_{\mathcal{K}}} (xR_{\preccurlyeq}y \quad \text{iff} \quad \exists n_1, n_2 \in \mathbb{N} \; ((x \in \mathcal{C}^{n_1}) \& \\ \& \; (y \in \mathcal{C}^{n_2}) \; \& \; (n_1 \leq n_2)));$$

$R_{\sim}$ is the equivalence relation on any $C^n \in W_{\mathcal{T}_{\mathcal{K}}}$:

$\forall x, y \in W_{\mathcal{T}_{\mathcal{K}}} (xR_{\sim}y \;$ iff $\; \exists n \in \mathbb{N} \; (x \in \mathcal{C}^n \; \& \; y \in \mathcal{C}^n)$;

Any R_i is some equivalence relation on any $\mathcal{C}^n$.

[1]See Fagin *et al.*, [2], pp. 127-128.

The informal meaning of these frames is as follows. Any cluster $\mathcal{C}^n$ contains a set of information nodes available at the time point n. The relation $R_\preccurlyeq$ is the connection of the information nodes by time current: $xR_\preccurlyeq y$ indicates that the node y is a node available in the same time as x, or y is an information node in a future time point. $xR_\sim y$ says that x and y are nodes in the same time point, and $xR_i y$ indicates that in the current time point y is accessible from x by of the agent i authorities. A model $\mathcal{M}_{\mathcal{T}_\mathcal{K}}$ on $\mathcal{T}_\mathcal{K}$ is a tuple $\mathcal{M}_{\mathcal{T}_\mathcal{K}} = \langle \mathcal{T}_\mathcal{K}, V\rangle$ where V is a valuation of a set P of propositional letters in $\mathcal{T}_\mathcal{K}$. That is, for any $p_i \in P$ $V(p_i) \subseteq W_{\mathcal{T}_\mathcal{K}}$.

The valuation V can be extended from the set P onto all wff's constructed from P in the standard way. In particular, $\forall x \in W_{\mathcal{T}_\mathcal{K}}$,

$x \Vdash_V \Box_\preccurlyeq A$ iff $\forall y \in W_{\mathcal{T}_\mathcal{K}}$ $(xR_\preccurlyeq y \implies y \Vdash_V A)$;

$x \Vdash_V \Box_\sim A$ iff $\forall y \in W_{\mathcal{T}_\mathcal{K}}$ $(xR_\sim y \implies y \Vdash_V A)$;

$x \Vdash_V K_i A$ iff $\forall y \in W_{\mathcal{T}_\mathcal{K}}$ $(xR_i y \implies y \Vdash_V A)$.

Let $\mathcal{M}_{\mathcal{T}_\mathcal{K}} := \langle \mathcal{T}_\mathcal{K}, V\rangle$ be a model on a frame $\mathcal{T}_\mathcal{K}$; a formula $A \in Fma(\mathcal{L}^{\mathcal{T}_\mathcal{K}})$ is said to be *true* in $\mathcal{M}_{\mathcal{T}_\mathcal{K}}$ at the point $a \in W_{\mathcal{T}_\mathcal{K}}$ if $a \Vdash_V A$. A formula A is *true* in the model $\mathcal{M}_{\mathcal{T}_\mathcal{K}}$, notation $\mathcal{M}_{\mathcal{T}_\mathcal{K}} \Vdash A$, if $\forall a \in W_{\mathcal{T}_\mathcal{K}}$, $a \Vdash_V A$. A is *valid* in the frame $\mathcal{T}_\mathcal{K}$, notation $\mathcal{T}_\mathcal{K} \Vdash A$, if, for any model $\mathcal{M}_{\mathcal{T}_\mathcal{K}}$ on $\mathcal{T}_\mathcal{K}$, $\mathcal{M}_{\mathcal{T}_\mathcal{K}} \Vdash A$.

Definition 2.1 *The logic $L(\mathcal{T}_\mathcal{K})$ is the set of all $\mathcal{T}_\mathcal{K}$-valid formulae:*
$L(\mathcal{T}_\mathcal{K}) := \{A \in Fma(\mathcal{L}^{\mathcal{T}_\mathcal{K}}) \mid \mathcal{T}_\mathcal{K} \Vdash A, \ \forall\ \mathcal{T}_\mathcal{K}\text{-}frame\}$

3 Decidability

The aim of our paper is to prove that the logic $L(\mathcal{T}_\mathcal{K})$ is decidable. Initially we will show that any formula A which is not a theorem of $L(\mathcal{T}_\mathcal{K})$ can be refused by a frame similar to f $\mathcal{T}_\mathcal{K}$ but of a finite size computable from the length of A. Consider and fix for the rest of this paper a formula A such that $A \notin L(\mathcal{T}_\mathcal{K})$. Then there is a frame $\mathcal{T}_\mathcal{K}$ and a model $\mathcal{M}_{\mathcal{T}_\mathcal{K}} := \langle \mathcal{T}_\mathcal{K}, V\rangle$ based on this frame such that, $\exists a \in W_{\mathcal{T}_\mathcal{K}}$, $(\mathcal{M}_{\mathcal{T}_\mathcal{K}}, a) \nVdash_V A$. Firstly we reduce the number of elements in any $\mathcal{C}^n$ to a finite number of ones effectively bounded from size of A. This can be easy done by a standard filtration on any separate $\mathcal{C}^n$. Below we briefly sketch this technique. Let $Sub(A)$ be the set of all the subformulae of A. Define the equivalence relation $\approx$ on $W_{\mathcal{T}_\mathcal{K}}$ as follows: $\forall a, b \in W_{\mathcal{T}_\mathcal{K}}[a \approx b$ iff $\exists n \in \mathbb{N}$ $(a, b \in C^n$ & $\forall \beta \in Sub(A)$ $(a \Vdash_V$

β iff $b \Vdash_V \beta))$]. Next, define the quotient set of the original model: $\forall a \in W_{\mathcal{T_K}}$ $[a]_\approx := \{b \mid a \approx b\}$, $\forall n \in \mathbb{N}$ $C^n_\approx := \{[a]_\approx \mid a \in C^n\}$, $W^{\approx}_{\mathcal{T_K}} := \bigcup_{n\in\mathbb{N}} C^n_\approx$.

The model resulting from this filtration is based on this quotient set and looks as follows: $\mathcal{M}^{\approx}_{\mathcal{T_K}} := \langle W^{\approx}_{\mathcal{T_K}}, R^{\approx}_{\preccurlyeq}, R^{\approx}_{\sim}, R^{\approx}_1, ..., R^{\approx}_k, V^{\approx}\rangle$ where: $\forall p \in Sub(A)$, $V^{\approx}(p) := \{[a]_\approx \mid a \in V(p)\}$; $\forall [a]_\approx, [b]_\approx \in W^{\approx}_{\mathcal{T_K}}$,

$[a]_\approx R^{\approx}_{\preccurlyeq}[b]_\approx$ iff $\exists n, m \in \mathbb{N}$ $([a]_\approx \in C^n_\approx$ & $[b]_\approx \in C^m_\approx$ & $n \leq m)$;

$[a]_\approx R^{\approx}_{\sim}[b]_\approx$ iff $\exists n, m \in \mathbb{N}$ $([a]_\approx \in C^n_\approx$ & $[b]_\approx \in C^m_\approx$ & $n = m)$;

$\forall i \in I$ $[a]_\approx R^{\approx}_i[b]_\approx$ iff $\exists n \in \mathbb{N}$ $([a]_\approx, [b]_\approx \in C^n_\approx$ &

&$\forall K_i\beta \in Sub(A)$ $((\mathcal{M}_{\mathcal{T_K}}, a) \Vdash_V K_i\beta$ iff $(\mathcal{M}_{\mathcal{T_K}}, b) \Vdash_V K_i\beta))$.

Since the model described is a result of filtration the standard filtration-lemma holds:

Lemma 3.1 *For any formula $\beta \in Sub(A)$, for any element $a \in W_{\mathcal{T_K}}$ $(\mathcal{M}_{\mathcal{T_K}}, a) \Vdash_V \beta \Leftrightarrow (\mathcal{M}^{\approx}_{\mathcal{T_K}}, [a]_\approx) \Vdash_{V^\approx} \beta$.*

Corollary 3.2 $\mathcal{M}^{\approx}_{\mathcal{T_K}} \nVdash A$.

Lemma 3.3 *If $\|Sub(A)\| := m$, then $\forall n \in \mathbb{N}$, $\|C^n_\approx\|$ is at most 2^m.*

Thus the model $\mathcal{M}^{\approx}_{\mathcal{T_K}}$ refutes A and has clusters $\mathcal{C}^n$ of effectively bounded size. Using $\mathcal{M}^{\approx}_{\mathcal{T_K}}$ we will construct a finite model refusing A. The clusters $C^n_\approx$ and $C^j_\approx$ are **isomorphic** (we will use in the sequel notation: $C^n_\approx \cong C^j_\approx$) if and only if there is a function f s.t.: $f: C^n_\approx \longrightarrow C^j_\approx$, (1) f is a bijection, (2) $\forall \xi \in \{\preccurlyeq, \sim, 1, ..., k\}$, $\forall a, b \in C^n_\approx$ $(aR^{\approx}_\xi b$ iff $f(a)R^{\approx}_\xi f(b))$, (3) $\forall p \in Sub(A)$, $\forall a \in C^n_\approx$ $(a \in V^{\approx}(p)$ iff $f(a) \in V^{\approx}(p))$. By Lemma 3.3 we conclude

Proposition 3.4 *There is only a finite, computable from A, number of non-isomorphic time-clusters $C^n_\approx \in W^{\approx}_{\mathcal{T_K}}$.*

For any time cluster $C^n_\approx$, $C^{n\preccurlyeq}_\approx$ is the set of all the $\preccurlyeq$-successor clusters of $C^n_\approx$: $\forall$ $C^n_\approx \in W^{\approx}_{\mathcal{T_K}}$, $C^{n\preccurlyeq}_\approx := \{C^j_\approx \mid n \leq j\}$, and $C^{n+}_\approx := \bigcup C^{n\preccurlyeq}_\approx$. In the sequel, $C^{n+}_\approx(M)$ or $C^{n\preccurlyeq}_\approx(M)$ are described sets from a frame M (we will alter these frames M).

Definition 3.5 *The time-cluster $C^n_\approx$ is a stabilizing cluster if and only if for any $C^j_\approx$, where $n \leq j$, the sets $C^{n\preccurlyeq}_\approx$ and $C^{j\preccurlyeq}_\approx$ coincide up to isomorphism of clusters.*

Lemma 3.6 *The model $\mathcal{M}^{\approxeq}_{\mathcal{T}_{\mathcal{K}}}$ has a stabilizing cluster C^s.*

Proof. By Proposition 3.4 the number of non-isomorphic time-clusters $C^n_{\approx} \in W^{\approxeq}_{\mathcal{T}_{\mathcal{K}}}$ is finite. The following also holds: $\forall n, j \in \mathbb{N}, \quad n \leq j \implies C^{n\preccurlyeq}_{\approx} \supseteq C^{j\preccurlyeq}_{\approx}$. Consider the sequence of all the time-clusters $C^1_{\approx}, C^2_{\approx}, \ldots$. We construct a subsequence $C^{n'}_{\approx}$ of the sequence $C^n_{\approx}, n \in \mathbb{N}$ as follows. Take $C^1_{\approx}$; if $C^1_{\approx}$ is a stabilizing cluster, then we stop, and the subsequence is chosen. Assume a subsequence $C^{1'}_{\approx}, \ldots, C^{n'}_{\approx}$ is chosen. If $C^{n'}_{\approx}$ is not a stabilizing cluster, then there is a cluster $C^k_{\approx}$, where, up to isomorphism, $C^{n'\preccurlyeq}_{\approx} \supset C^{k\preccurlyeq}_{\approx}$. Take the $\preccurlyeq$-smallest $C^k_{\approx}$ with this property and set $C^{(n+1)'}_{\approx} := C^k_{\approx}$. Since $C^{n'\preccurlyeq}_{\approx} \supset C^{(n+1)'\preccurlyeq}_{\approx}$, this procedure must terminate, and it terminates at a stabilizing cluster. ⊠

We denote by C^s the $\preccurlyeq$-smallest stabilizing cluster.

Lemma 3.7 *If C^s is a stabilizing cluster, then $\forall n, j \in \mathbb{N}$, where $n, j \geq s$, the following holds. If $C^n_{\approx}$ is isomorphic to $C^j_{\approx}$ by a mapping f, then $\forall \beta \in Sub(A)$, $\forall a \in C^n_{\approx}$ $((C^{n+}_{\approx}, a) \Vdash_{V^{\approx}} \beta$ iff $(C^{j+}_{\approx}, f(a)) \Vdash_{V^{\approx}} \beta)$.*

Proof may be given by induction on the length of β. The only non-trivial steps are the ones for the modal operations. If β is $\Box_{\sim} B$ or $K_i B$ for $i \in I$ the claim holds by the induction hypotheses and the definition of isomorphism. Let β be $\Box_{\preccurlyeq} B$. Assume $(C^{n+}_{\approx}, a) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$. We can have 3 cases: (i) $n = j$ where the proof is trivial, (ii) $n < j$, and (iii) $n > j$. If $n < j$, $(C^{n+}_{\approx}, a) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$ implies that for any $b \in C^{n+}_{\approx}$ $(b \Vdash_{V^{\approx}} B)$. Since $n < j$, $C^{n+}_{\approx} \supseteq C^{j+}_{\approx}$ holds and $\forall c \in C^{j+}_{\approx}, \quad (\mathcal{M}^{\approxeq}_{\mathcal{T}_{\mathcal{K}}}, c) \Vdash_{V^{\approx}} B$. Consequently $(\mathcal{M}^{\approxeq}_{\mathcal{T}_{\mathcal{K}}}, f(a)) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$ and $(C^{j+}_{\approx}, f(a)) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$. The proof of the converse is similar to the case (iii) below. Consider the case (iii) when $n > j$. Assume $(C^{n+}_{\approx}, a) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$. This implies that, for any $b \in C^{n+}_{\approx}$, $((\mathcal{M}^{\approxeq}_{\mathcal{T}_{\mathcal{K}}}, b) \Vdash_{V^{\approx}} B)$. Since $n, j \geq s$ and C^s is the stabilizing cluster, for any $C^m_{\approx} \in C^{j\preccurlyeq}_{\approx}$ there is some $C^{m'}_{\approx} \in C^{n\preccurlyeq}_{\approx}$ such that $C^m_{\approx} \cong C^{m'}_{\approx}$. Therefore by induction hypothesis we conclude $\forall C^m_{\approx} \in C^{j\preccurlyeq}_{\approx}$, $\forall c \in C^m_{\approx}$ $(C^{m+}_{\approx}, c) \Vdash_{V^{\approx}} B$. Then $(\mathcal{M}^{\approxeq}_{\mathcal{T}_{\mathcal{K}}}, f(a)) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$ and $(C^{j+}_{\approx}, f(a)) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$. The proof of the converse is similar to the previous case. ⊠

For any time-cluster $C^n_{\approx}$, where $n \geq s$, $[C^n_{\approx}]_{\cong}$ is the set of all the time-clusters isomorphic to $C^n_{\approx}$: $\forall n, j \geq s$ $[C^n_{\approx}]_{\cong} := \{C^j_{\approx} \mid C^n_{\approx} \cong C^j_{\approx}\}$.

Take and fix, for any $[C^n_{\approx}]_{\cong}$ a unique representative cluster C^*_n. Let $St := \bigcup_{n \geq s} C^*_n$ be the set of all the elements of such clusters. We define a new finite model as follows: $\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}} := \langle W^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, R^{\mathcal{B}}_{\preccurlyeq}, R^{\mathcal{B}}_{\sim}, R^{\mathcal{B}}_1, ..., R^{\mathcal{B}}_k, V^{\mathcal{B}} \rangle$, where $W^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}} := \{C^1_{\approx}, C^1_{\approx}, ..., C^s, St\}$, $\forall p \in Sub(A)$ $V^{\mathcal{B}}(p) := \{a \in W^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}} \mid a \in V^{\approx}(p)\}$, $\forall a, b \in W^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}$ $\forall n, j \leq s$ $((a \in C^n_{\approx}$ & $b \in C^j_{\approx}) \implies (aR^{\mathcal{B}}_{\preccurlyeq}b$ iff $aR^{\approx}_{\preccurlyeq}b))$, otherwise, if $n, j > s$, $R^{\mathcal{B}}_{\preccurlyeq}$ is a universal relation on St: $\forall a, b \in St$ $(aR^{\mathcal{B}}_{\preccurlyeq}b)$. And $aR^{\mathcal{B}}_{\sim}b$ iff $aR^{\approx}_{\sim}b$, $\forall i \in I$ $aR^{\mathcal{B}}_i b$ iff $aR^{\approx}_i b$.

Lemma 3.8 *For any formula $\beta \in Sub(A)$ and, for any $a \in W^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}$, $(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\approx}} \beta$ iff $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\mathcal{B}}} \beta$.*

Proof is given by induction on the length of β. The steps for the boolean operations are standard. Let β be $\Box_{\preccurlyeq}B$. Assume $(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\approx}} \Box_{\preccurlyeq}B$. Since $a \in C^n_{\approx}$ for some $n \in \mathbb{N}$, we have 2 cases: (A): $n \leq s$ and (B): $n > s$.

Consider (A). Then $(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\approx}} \Box_{\preccurlyeq}B$ implies that for all elements $b \in C^{n+}_{\approx}$ $(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, b) \Vdash_{V^{\approx}} B$. $C^n_{\approx}$ belongs to $\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}$ by assumption. Let $C^{n+}_{\approx}(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}})$ be the set of all $\preccurlyeq$-successors of $C^n_{\approx}$ in $\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}$ and $C^{n+}_{\approx}(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}})$ be the set of all $\preccurlyeq$-successors of $C^n_{\approx}$ in $\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}$. Then $C^{n+}_{\approx}(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}) \supseteq C^{n+}_{\approx}(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}))$. Therefore by IH we have $\forall c \in C^{n+}_{\approx}(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}})$, $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, c) \Vdash_{V^{\mathcal{B}}} B$, and so it follows $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\mathcal{B}}} \Box_{\preccurlyeq}B$.

Consider the case (B): $n > s$. Then $(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\approx}} \Box_{\preccurlyeq}B$ implies that $\forall b \in C^{n+}_{\approx}$ $(\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, b) \Vdash_{V^{\approx}} B$. Consider all the clusters between C^s and $C^{\approx}_n$: by the definition of stabilizing cluster, each of them is isomorphic to some cluster belonging to $C^{n\preccurlyeq}_{\approx}$. Therefore, by Lemma 3.7 we have that $\forall c \in C^j_{\approx}$ $(s \leq j \leq n \implies (\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, c) \Vdash_{V^{\approx}} B)$. So we have $\forall b \in C^{s\preccurlyeq}$ $((\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, b) \Vdash_{V^{\approx}} B)$. Since $St \subseteq C^{s+}$, $\forall c \in St$ $((\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, b) \Vdash_{V^{\approx}} B)$ holds. Applying IH we conclude $\forall c \in St$ $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, b) \Vdash_{V^{\mathcal{B}}} B$ and it follows $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\mathcal{B}}} \Box_{\preccurlyeq}B$.

Assume now that $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\mathcal{B}}} \Box_{\preccurlyeq}B$. Since $a \in C^n$ for some $n \in \mathbb{N}$, we still have 2 cases: (C): $n \leq s$ and (D): $n > s$. In the case (C), when $n \leq s$, $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, a) \Vdash_{V^{\mathcal{B}}} \Box_{\preccurlyeq}B$ implies that $\forall b \in C^{n+}_{\approx}(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}})$, $(\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}, b) \Vdash_{V^{\mathcal{B}}} B$. Since up to the stabilizing cluster C^s, $\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}$ and $\mathcal{M}^{\mathcal{B}}_{\mathcal{I}_{\mathcal{K}}}$ have exactly the same clusters, applying the induction hypotheses we have $\forall c \in C^m_{\approx}$ $(n \leq m \leq s \implies (\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}, c) \Vdash_{V^{\approx}} B)$.

We have now to analyze the case when $c \in St$. First, B is true w.r.t. $V^{\mathcal{B}}$ in any element $C^*_j \in St$. Any C^*_j belongs to $\mathcal{M}^{\approx}_{\mathcal{I}_{\mathcal{K}}}$ as well, and applying the induction hypotheses we conclude: $\forall C^*_j \in St$ $\forall b \in$

C_j^* $(\mathcal{M}^{\approx}_{\mathcal{T}_\mathcal{K}}, b) \Vdash_{V^{\approx}} B$. By Lemma 3.7, we have $\forall C^m_{\approx} \in [C_j^*]_{\cong}$ $\forall c \in C^m_{\approx}$, $(\mathcal{M}^{\approx}_{\mathcal{T}_\mathcal{K}}, c) \Vdash_{V^{\approx}} B$ and so we can conclude $\forall c \in C^{s+}$, $(\mathcal{M}^{\approx}_{\mathcal{T}_\mathcal{K}}, c) \Vdash_{V^{\approx}} B$. Consequently $(\mathcal{M}^{\approx}_{\mathcal{T}_\mathcal{K}}, a) \Vdash_{V^{\approx}} \Box_{\preccurlyeq} B$.

Consider now the case (D): $n > s$. $(\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, a) \Vdash_{V^{\mathcal{B}}} \Box_{\preccurlyeq} B$ implies that $\forall b \in St$ $(\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, b) \Vdash_{V^{\mathcal{B}}} B$, because $R^{\mathcal{B}}_{\preccurlyeq}$ is an equivalence relation on $St \times St$. The rest of the proof for this case is similar to the final part of the case (C).

The inductive step for the case when β is $\Box_{\sim} B$ or β is $K_i B$, $i \in I$ is immediate because the relations $R^{\approx}_{\sim}$ $(R^{\approx}_i)$ and $R^{\mathcal{B}}_{\sim}$ $(R^{\mathcal{B}}_i)$ are the same in $\mathcal{M}^{\approx}_{\mathcal{T}_\mathcal{K}}$ and $\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}$. $\boxtimes$

Thus, by this lemma, the model, $\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}$, is finite and refuses the formula A. Since the number of elements in this model is not effectively bounded, we do not have yet decidability of the logic $L(\mathcal{T}_\mathcal{K})$. Below we will construct a new model by dropping some $\preccurlyeq$-clusters from $\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}$.

For any sub-formula β of A, C_β is the $\preccurlyeq$-maximal $\preccurlyeq$-cluster among $C^1_{\approx}, C^2_{\approx}, ..., C^s_{\approx}$ s.t. $\exists b \in C_\beta (\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, b) \Vdash_{V^{\mathcal{B}}} \beta$, if such clusters exists. $C_{\neg\beta}$ is the analogous cluster for $\neg\beta$. The new model is as follows:

$$W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}} := \bigcup_{\beta \in Sub(A)} C_\beta \cup \bigcup_{\beta \in Sub(A)} C_{\neg\beta} \cup St,$$

$$\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}} := \langle W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, R^{\mathcal{F}}_{\preccurlyeq}, R^{\mathcal{F}}_{\sim}, R^{\mathcal{F}}_1, ..., R^{\mathcal{F}}_k, V^{\mathcal{F}} \rangle$$

where: $\forall p \in Sub(A)$ $V^{\mathcal{F}}(p) := \{a \in W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}} \mid a \in V^{\mathcal{B}}(p)\}$, $\forall a, b \in W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}$, $\forall R_\xi \in \{R_{\preccurlyeq}, R_{\sim}, R_1, ..., R_k\}$, $aR^{\mathcal{F}}_\xi b$ iff $aR^{\mathcal{B}}_\xi b$.

Lemma 3.9 *For any formula $\beta \in Sub(A)$, for any element $a \in W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}$, $(\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, a) \Vdash_{V^{\mathcal{F}}} \beta$ iff $(\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, a) \Vdash_{V^{\mathcal{B}}} \beta$.*

Proof is by induction on the length of β. Evidently we only need to consider the steps for modal operations. If β is $\Box_{\approx} B$ or β is $K_i B$, the steps are evident because all the relations $R^{\mathcal{F}}_{\sim}$ and $R^{\mathcal{F}}_i$ are the same in $\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}$ and $\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}$. Consider the case when β is $\Box_{\preccurlyeq} B$. If $(\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, a) \Vdash_{V^{\mathcal{B}}} \Box_{\preccurlyeq} B$ then $\forall b \in W^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}$ $(aR_{\preccurlyeq} b \implies (\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, b) \Vdash_{V^{\mathcal{B}}} B)$. Since $W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}} \subseteq W^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}$, by induction hypothesis we have $\forall c \in W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}$ $(aR_{\preccurlyeq} b \implies (\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, b) \Vdash_{V^{\mathcal{F}}} B)$ and so $(\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, a) \Vdash_{V^{\mathcal{F}}} \Box_{\preccurlyeq} B$.

If $(\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, a) \nVdash_{V^{\mathcal{F}}} \Box_{\preccurlyeq} B$ then there is an element $b \in W^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}$ such that $aR_{\preccurlyeq} b$ and $(\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, b) \nVdash_{V^{\mathcal{B}}} B$.

If $b \in St$, then clearly $(\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, b) \nVdash_{V^\mathcal{F}} B$ and $(\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, a) \nVdash_{V^\mathcal{F}} \Box_{\preccurlyeq} B$. Otherwise there is an $R_{\preccurlyeq}$-maximal cluster $C_{\neg B}$ among $C^1_{\approx}, C^2_{\approx}, ..., C^s_{\approx}$ and a $c \in C_{\neg B}$ s.t. $(\mathcal{M}^{\mathcal{B}}_{\mathcal{T}_\mathcal{K}}, c) \nVdash_{V^\mathcal{B}} B$. Since $C_{\neg B}$ belongs to $W^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}$ by IH we conclude $(\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, c) \nVdash_{V^\mathcal{F}} B$. Since $aR_{\preccurlyeq}b$, it follows $(\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}, a) \nVdash_{V^\mathcal{F}} \Box_{\preccurlyeq} B$. $\boxtimes$

So, by this lemma A is refused by the model $\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}$ with effectively bounded size. Take an arbitrary frame F with the structure as the frame of a model $\mathcal{M}^{\mathcal{F}}_{\mathcal{T}_\mathcal{K}}$. It is easy to show that F is a p-morphic image of a frame $\mathcal{T}_\mathcal{K}$ based on $\sim$-clusters from the $\preccurlyeq$-linear part of F which $\preccurlyeq$-followed by infinite chain of $\sim$-clusters subsequently doubling the remaining part of $\sim$-clusters from F. Therefore all theorems of $L(\mathcal{T}_\mathcal{K})$ are true in F, and we have the following

Theorem 3.10 *The logic $L(\mathcal{T}_\mathcal{K})$ has the finite model property with computable size of refusing models, and hence $L(\mathcal{T}_\mathcal{K})$ is decidable.*

References

[1] R. Bull, K. Segerberg, Basic Modal Logic. In D. Gabbay and F.Guenthner (Eds), *Handbook of Philosophical Logic, Vol. II*, pp. 135-165. Reidel, Dordrechtn The Netherlands: 1984.

[2] R. Fagin, J.Y. Halpern, Y. Moses, M.Y. Vardi, *Reasoning About Knowledge*, MIT Press, Cambridge, Massachussets:1995.

[3] R. Goldblatt, *Logics of Time and Computation*, second edition, CSLI: 1992.

[4] J.Y. Halpern, R. Van Der Meyden, M.Y. Vardi, Complete Axiomatization for Reasoning About Knowledge and Time. *SIAM Journal on Computing* 33(3), 2004, pp. 674-703.

[5] V.V. Rybakov, *Admissibility of Logical Inference Rules*. Elsevier Science, Amsterdam: 1997.

[6] R.H. Thomason, Combination of tense and modality. In D. Gabbay and F.Guenthner (Eds), *Handbook of Philosophical Logic, Vol. II*, pp. 135-165. Reidel Dordrechtn, The Netherlands: 1984.

Admissible Inference Rules in the Linear Logic of Knowledge and Time LTK

ERICA CALARDO, *Department of Computing and Mathematics, Manchester Metropolitan University, John Dalton Building, Chester Street, Manchester M1 5GD, UK. E-mail: E.Calardo@mmu.ac.uk*

Abstract

The paper investigates admissible inference rules for the multi-modal logic LTK, which describes a combination of linear time and knowledge. This logic is semantically defined as the set of all $\mathcal{LTK}$-valid formulae, where $\mathcal{LTK}$-frames are multi-modal Kripke-frames combining a linear and discrete representation of the flow of time with special $S5$-like modalities, defined at each time cluster and representing knowledge. We start by revising the effective finite model property in this particular case, while the central part of the paper is devoted to constructing special n-characterising models for LTK. Such structeres allow us to find an algorithm determining admissible inference rules in LTK; the main result of this work is that LTK is decidable with respect to inference rules.

Keywords: Modal logic, Multi-modal logic, Epistemic logic, Tense logic, Inference rules, Admissible rules

1 Introduction

Modal and multi-modal propositional logics are among the most promising tools that have been developed so far to describe human reasoning. Modalities have been investigated since the dawn of philosophical and logical research. They are flexible by nature: modal operators can be interpreted in many ways. Depending on the chosen interpretation, we can generate different languages which are useful to describe distinct aspects of human reasoning. It is well known that the combination of temporal and knowledge modalities provides an highly expressive language, (cf. Fagin et al. [3], Thomason [23]). Multi-modal logics generated by adjoining operators representing time and knowledge to the classical propositional calculus PC are particularly effective for representing a state in which agents, who possess a certain knowledge, are operating in the flow of time (see for instance Dixon et al. [2], Fagin et al. [3], Gabbay et al. [5], Halpern et al. [12], Thomason [23], Wooldridge and Lomuscio [24]). These logics have many applications both in AI and in CS.

Although such techniques work fine in numerous applications, it is reasonable to ask whether and how the inference machinery could be enlarged. Many variations of axiomatic systems have been presented so far (see for instance

Halpern et al. [12]). But there are also other important components in derivations: inference rules.

Inference rules, or logical consecutions, are an important instrument of non-standard logics. For instance, rules can describe properties of modal frames in some cases in which using formulae may be difficult. A good example is Gabbay's *irreflexive rule* (cf. [6]):

$$\mathbf{ir} := \frac{\neg(p \to \Diamond p) \to \mathtt{A}}{\mathtt{A}}$$

(where p does not occur in the formula $\mathtt{A}$). This rule states that each world of a model, where $\mathtt{A}$ is not valid, should be irreflexive. Admissible consecutions have been deeply investigated for many modal and superintuitionistic logics (see, for instance, Ghilardi [7, 8, 9], Golovanov et al. [11], Iemhoff [15, 16], Jeřábek [17], Rybakov [20, 21, 22]). Their investigation began with Harrop's observation (cf. [13]) that we can enlarge an axiomatic system by adding admissible, though not derivable, inference rules.This approach led Friedman (see [4]) to ask whether there is an algorithm to recognise the rules admissible in IPC, the intuitionistic propositional calculus. This question and its analogues for modal logic has been solved by Rybakov [18, 19, 22], and a robust mathematical theory has been developed[1].

However, for the case of multi-modal logics, not much is known concerning admissible inference rules, though there have been some attempts to approach the problem (cf. for instance Golovanov et al. [10]). Nowadays, logics of this kind are an active research area and the axiomatic systems that have been constructed and examined are numerous (cf. Halpern et al. [12]). In our paper, we extend the investigation of this area to a multi-modal logic, LTK (Linear Time and Knowledge), which combines tense and knowledge modalities. This logic is semantically defined as the set of all $\mathcal{LTK}$-valid formulae, where $\mathcal{LTK}$-frames are multi-modal Kripke-frames combining a linear and discrete representation of the flow of time with special $S5$-like modalities, defined at each time cluster and representing knowledge.

The aim of this paper is to show that LTK is decidable with respect to admissible inference rules, i.e. to find an algorithm which recognises, given a rule $\mathbf{r}$, if $\mathbf{r}$ is admissible for LTK. We start by proving that LTK has the *effective finite model property* and hence it is decidable with respect to theorems (cf. Section 3). Although this result follows from Calardo and Rybakov [1], we will briefly sketch the proof in Section 3, because in the sequel we will need the techniques used. Section 4 is the core of this work and it is devoted to the construction of special countable n-characterising models for LTK. In Section

[1] For a more detailed historical account see Rybakov [22], Iemhoff [14].

5 we prove several technical lemmas. Finally, in Section 6, we present the main contribution of this paper. We show that an inference rule **r** is admissible in LTK if and only if it is valid in all the frames of a special kind, whose size is computable and bounded by the size of **r**. Hence, we prove that LTK is decidable w.r.t. inference rules.

2 Preliminaries

Before presenting our account of the semantic tools we will use in this paper, we recall some necessary basic definitions, so that this paper will be largely self contained.

The language $\mathcal{L}^{\mathsf{LTK}}$ is as follows: its alphabet consists of a countable set of propositional letters $P := \{p_1, \dots, p_n, \dots\}$, round brackets $(,)$, the standard boolean operations and the set of modal operations $\{\Box_{\preccurlyeq}, \mathsf{K_e}, \mathsf{K_a}\}$. Well formed formulae (wff's) are defined in the standard way, in particular, if $\mathtt{A}$ is a wff, than $\Box_{\preccurlyeq}\mathtt{A}$, $\mathsf{K_e}\mathtt{A}$, $\mathsf{K_a}\mathtt{A}$ are wff's; $Fma(\mathcal{L}^{\mathsf{LTK}})$ is the set of all the wff's of $\mathcal{L}^{\mathsf{LTK}}$ (in the rest of the paper, by the expression *formula* we always refer to a formula from $Fma(\mathcal{L}^{\mathsf{LTK}})$). The intended meaning of the modal operations is: (a) $\Box_{\preccurlyeq}\mathtt{A}$ means that *the formula* $\mathtt{A}$ *will always be true*; (b) $\mathsf{K_a}\mathtt{A}$ stands for *the agent operating in the system knows* $\mathtt{A}$ *in the current moment*; (c) $\mathsf{K_e}\mathtt{A}$ means that $\mathtt{A}$ *is known everywhere in the present time-cluster (i.e.* $\mathtt{A}$ *is part of the environmental knowledge).*

Definition 2.1
A k-modal Kripke-frame is a tuple $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_1, \dots, \mathsf{R}_k\rangle$ *where* $W_{\mathcal{F}}$ *is a non-empty set of worlds and each* R_i *is some binary relation on* $W_{\mathcal{F}}$*. Given a frame* $\mathcal{F}$*, by* $W_{\mathcal{F}}$ *we denote its base set.*

Definition 2.2
Given a Kripke-frame $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_1, \dots, \mathsf{R}_k\rangle$*, for any* R_i*, an* R_i*-cluster of worlds is a subset* $\mathcal{C}_{\mathsf{R}_i}$ *of* $W_{\mathcal{F}}$ *s.t.:* $\forall w \forall z \in \mathcal{C}_{\mathsf{R}_i}(w\mathsf{R}_i z \ \&\ z\mathsf{R}_i w)$ *and* $\forall z \in W_{\mathcal{F}} \forall w \in \mathcal{C}_{\mathsf{R}_i}((w\mathsf{R}_i z \ \&\ z\mathsf{R}_i w) \Rightarrow z \in \mathcal{C}_{\mathsf{R}_i})$*. For any* R_i*,* $\mathcal{C}_{\mathsf{R}_i}(w)$ *is the* R_i*-cluster s.t.* $w \in \mathcal{C}_{\mathsf{R}_i}(w)$*. Given two* R_i*-clusters* $\mathcal{C}_m$ *and* $\mathcal{C}_j$ *the expression* $\mathcal{C}_m\mathsf{R}_i\mathcal{C}_j$ *is an abbreviation for* $\forall w \in \mathcal{C}_m \forall z \in \mathcal{C}_j(w\mathsf{R}_i z)$*.*

Semantics for the language $\mathcal{L}^{\mathsf{LTK}}$ is based on a linear and discrete flow of time, associating a time point with any natural number n. The semantic tools we will use are a particular kind of 3-modal Kripke-frames:

Definition 2.3
An $\mathcal{LTK}$*-frame (*Linear Time and Knowledge frame*) is a 3-modal Kripke-frame* $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R_e}, \mathsf{R_a}\rangle$*, where* $W_{\mathcal{F}}$ *is the disjoint union of certain non empty sets* $\mathcal{C}_n$*, for* $n \in \mathbb{N}$*:* $W_{\mathcal{F}} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$*. The binary relations* $\mathsf{R}_{\preccurlyeq}$*,* $\mathsf{R_e}$*, and* $\mathsf{R_a}$ *are as follows:*

(a) $\mathsf{R}_{\preccurlyeq}$ *is the linear, reflexive and transitive relation on* $W_{\mathcal{F}}$ *such that:* $\forall w \forall z \in W_{\mathcal{F}}(w\mathsf{R}_{\preccurlyeq}z \;\; \textit{iff} \;\; \exists i, j \in \mathbb{N} \; ((w \in \mathcal{C}_i) \; \& \; (z \in \mathcal{C}_j) \; \& \; (i \leq j)))$

(b) R_{e} *is a universal relation on any* $\mathcal{C}_i \in W_{\mathcal{F}}$*:*
$\forall w \forall z \in W_{\mathcal{F}}(w\mathsf{R}_{\mathsf{e}}z \;\Leftrightarrow\; \exists i \in \mathbb{N} \; (w \in \mathcal{C}_i \; \& \; z \in \mathcal{C}_i))$*;*

(c) for all i*,* R_{a} *is some equivalence relation on* C_i*.*

The number of sets $\mathcal{C}_n$ can be either finite or infinite. The intended meaning of these frames is to represent a situation in which one agent, having a certain knowledge background at any moment, is operating in the linear flow of time. Each time-cluster (i.e. an $\mathsf{R}_{\preccurlyeq}$-cluster) $\mathcal{C}_n$ consists of a set of information points that are available at the moment n. The relation $\mathsf{R}_{\preccurlyeq}$ is the connection of such information points by the flow of time, that is, given two information points w and z, the expression $w\mathsf{R}_{\preccurlyeq}z$ means either that w and z are both available at a moment n, or that z will be available in the future with respect to w. Since the relation R_{e} connects all the information-points available at the same moment, it is intended to represent a sort of *environmental* knowledge, that is the whole information potentially available for the agent at a given time. Moreover R_{a} says which information points are effectively available for the agent: it specifies the piece of information the agent has access to at any given moment.

Definition 2.4

Given a Kripke-frame $\mathcal{F}$*, a model* $\mathcal{M}_{\mathcal{F}}$ *on* $\mathcal{F}$ *is a tuple* $\mathcal{M}_{\mathcal{F}} = \langle \mathcal{F}, V \rangle$ *where* V *is a valuation of a set* P *of propositional letters in* $\mathcal{F}$*. That is, for any* $p \in P$ $(V(p) \subseteq W_{\mathcal{F}})$*.*

Given a model $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is an $\mathcal{LTK}$-frame, the valuation V can be extended in the standard way from the set P onto all well formed formulae constructed from P. In particular, $\forall w \in W_{\mathcal{F}}$,

(a) $(\mathcal{F}, w) \Vdash_V p \;\;\Leftrightarrow\;\; w \in V(p)$;

(b) $(\mathcal{F}, w) \Vdash_V \Box_{\preccurlyeq}\mathtt{A} \;\;\Leftrightarrow\;\; \forall z \in W_{\mathcal{F}} \; (w\mathsf{R}_{\preccurlyeq}z \;\Rightarrow\; (\mathcal{F}, z) \Vdash_V \mathtt{A})$;

(c) $(\mathcal{F}, w) \Vdash_V \mathsf{K}_{\mathsf{e}}\mathtt{A} \;\;\Leftrightarrow\;\; \forall z \in W_{\mathcal{F}} \; (w\mathsf{R}_{\mathsf{e}}z \;\Rightarrow\; (\mathcal{F}, z) \Vdash_V \mathtt{A})$;

(d) $(\mathcal{F}, w) \Vdash_V \mathsf{K}_{\mathsf{a}}\mathtt{A} \;\;\Leftrightarrow\;\; \forall z \in W_{\mathcal{F}} \; (w\mathsf{R}_{\mathsf{a}}z \;\Rightarrow\; (\mathcal{F}, z) \Vdash_V \mathtt{A})$.

Definition 2.5

If $\mathcal{M} = \langle \mathcal{F}, V \rangle$ *is a model on a frame* $\mathcal{F}$*, a formula* $\mathtt{A}$ *is said to be* true in the model $\mathcal{M}$ at the world w *if* $(\mathcal{F}, w) \Vdash_V \mathtt{A}$*;* $\mathtt{A}$ *is* true in the model $\mathcal{M}$, *notation* $\mathcal{F} \Vdash_V \mathtt{A}$*, if* $\forall w \in W_{\mathcal{F}},\; (\mathcal{F}, w) \Vdash_V \mathtt{A}$*;* $\mathtt{A}$ *is* valid in the frame $\mathcal{F}$, *notation* $\mathcal{F} \Vdash \mathtt{A}$*, if, for any valuation* V *for* $\mathcal{F}$ *(that is for any model* $\mathcal{M}_{\mathcal{F}}$ *on* $\mathcal{F}$*),* $\mathcal{F} \Vdash_V \mathtt{A}$*. Given a class of frames* $\mathbb{F}$*,* $\mathtt{A}$ is valid on $\mathbb{F}$ *(and we say* $\mathtt{A}$ *to be* $\mathbb{F}$*-valid) if* $\forall \mathcal{F} \in \mathbb{F},\; \mathcal{F} \Vdash \mathtt{A}$*. The expression* $V(\mathtt{A})$ *is an abbreviation for the set* $\{w \mid w \Vdash_V \mathtt{A}\}$*.*

Definition 2.6

Let $\mathcal{LTK}$ *be the class of all* $\mathcal{LTK}$*-frames. The logic* LTK *is the set of all* $\mathcal{LTK}$*-*

valid formulae: $\mathsf{LTK} := \{\mathtt{A} \in Fma(\mathcal{L}^{\mathsf{LTK}}) \mid \ \mathcal{F} \Vdash \mathtt{A} \ \& \ \mathcal{F} \in \mathcal{LTK}\}$. *If* A *belongs to* LTK*, then* A *is a theorem of* LTK.

3 *Effective finite model property* for LTK

The first question we will give an answer to is whether LTK has the *effective finite model property (efmp)*. A logic L has the *efmp* if for every formula $\mathtt{A} \notin \mathsf{L}$ there is a finite model $\langle \mathcal{F}, V\rangle$ such that $\mathcal{F} \Vdash_V \mathtt{B}$ for each $\mathtt{B} \in \mathsf{L}$, $\mathcal{F} \nVdash_V \mathtt{A}$ and $\|W_{\mathcal{F}}\| \leq f(\|\mathtt{A}\|)$, where f is computable. We will prove below that LTK has the *efmp* and hence it is decidable. Though this result follows from Calardo and Rybakov [1], we will give an enlightened version of the proof, because we will need such technique in the sequel.

Definition 3.1
Given a Kripke-frame $\mathcal{F} = \langle W, \mathsf{R}_1, \dots, \mathsf{R}_k\rangle$ *and a world* w *in* $W_{\mathcal{F}}$, $w^{\mathsf{R}_i\leq} := \{z \mid w\mathsf{R}_i z\}$ *and* $w^{\mathsf{R}_i<} := \{z \mid w\mathsf{R}_i z \ \& \ \neg(z\mathsf{R}_i w)\}$. *Given a* R_i*-cluster* $\mathcal{C}$, $\mathcal{C}^{\mathsf{R}_i\leq} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{R}_i\mathcal{C}_j\}$ *and* $\mathcal{C}^{\mathsf{R}_i<} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{R}_i\mathcal{C}_j \ \& \ \neg(\mathcal{C}_j\mathsf{R}_i\mathcal{C})\}$ *(In what follows we will always use the expression* $w^{\preccurlyeq}$ *and* $\mathcal{C}^{\preccurlyeq}$ *as abbreviations for* $w^{\mathsf{R}_{\preccurlyeq}\leq}$ *and* $\mathcal{C}^{\mathsf{R}_{\preccurlyeq}\leq}$ *respectively. We will also use* $w^{<}$ *and* $\mathcal{C}^{<}$ *instead of* $w^{\mathsf{R}_{\preccurlyeq}<}$ *and* $\mathcal{C}^{\mathsf{R}_{\preccurlyeq}<}$*).*

Theorem 3.2
The logic LTK *has the* efmp *and hence it is decidable.*

Proof. Take a formula A such that $\mathtt{A} \notin \mathsf{LTK}$; then there are an $\mathcal{LTK}$-frame $\mathcal{F}_1 := \langle W_{\mathcal{F}_1}, \mathsf{R}^1_{\preccurlyeq}, \mathsf{R}^1_{\mathsf{e}}.\mathsf{R}^1_{\mathsf{a}}\rangle$, a model $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1\rangle$ and a world $w \in W_{\mathcal{F}_1}$ such that $(\mathcal{F}_1, w) \nVdash_{V_1} \mathtt{A}$. Suppose $\mathcal{F}_1$ is infinite.

Step 1. We start by reducing the number of worlds belonging to each $\mathsf{R}^1_{\preccurlyeq}$-cluster $\mathcal{C}$ of worlds from $W_{\mathcal{F}_1}$ using the standard filtration technique, briefly sketched below. Let $Sub(\mathtt{A})$ be the set of all the sub-formulae of A. Define the equivalence relation $\approx$ on $W_{\mathcal{F}_1}$ as follows:
$\forall w \forall z \in W_{\mathcal{F}_1}(w \approx z \quad \Leftrightarrow \quad w\mathsf{R}^1_{\preccurlyeq}z \ \& \ z\mathsf{R}^1_{\preccurlyeq}w \ \& \ \forall \mathtt{B} \in Sub(\mathtt{A}) \ ((\mathcal{F}_1, w) \Vdash_{V_1} \mathtt{B} \Leftrightarrow (\mathcal{F}_1, z) \Vdash_{V_1} \mathtt{B})))$ (Recall that the condition $w\mathsf{R}^1_{\preccurlyeq}z \ \& \ z\mathsf{R}^1_{\preccurlyeq}w$ is equivalent to $\exists i(w \in \mathcal{C}_i \ \& \ z \in \mathcal{C}_i)$, that is the worlds w and z belong to the same time-cluster and hence $w\mathsf{R}^1_{\mathsf{e}}z$).

Next, define the quotient set of the original model: $\forall w \in W_{\mathcal{F}_1} \ \ [w] := \{z \mid w \approx z\}$, $\forall n \in \mathbb{N} \ \ [\mathcal{C}_n] := \{[w] \mid w \in \mathcal{C}_n\}$. Let $\mathcal{F}_2 := \langle W_{\mathcal{F}_2}, \mathsf{R}^2_{\preccurlyeq}, \mathsf{R}^2_{\mathsf{e}}, \mathsf{R}^2_{\mathsf{a}}\rangle$ be a frame where:

(a) $W_{\mathcal{F}_2} := \bigcup_{n\in\mathbb{N}}[\mathcal{C}_n]$;
(b) $[w]\mathsf{R}^2_{\preccurlyeq}[z] \Leftrightarrow w\mathsf{R}^1_{\preccurlyeq}z$;
(c) $[w]\mathsf{R}^2_{\mathsf{e}}[z] \Leftrightarrow w\mathsf{R}^1_{\mathsf{e}}z$;
(d) $[w]\mathsf{R}^2_{\mathsf{a}}[z] \Leftrightarrow ([w] \in [\mathcal{C}_n] \ \& \ [z] \in [\mathcal{C}_n] \ \& \ \forall \mathtt{B} \in Sub(\mathtt{A})((\mathcal{F}_1, w) \Vdash_{V_1} \mathsf{K}_{\mathsf{a}}\mathtt{B} \Leftrightarrow (\mathcal{F}_1, z) \Vdash_{V_1} \mathsf{K}_{\mathsf{a}}\mathtt{B}))$.

Let $\mathcal{M}_2 := \langle \mathcal{F}_2, V_2 \rangle$ be a model on $\mathcal{F}_2$ where V_2 is defined as:

$$\forall p \in Sub(\mathtt{A}) \;\; V_2(p) := \{[w] \mid w \in V_1(p)\}$$

Since the model described is the result of a filtration, the standard filtration-lemma holds:

LEMMA 3.3
For any formula $\mathtt{B} \in Sub(\mathtt{A})$*, for any world* $w \in W_1$*,* $(\mathcal{F}_1, w) \Vdash_{V_1} \mathtt{B} \Leftrightarrow (\mathcal{F}_2, [w]) \Vdash_{V_2} \mathtt{B}$.

COROLLARY 3.4
$\mathcal{F}_2 \nVdash_{V_2} \mathtt{A}$.

Thus the model $\mathcal{M}_2$ refutes $\mathtt{A}$ as well. Moreover, each $\mathsf{R}^2_{\preccurlyeq}$-cluster contains a finite number of worlds, bounded by the size of $\mathtt{A}$, namely $\|\mathcal{C}\| \leq 2^{\|Sub(\mathtt{A})\|}$ for each $\mathsf{R}^2_{\preccurlyeq}$-cluster $\mathcal{C}$.

STEP 2. We will reduce, now, the amount of time-clusters (i.e. $\mathsf{R}^2_{\preccurlyeq}$-clusters) to a finite one. We need few preliminary facts. Evidently, the following holds:

PROPOSITION 3.5
There is only a finite, computable from the size of $\mathtt{A}$*, number of non-isomorphic w.r.t.* $Sub(\mathtt{A})$ *time-clusters* $\mathcal{C}$ *from* $W_{\mathcal{F}_2}$.

DEFINITION 3.6
Given an $\mathcal{LTK}$*-frame* $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}} \rangle$ *and a model* $\mathcal{M} := \langle \mathcal{F}, V \rangle$*, an* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_s$ *is a* stabilizing cluster *if and only if for any* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_i \in \mathcal{C}_s^{\preccurlyeq}$*, for any* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_j \in \mathcal{C}_s^{\preccurlyeq}$ *there is an* $\mathsf{R}_{\preccurlyeq}$*-cluster* $\mathcal{C}_k \in \mathcal{C}_i^{\preccurlyeq}$ *such that* $\mathcal{C}_j \cong \mathcal{C}_k$*, i.e. the sets* $\mathcal{C}_s^{\preccurlyeq}$ *and* $\mathcal{C}_i^{\preccurlyeq}$ *coincide up to isomorphism between* $\mathsf{R}_{\preccurlyeq}$*-clusters.*

LEMMA 3.7
The model $\mathcal{M}_2$ *has a stabilizing* $\mathsf{R}^2_{\preccurlyeq}$*-cluster* $\mathcal{C}_s$.

PROOF. By Proposition 3.5 the number of non-isomorphic $\mathsf{R}^2_{\preccurlyeq}$-clusters $\mathcal{C}$ is finite. Moreover, we have that for all the $\mathsf{R}^2_{\preccurlyeq}$-clusters $\mathcal{C}_i$, $\mathcal{C}_j$ from $W_{\mathcal{F}_2}$, $\mathcal{C}_i \mathsf{R}^2_{\preccurlyeq} \mathcal{C}_j \Rightarrow \mathcal{C}_i^{\preccurlyeq} \supseteq \mathcal{C}_j^{\preccurlyeq}$. Consider the sequence of all the time-clusters $\mathcal{C}_1, \mathcal{C}_2, \ldots$. We construct a subsequence $\mathcal{C}'_n$ of the sequence $\mathcal{C}_n$, $n \in \mathbb{N}$ as follows. Take $\mathcal{C}_1$; if $\mathcal{C}_1$ is a stabilizing cluster, then we stop, and the subsequence is chosen. Otherwise, assume that a subsequence $\mathcal{C}'_1, \ldots, \mathcal{C}'_n$ is chosen. If $\mathcal{C}'_n$ is not a stabilizing cluster, then there is a cluster $\mathcal{C}_k$, where, up to isomorphism, $\mathcal{C}'^{\preccurlyeq}_n \supset \mathcal{C}^{\preccurlyeq}_k$. Take the $\mathsf{R}^2_{\preccurlyeq}$-smallest $\mathcal{C}_k$ with this property and set $\mathcal{C}'_{(n+1)} := \mathcal{C}_k$. Since $\mathcal{C}'^{\preccurlyeq}_n \supset \mathcal{C}'^{\preccurlyeq}_{(n+1)}$, this procedure must terminate, and it stops at a stabilizing cluster. ∎

LEMMA 3.8
If $\mathcal{C}_s$ is a stabilizing cluster, then, for all the $\mathsf{R}^2_{\preccurlyeq}$-clusters $\mathcal{C}_i$, $\mathcal{C}_j$ of worlds from $W_{\mathcal{F}_2}$ such that $\mathcal{C}_s \mathsf{R}^2_{\preccurlyeq} \mathcal{C}_i$ and $\mathcal{C}_s \mathsf{R}^2_{\preccurlyeq} \mathcal{C}_j$, if $\mathcal{C}_i$ is isomorphic to $\mathcal{C}_j$ by a mapping f, then $\forall \mathtt{B} \in Sub(\mathtt{A})$, $\forall w \in \mathcal{C}_i$ $(\mathcal{F}_2, w) \Vdash_{V_2} \mathtt{B} \Leftrightarrow (\mathcal{F}_2, f(w)) \Vdash_{V_2} \mathtt{B}$.

PROOF. It may be given by an easy induction on the length of $\mathtt{B}$. Both the basis of the induction and the inductive steps regarding the boolean operations and the modal operators $\mathsf{K_e}$ and $\mathsf{K_a}$ are evident. Hence, we turn our attention only to the case $\mathtt{B}$ is $\Box_{\preccurlyeq}\mathtt{D}$, $(\mathcal{F}_2, w) \Vdash_{V_2} \Box_{\preccurlyeq}\mathtt{D}$ and $w^{\preccurlyeq} \subset f(w)^{\preccurlyeq}$. It follows that $\forall z \in w^{\preccurlyeq}$, $(\mathcal{F}_2, z) \Vdash_{V_2} \mathtt{D}$. Since each $\mathsf{R}^2_{\preccurlyeq}$-cluster from $f(w)^{\preccurlyeq}$ is isomorphic to some $\mathsf{R}^2_{\preccurlyeq}$-cluster from $w^{\preccurlyeq}$, by Inductive Hypothesis, we have $\forall v \in f(w)^{\preccurlyeq}$, $(\mathcal{F}_2, v) \Vdash_{V_2} \mathtt{D}$, therefore $(\mathcal{F}_2, f(w)) \Vdash_{V_2} \Box_{\preccurlyeq}\mathtt{D}$. ∎

STEP 3. Consider the set $\mathcal{C}_s^{\preccurlyeq}$. We want to reduce the number of its elements to a finite one. Firstly, we make a partition of this set into equivalence classes. We take each time-cluster of worlds from $\mathcal{C}_s^{\preccurlyeq}$ and we define its equivalence class w.r.t. isomorphic time-clusters $[\mathcal{C}]_{\cong} := \{\mathcal{C}_j \mid \mathcal{C}_s \mathsf{R}^2_{\preccurlyeq} \mathcal{C}_j \ \& \ \mathcal{C} \cong \mathcal{C}_j\}$. We take and fix, for each $\mathsf{R}^2_{\preccurlyeq}$-cluster $\mathcal{C}$ from $\mathcal{C}_s^{\preccurlyeq}$, a representative $\mathsf{R}^2_{\preccurlyeq}$-cluster $\mathsf{Rep}(\mathcal{C}) \in [\mathcal{C}]_{\cong}$. Next we set $\mathsf{REP} := \bigcup_{\mathcal{C} \in \mathcal{C}_s^{\preccurlyeq}} \mathsf{Rep}(\mathcal{C})$. Now we introduce a new frame $\mathsf{St} := \langle W_{\mathsf{St}}, \mathsf{R}^{\mathsf{St}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{St}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{St}}_{\mathsf{a}} \rangle$ where:

(a) $W_{\mathsf{St}} := \bigcup_{\mathcal{C} \in \mathsf{REP}} \mathcal{C}$
(b) $\mathsf{R}^{\mathsf{St}}_{\preccurlyeq} := W_{\mathsf{St}} \times W_{\mathsf{St}}$
(c) $\mathsf{R}^{\mathsf{St}}_{\mathsf{e}} := \mathsf{R}^2_{\mathsf{e}} \upharpoonright W_{\mathsf{St}}$ (i.e. $\mathsf{R}^{\mathsf{St}}_{\mathsf{e}}$ is the restriction of $\mathsf{R}^2_{\mathsf{e}}$ on W_{St}.)
(d) $\mathsf{R}^{\mathsf{St}}_{\mathsf{a}} := \mathsf{R}^2_{\mathsf{a}} \upharpoonright W_{\mathsf{St}}$

We consider, now, the linear part of $\mathcal{M}_2$ up to the stabilising cluster $\mathcal{C}_s$ and we define a subframe $\mathcal{F}_l \sqsubseteq \mathcal{F}_2$, $\mathcal{F}_l := \langle W_l, \mathsf{R}^l_{\preccurlyeq}, \mathsf{R}^l_{\mathsf{e}}, \mathsf{R}^l_{\mathsf{a}} \rangle$, where $W_{\mathcal{F}_l} := W_{\mathcal{F}_2} - \bigcup \mathcal{C}_s^{\preccurlyeq}$. The $\mathcal{LTK}$-frame $\mathcal{F}_3 := \langle W_{\mathcal{F}_3}, \mathsf{R}^3_{\preccurlyeq}, \mathsf{R}^3_{\mathsf{e}}, \mathsf{R}^3_{\mathsf{a}} \rangle$ has the following structure (see Figure 1):

(a) $W_{\mathcal{F}_3} := W_{\mathsf{St}} \cup W_{\mathcal{F}_l}$
(b) $\mathsf{R}^3_{\preccurlyeq} := \mathsf{R}^{\mathsf{St}}_{\preccurlyeq} \cup \mathsf{R}^l_{\preccurlyeq} \cup \{\langle w, z \rangle \mid w \in W_{\mathcal{F}_l} \ \& \ z \in W_{\mathsf{St}}\}$
(c) $\mathsf{R}^3_{\mathsf{e}} := \mathsf{R}^{\mathsf{St}}_{\mathsf{e}} \cup \mathsf{R}^l_{\mathsf{e}}$
(d) $\mathsf{R}^3_{\mathsf{a}} := \mathsf{R}^{\mathsf{St}}_{\mathsf{a}} \cup \mathsf{R}^l_{\mathsf{a}}$

Let $\mathcal{M}_{\mathcal{F}_3} := \langle \mathcal{F}_3, V_3 \rangle$ be the model in which V_3 is the restriction of V_2 on $W_{\mathcal{F}_3}$.

LEMMA 3.9
For any formula $\mathtt{B} \in Sub(\mathtt{A})$, *for any world* $w \in W_{\mathcal{F}_3}$, $(\mathcal{F}_3, w) \Vdash_{V_3} \mathtt{B} \Leftrightarrow (\mathcal{F}_2, w) \Vdash_{V_2} \mathtt{B}$.

PROOF. The proof can be given by induction on the lenght of $\mathtt{B}$. We consider only the case in which $\mathtt{B}$ is $\Box_{\preccurlyeq}\mathtt{D}$, $(\mathcal{F}_3, w) \Vdash_{V_3} \Box_{\preccurlyeq}\mathtt{D}$ and $w \in W_{\mathsf{St}}$. This means that $\mathtt{D}$ is true at all those worlds $z \in W_{\mathcal{F}_3}$ s.t. $w \mathsf{R}^3_{\preccurlyeq} z$, i.e. all the worlds belonging to W_{St} (recall that $\mathsf{R}^{\mathsf{St}}_{\preccurlyeq}$ is an equivalence relation on W_{St}). By

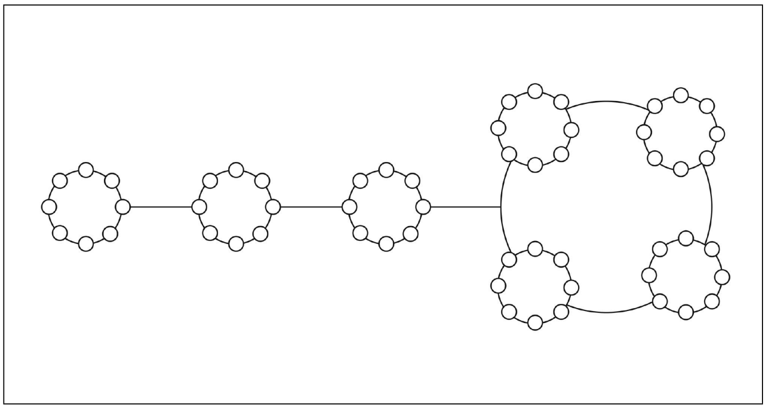

FIG. 1. Scheme of the structure of the frame $\mathcal{F}_3$.

Inductive Hypothesis we have that $(\mathcal{F}_2, z) \Vdash_{V_2} \mathtt{D}$ for any z belonging both to $W_{\mathcal{F}_3}$ and to $W_{\mathcal{F}_2}$. Consider a world $v \in W_{\mathcal{F}_2}$ such that $v \in \mathcal{C}_s^{\preccurlyeq}$. We can have two cases: either v belongs to W_{St} or v does not. In the former case $(\mathcal{F}_2, v) \Vdash_{V_2} \mathtt{D}$ holds by Inductive Hypothesis, while in the latter, since v belongs to an $\mathsf{R}^2_{\preccurlyeq}$-cluster isomorphic to an $\mathsf{R}^{\mathsf{St}}_{\mathsf{e}}$-cluster from W_{St}, $(\mathcal{F}_2, v) \Vdash_{V_2} \mathtt{D}$ holds by Lemma 3.8. Therefore $(\mathcal{F}_2, w) \Vdash_{V_2} \Box_{\preccurlyeq}\mathtt{D}$. ∎

The base set of $\mathcal{M}_{\mathcal{F}_3}$ contains a finite number of worlds, but, since we do not know how many they are, we need to contract it again.

STEP 4. For each $\mathtt{B} \in Sub(\mathtt{A})$, we consider the $\mathsf{R}^3_{\preccurlyeq}$-maximal $\mathsf{R}^3_{\mathsf{e}}$-cluster $\mathcal{C}$ of worlds from $W_{\mathcal{F}_3}$ such that $\exists w \in \mathcal{C},\ (\mathcal{F}_3, w) \Vdash_{V_3} \mathtt{B}$ and we denote it by $\mathcal{C}_{\mathtt{B}}$. Likewise, by $\mathcal{C}_{\neg\mathtt{B}}$ we denote the $\mathsf{R}^3_{\preccurlyeq}$-maximal $\mathsf{R}^3_{\mathsf{e}}$-cluster containing a world z refuting $\mathtt{B}$. Then we introduce a new frame $\mathcal{F}_4 := \langle W_{\mathcal{F}_4}, \mathsf{R}^4_{\preccurlyeq}, \mathsf{R}^4_{\mathsf{e}}, \mathsf{R}^4_{\mathsf{a}} \rangle$ where:

$$W_{\mathcal{F}_4} := \bigcup_{\mathtt{B} \in Sub(\mathtt{A})} \mathcal{C}_{\mathtt{B}} \cup \bigcup_{\mathtt{B} \in Sub(\mathtt{A})} \mathcal{C}_{\neg\mathtt{B}} \cup W_{\mathsf{St}}$$

and all the binary relations are the restriction of the ones from $\mathcal{F}_3$ on $W_{\mathcal{F}_4}$. Let $\mathcal{M}_4 := \langle W_{\mathcal{F}_4}, V_4 \rangle$ be a model on $\mathcal{F}_4$ where V_4 is nothing but the restriction of V_3 on $W_{\mathcal{F}_4}$.

LEMMA 3.10
For any formula $\mathtt{B} \in Sub(\mathtt{A})$*, for any world* $w \in W_4$*,* $(\mathcal{F}_4, w) \Vdash_{V_4} \mathtt{B} \Leftrightarrow (\mathcal{F}_3, w) \Vdash_{V_3} \mathtt{B}$.

PROOF. We conduct an easy induction on the length of $\mathtt{B}$, and we illustrate only the case $\mathtt{B}$ is $\Box_{\preccurlyeq}\mathtt{D}$, $(\mathcal{F}_4, w) \Vdash_{V_4} \Box_{\preccurlyeq}\mathtt{D}$ and $w \notin W_{\mathsf{St}}$. Suppose $(\mathcal{F}_3, w) \nVdash_{V_3} \Box_{\preccurlyeq}\mathtt{D}$. Then there is a world $z \in W_{\mathcal{F}_3}$ such that $w\mathsf{R}^3_{\preccurlyeq}z$, $(\mathcal{F}_3, z) \nVdash_{V_3} \mathtt{D}$ and

$z \notin W_{\mathcal{F}_4}$. By construction of $\mathcal{M}_{\mathcal{F}_4}$, there must be an $\mathsf{R}^4_{\preccurlyeq}$-maximal $\mathsf{R}^4_{\mathsf{e}}$-cluster $\mathcal{C}_{\neg\mathtt{D}}$ in $\mathcal{F}_4$ such that there exists a world $v \in \mathcal{C}_{\neg\mathtt{D}}$, $(\mathcal{F}_4, v) \not\Vdash_{V_4} \mathtt{D}$. Since $\mathcal{C}_{\neg\mathtt{D}}$ is $\mathsf{R}^4_{\preccurlyeq}$-maximal, we also have $w\mathsf{R}^4_{\preccurlyeq}v$. This is a contradiction, hence $(\mathcal{F}_3, w) \Vdash_{V_3} \Box_{\preccurlyeq}\mathtt{D}$. ■

Now the number of worlds from $W_{\mathcal{F}_4}$ is finite and it is $f(\|\mathtt{A}\|)$, where f is a computable function and $f(\|\mathtt{A}\|) \leq (2^{\|Sub(\mathtt{A})\|}(2\|Sub(\mathtt{A})\| + 2^{2^{\|Sub(\mathtt{A})\|}}))$.

Step 5. Our next step is to show that $\mathcal{F}_4$ is the p-morphic image of an $\mathcal{LTK}$-frame and hence $\forall \mathtt{B} \in \mathsf{LTK}$, $\mathcal{F}_4 \Vdash \mathtt{B}$. Let $\mathcal{C}^{\mathsf{St}}_1, \ldots, \mathcal{C}^{\mathsf{St}}_i$ be an enumeration of all the $\mathsf{R}^4_{\mathsf{e}}$-clusters of worlds from W_{St} and let $\mathcal{F}_5 := \langle W_{\mathcal{F}_5}, \mathsf{R}^5_{\preccurlyeq}, \mathsf{R}^5_{\mathsf{e}}, \mathsf{R}^5_{\mathsf{a}}\rangle$ be a frame such that:

(a) $W_{\mathcal{F}_5} := \bigcup_{1 \leq j \leq i} \mathcal{C}^{\mathsf{St}}_j$

(b) $\forall w \forall z \in W_{\mathcal{F}_5}$ $(w\mathsf{R}^5_{\preccurlyeq}z \Leftrightarrow (w \in \mathcal{C}^{\mathsf{St}}_j \;\&\; z \in \mathcal{C}^{\mathsf{St}}_k \;\&\; j \leq k))$

(c) $\forall w \forall z \in W_{\mathcal{F}_5}$ $(w\mathsf{R}^5_{\mathsf{e}}z \quad \Leftrightarrow \quad w\mathsf{R}^4_{\mathsf{e}}z)$

(d) $\forall w \forall z \in W_{\mathcal{F}_5}$ $(w\mathsf{R}^5_{\mathsf{a}}z \quad \Leftrightarrow \quad w\mathsf{R}^4_{\mathsf{a}}z)$

Let $\mathcal{F}_\infty = \langle W_{\mathcal{F}_\infty}, \mathsf{R}^\infty_{\preccurlyeq}, \mathsf{R}^\infty_{\mathsf{e}}, \mathsf{R}^\infty_{\mathsf{a}}\rangle$ be an $\mathcal{LTK}$-frame consisting of an infinite repetition of $\mathcal{F}_5$ and let $\mathcal{F}_6 = \langle W_{\mathcal{F}_6}, \mathsf{R}^6_{\preccurlyeq}, \mathsf{R}^6_{\mathsf{e}}, \mathsf{R}^6_{\mathsf{a}}\rangle$ be a subframe of $\mathcal{F}_4$ such that $W_{\mathcal{F}_6} = W_{\mathcal{F}_4} - \bigcup \mathcal{C}^{<}_s$ (recall that $\mathcal{C}_s$ is the stabilizing cluster of $\mathcal{F}_4$). Let $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}\rangle$ be an $\mathcal{LTK}$-frame such that:

(a) $W = W_{\mathcal{F}_\infty} \cup W_6$

(b) $\mathsf{R}_{\preccurlyeq} = \mathsf{R}^\infty_{\preccurlyeq} \cup \mathsf{R}^6_{\preccurlyeq} \cup \{\langle w, z\rangle \mid w \in W_{\mathcal{F}_6} \;\&\; z \in W_{\mathcal{F}_\infty}\}$

(c) $\mathsf{R}_{\mathsf{e}} = \mathsf{R}^\infty_{\mathsf{e}} \cup \mathsf{R}^6_{\mathsf{e}}$

(d) $\mathsf{R}_{\mathsf{a}} = \mathsf{R}^\infty_{\mathsf{a}} \cup \mathsf{R}^6_{\mathsf{a}}$.

It is easy to see that $\mathcal{F}_4$ is a p-morphic image of $\mathcal{F}$. ■

Notice that in this proof we have examined only the general case in which the formula $\mathtt{A}$ is not valid in an *infinite* $\mathcal{LTK}$-frame. If such frame is a *finite* one, we do not need to go through steps 2, 3 and 5.

4 Construction of $Ch_{\mathsf{LTK}}(n)$

In this section we will construct special countable n-characterizing models for the logic LTK (see Definition 4.2) based on the tecniques presented in Rybakov [22]. This construction is the ground on which we will base our main result.

Given an $\mathcal{LTK}$-frame $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}\rangle$, a world w (or an $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}$) from $W_{\mathcal{F}}$ has $\mathsf{R}_{\preccurlyeq}$-depth n, in symbols $depth_{\mathsf{R}_{\preccurlyeq}}(w) = n$, if the number of $\mathsf{R}_{\preccurlyeq}$-clusters in $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)^{\preccurlyeq}$ is n (in what follows, we will always use the expression *depth* instead of $\mathsf{R}_{\preccurlyeq}$-*depth* or $depth_{\mathsf{R}_{\preccurlyeq}}$). The expression $Sl_n(\mathcal{F})$ denotes the n-slice of $\mathcal{F}$, i.e. the family of all the elements of depth n from $W_{\mathcal{F}}$. $S_n(\mathcal{F})$ is the set of all the elements from $W_{\mathcal{F}}$ with depth at most n. Given a model

$\mathcal{M} := \langle \mathcal{F}, V \rangle$ and a world $w \in W_{\mathcal{F}}$, by $Val_V(w)$ we will denote the set $\{p_i \mid w \Vdash_V p_i\}$. For any valuation V, $\mathsf{Dom}(V)$ denotes the domain of V.

DEFINITION 4.1
Let $\mathcal{F}_i = \langle W_{\mathcal{F}_i}, \mathsf{R}^i_1, \ldots, \mathsf{R}^i_k \rangle$, for $i \in I$, be a family of k-modal Kripke-frames with pairwise disjoint base sets, i.e. $W_{\mathcal{F}_i} \cap W_{\mathcal{F}_j} = \emptyset$ for each $i, j \in I$. The disjoint union of $\mathcal{F}_i$ is the frame:

$$\bigsqcup_{i \in I} \mathcal{F}_i = \langle \bigcup_{i \in I} W_{\mathcal{F}_i}, \bigcup_{i \in I} \mathsf{R}^i_1, \ldots, \bigcup_{i \in I} \mathsf{R}^i_k \rangle$$

Given a family $\mathcal{M}_i = \langle \mathcal{F}_i, V_i \rangle$ of Kripke-models on the family of frames $\mathcal{F}_i$, the disjoint union of $\mathcal{M}_i$ is the model:

$$\bigsqcup_{i \in I} \mathcal{M}_i = \langle \bigsqcup_{i \in I} \mathcal{F}_i, \bigcup_{i \in I} V_i \rangle$$

DEFINITION 4.2
Given a logic L, a Kripke-model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ is an n-characterizing model for L iff: (a) $\mathsf{Dom}(V) := \{p_1, \ldots, p_n\}$ (b) for any formula $\mathtt{A}$ built up from $\mathsf{Dom}(V)$, $\mathcal{F} \Vdash_V \mathtt{A} \Leftrightarrow \mathtt{A} \in L$.

Let $\mathbb{F}$ be a class of finite $\mathcal{LTK}$-frames (i.e. $\mathcal{LTK}$-frames whose base sets are finite) such that, for any frame $\mathcal{F} \in \mathbb{F}$, $\forall w \forall z \in W_{\mathcal{F}}(w\mathsf{R}_{\preccurlyeq} z \ \&\ w\mathsf{R}_{\mathsf{e}} z)$. Let $\mathbb{C}(\mathbb{F})_n$ be the class of all the possible different, non isomorphic models $\mathcal{C} := \langle \mathcal{F}, V \rangle$, where:

(a) $\mathcal{F} \in \mathbb{F}$;
(b) $\mathsf{Dom}(V) = \{p_1, \ldots, p_n\}$;
(c) $\forall w \forall z \in W_{\mathcal{F}}\Big(((Val_V(w) = Val_V(z))\&(\{Val_V(w') \mid w\mathsf{R}_{\mathsf{a}} w'\} = \{Val_V(z') \mid z\mathsf{R}_{\mathsf{a}} z'\})) \Rightarrow (w = z)\Big)$.

It is easy to notice that the size of $\mathbb{C}(\mathbb{F})_n$ is computable and bounded by n.

STEP 1. Let $\mathcal{P}(\mathbb{C}(\mathbb{F})_n)$ be the set of all the subsets of $\mathbb{C}(\mathbb{F})_n$.

Given a set $\mathbb{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_j\}$ from $\mathcal{P}(\mathbb{C}(\mathbb{F})_n)$, for each $\mathcal{C}_i \in \mathbb{S}$, we display the model $\mathcal{C}_i$ as $\mathcal{C}_i := \langle W_i, \mathsf{R}^i_{\preccurlyeq}, \mathsf{R}^i_{\mathsf{e}}, \mathsf{R}^i_{\mathsf{a}}, V_i \rangle$.

For any set $\mathbb{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_j\}$ from $\mathcal{P}(\mathbb{C}(\mathbb{F})_n)$, $\mathcal{T}_{\mathbb{S}}$ is the Kripke-model $\mathcal{T}_{\mathbb{S}} := \langle W_{\mathbb{S}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}, V \rangle$, where:

(a) $W_{\mathbb{S}} := \bigcup_{1 \leq i \leq j} W_i$
(b) $\mathsf{R}_{\preccurlyeq} := W_{\mathbb{S}} \times W_{\mathbb{S}}$
(c) $\mathsf{R}_{\mathsf{e}} := \bigcup_{1 \leq i \leq j} \mathsf{R}^i_{\mathsf{e}}$
(d) $\mathsf{R}_{\mathsf{a}} := \bigcup_{1 \leq i \leq j} \mathsf{R}^i_{\mathsf{a}}$

(e) $\mathsf{Dom}(V) := \{p_1, \ldots, p_n\}$
(f) $\forall p \in \mathsf{Dom}(V)(V(p) := \bigcup_{1 \leq i \leq j} V_i(p))$

Since the temporal relation $\mathsf{R}_{\preccurlyeq}$ is universal, each $\mathcal{T}_{\mathbb{S}}$ is an $\mathsf{R}_{\preccurlyeq}$-cluster of R_{e}-clusters.
Let $S_1(Ch_{\mathsf{LTK}}(n)) := \bigsqcup_{\mathbb{S} \in \mathcal{P}(\mathbb{C}(\mathbb{F})_n)} \mathcal{T}_{\mathbb{S}}$.

Hence the first slice contains a finite number of pairwise disjoint models, where each model is an $\mathsf{R}_{\preccurlyeq}$-cluster of R_{e}-clusters.

Step 2. Consider any $\mathcal{T}_{\mathbb{S}}$ from $S_1(Ch_{\mathsf{LTK}}(n))$, and any R_{e}-cluster $\mathcal{C}_i$ from $\mathbb{C}(\mathbb{F})_n$ s.t. $\forall \mathcal{C} \in \mathcal{T}_{\mathbb{S}}$, $\mathcal{C}_i$ is not isomorphic to a submodel of $\mathcal{C}$.

We adjoin all such models $\mathcal{C}_i$ to $S_1(Ch_{\mathsf{LTK}}(n))$ assuming $\mathcal{C}_i$ to be the immediate $\mathsf{R}_{\preccurlyeq}$-predecessor of all the R_{e}-clusters from $\mathcal{T}_{\mathbb{S}}$. The resulting model is defined as $S_2(Ch_{\mathsf{LTK}}(n))$.

Step 3. Suppose we have already constructed the model $S_i(Ch_{\mathsf{LTK}}(n))$ for $i \geq 2$ such that its frame is an $\mathcal{LTK}$-frame and given two different $\mathsf{R}_{\preccurlyeq}$-clusters $\mathcal{C}_j$, $\mathcal{C}_k$ from this frame, if $\mathcal{C}_j$ is an immediate $\mathsf{R}_{\preccurlyeq}$-predecessor of $\mathcal{C}_k$, then $\mathcal{C}_j$ is not isomorphic to a submodel of $\mathcal{C}_k$. To construct $Sl_{i+1}(Ch_{\mathsf{LTK}}(n))$ we add R_{e}-clusters from $\mathbb{C}(\mathbb{F})_n$ in the following way. We take each R_{e}-cluster $\mathcal{C}$ of depth i and we add as its immediate $\mathsf{R}_{\preccurlyeq}$-predecessors all the possible different R_{e}-clusters $\mathcal{C}_j$ from $\mathbb{C}(\mathbb{F})_n$, but only provided that $\mathcal{C}_j$ is not isomorphic to a submodel of $\mathcal{C}$.
Let $S_{i+1}(Ch_{\mathsf{LTK}}(n))$ be the model resulting from all such additions.

The final model $Ch_{\mathsf{LTK}}(n) := \langle W_{Ch(n)}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}, V\rangle$ is given by

$$\bigcup_{i \in \mathbb{N}} S_i(Ch_{\mathsf{LTK}}(n))$$

Let $Ch(n)$ be the name for the frame on which $Ch_{\mathsf{LTK}}(n)$ is based.

Lemma 4.3
The model $Ch_{\mathsf{LTK}}(n) = \langle Ch(n), V\rangle$ is n-characterizing for LTK.

Proof. Since $Ch(n) \Vdash \mathsf{LTK}$ by construction, the claim $\mathtt{A} \in \mathsf{LTK} \Rightarrow Ch(n) \Vdash_V \mathtt{A}$, for any formula $\mathtt{A}$ built up from the propositional letters $p_1, \ldots, p_n$, follows immediately.

Suppose there is a formula $\mathtt{A}$ built up from $p_1, \ldots, p_n$ s.t. $\mathtt{A} \notin \mathsf{LTK}$. In order to prove that $\mathtt{A}$ is not true in $Ch_{\mathsf{LTK}}(n)$, we will construct a model refuting $\mathtt{A}$, which is isomorphic to an open submodel of $Ch_{\mathsf{LTK}}(n)$.

By Theorem 3.2, there are a finite frame $\mathcal{F}_1 = \langle W_{\mathcal{F}_1}, \mathsf{R}^1_{\preccurlyeq}, \mathsf{R}^1_{\mathsf{e}}, \mathsf{R}^1_{\mathsf{a}}\rangle$ (whose size is computable and bounded by the size of $\mathtt{A}$) and a model $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1\rangle$ such that $\mathcal{F}_1 \nVdash_{V_1} \mathtt{A}$. $\forall w, z \in \mathcal{C}$, if the following two conditions hold:

(a) $Val_{V_1}(w) = Val_{V_1}(z)$

(b) $\{Val_{V_1}(w') \mid w\mathsf{R}^1_{\mathsf{a}}w'\} = \{Val_{V_1}(z') \mid z\mathsf{R}^1_{\mathsf{a}}z'\}$

then we glue w and z together. The resulting model $\mathcal{M}_2 := \langle \mathcal{F}_2, V_2\rangle$ is a p-morphic image of $\mathcal{M}_1 := \langle \mathcal{F}_1, V_1\rangle$, thus it still refutes $\mathtt{A}$.

Let St_1 be the set of $\mathsf{R}^1_{\mathsf{e}}$-clusters of depth 1 from $\mathcal{F}_1$, and let St_2 be the set of $\mathsf{R}^2_{\mathsf{e}}$-clusters of depth 1 from $\mathcal{F}_2$ (cf. Section 3 and Figure 1). We delete $\mathsf{R}^2_{\mathsf{e}}$-clusters from St_2 as follows: for any $\mathcal{C}_1$, $\mathcal{C}_2$ from St_2 s.t. $\mathcal{C}_1 \neq \mathcal{C}_2$, if $\mathcal{C}_1$ is a submodel of $\mathcal{C}_2$, then we delete $\mathcal{C}_1$. Let $\mathcal{M}^*_1$ be the resulting model. Clearly, $\mathcal{M}^*_1$ is a p-morphic image of St_1 and it is also isomorphic to an open submodel of $Ch_{\mathsf{LTK}}(n)$.

Suppose we have already constructed the model $\mathcal{M}^*_i := \langle \mathcal{F}^*_i, V^*_i\rangle$ s.t.:

(a) $\forall w \in W_{\mathcal{F}^*_i},\ \ depth(w) \leq i$

(b) $\mathcal{M}^*_i$ is a p-morphic image of the open submodel of $\mathcal{M}_2$ generated by the set $\bigcup \mathcal{C}^{\preccurlyeq}$, where $\mathcal{C}$ is an $\mathsf{R}^2_{\preccurlyeq}$-cluster of depth i.

(c) $\mathcal{M}^*_i$ is isomorphic to some open submodel of $Ch_{\mathsf{LTK}}(n)$.

The following procedure will explain how to obtain the model $\mathcal{M}^*_{i+1}$. Let $\mathcal{C}$ be the $\mathsf{R}^*_{\preccurlyeq}$-deepest $\mathsf{R}^*_{\preccurlyeq}$-cluster in $\mathcal{M}^*_i$. Consider the $\mathsf{R}^2_{\preccurlyeq}$-cluster $\mathcal{C}_{i+1}$ in $\mathcal{M}_2$ of depth $i+1$. If $\mathcal{C}_{i+1}$ is not a submodel of $\mathcal{C}$, then we adjoin $\mathcal{C}_{i+1}$ as the immediate $\mathsf{R}^*_{\preccurlyeq}$-predecessor of $\mathcal{C}$, otherwise we do not add anything. This procedure ends when we reach the $\mathsf{R}^2_{\preccurlyeq}$-deepest $\mathsf{R}^2_{\preccurlyeq}$-cluster $\mathcal{C}$ in $\mathcal{M}_2$. We denote the resulting model by $\mathcal{M}^*$. Clearly, $\mathcal{M}^*$ is a p-morphic image of the original model $\mathcal{M}_1$, therefore it refutes $\mathtt{A}$. Since $\mathcal{M}^*$ is also isomorphic to some open submodel of $Ch_{\mathsf{LTK}}(n)$, it follows $Ch(n) \nVdash_V \mathtt{A}$. ∎

5 Definability of worlds

Definition 5.1

Given a model $\mathcal{M} = \langle \mathcal{F}, V\rangle$, a world $w \in W_{\mathcal{F}}$ is definable if and only if there is a formula $\beta(w)$ such that $\forall z \in W_{\mathcal{F}}\ ((\mathcal{F}, z) \Vdash_V \beta(w) \quad \Leftrightarrow \quad w = z)$.

Lemma 5.2

For any n-characterising model $Ch_{\mathsf{LTK}}(n)$, each world w from $W_{Ch(n)}$ is definable.

Proof. We will use the following abbreviations: S_i for $S_i(Ch_{\mathsf{LTK}}(n))$; $\Diamond_{\preccurlyeq}$ for $\neg\Box_{\preccurlyeq}\neg$ and $\Diamond_{\mathsf{e}}$, $\Diamond_{\mathsf{a}}$ for $\neg\mathsf{K}_{\mathsf{e}}\neg$, $\neg\mathsf{K}_{\mathsf{a}}\neg$ respectively. If $depth(w) = 1$, the expression $\mathcal{T}_{\mathbb{S}}(w)$ will denote the $\mathsf{R}_{\preccurlyeq}$-circle of R_{e}-clusters to which w belongs.

Step 1. We start by analysing the case $depth(w) = 1$, that is w belongs to some $\mathcal{T}_{\mathbb{S}}(w) \in S_1(Ch_{\mathsf{LTK}}(n))$. We will use the following formulae:

$$\alpha(w) := \bigwedge_{w\in V(p_i)} p_i \quad \wedge \quad \bigwedge_{w\notin V(p_i)} \neg p_i$$

$$\rho_{\mathsf{a}}(w) := \bigwedge_{w\mathsf{R_a}z} \Diamond_{\mathsf{a}}\alpha(z) \quad \wedge \quad \mathsf{K_a} \bigvee_{w\mathsf{R_a}z} \alpha(z)$$

$$\rho_{\mathsf{e}}(w) := \bigwedge_{z\in\mathcal{C}_{\mathsf{R_e}}(w)} \Diamond_{\mathsf{e}}\Big(\alpha(z)\wedge\rho_{\mathsf{a}}(z)\Big) \quad \wedge \quad \mathsf{K_e} \bigvee_{z\in\mathcal{C}_{\mathsf{R_e}}(w)} \Big(\alpha(z)\wedge\rho_{\mathsf{a}}(z)\Big)$$

$$\rho_{\preccurlyeq}(w) := \bigwedge_{z\in\mathcal{T}_{\mathbb{S}}(w)} \Diamond_{\preccurlyeq}\Big(\alpha(z)\wedge\rho_{\mathsf{a}}(z)\wedge\rho_{\mathsf{e}}(z)\Big)\wedge\Box_{\preccurlyeq} \bigvee_{z\in\mathcal{T}_{\mathbb{S}}(w)} \Big(\alpha(z)\wedge\rho_{\mathsf{a}}(z)\wedge\rho_{\mathsf{e}}(z)\Big)$$

$$\rho_{\succcurlyeq}(w) := \bigwedge_{z\in\mathcal{T}_{\mathbb{S}}(w)} \Box_{\preccurlyeq}\Diamond_{\preccurlyeq}\Big(\alpha(z)\wedge\rho_{\mathsf{a}}(z)\wedge\rho_{\mathsf{e}}(z)\Big)$$

We set the formula $\beta(w)$ to be:

$$\beta(w) := \alpha(w) \quad \wedge \quad \rho_{\mathsf{a}}(w) \quad \wedge \quad \rho_{\mathsf{e}}(w) \quad \wedge \quad \rho_{\preccurlyeq}(w) \quad \wedge \quad \rho_{\succcurlyeq}(w)$$

The intended meaning of the defined formulae is:

(a) $\rho_{\mathsf{a}}(w)$ specifies the structure of the $\mathsf{R_a}$-cluster generated by w;

(b) $\rho_{\mathsf{e}}(w)$ describes the $\mathsf{R_e}$-cluster generated by w;

(c) $\rho_{\preccurlyeq}(w)$ indicates all the $\mathsf{R}_{\preccurlyeq}$-accessible worlds from w and it also specifies that they are the only ones $\mathsf{R}_{\preccurlyeq}$-seen by w;

(d) $\rho_{\succcurlyeq}(w)$, finally, says that the $\mathsf{R}_{\preccurlyeq}$-maximal time-cluster that is $\mathsf{R}_{\preccurlyeq}$-accessible from w consists of all the $\mathsf{R_e}$-clusters from $\mathcal{T}_{\mathbb{S}}(w)$.

STEP 2. Suppose w is an element of depth $i+1$. The formulae $\alpha(w)$, $\rho_{\mathsf{a}}(w)$ and $\rho_{\mathsf{e}}(w)$ are defined in the same way as the former case. Recall that $w^{<} := \{z \mid w\mathsf{R}_{\preccurlyeq}z \;\&\; \neg(z\mathsf{R}_{\preccurlyeq}w)\}$.

$$\gamma(i) := \bigwedge_{z\in S_i} \neg\beta(z)$$

$$\rho'_{\preccurlyeq}(w) := \bigwedge_{z\in w^{<}} \Diamond_{\preccurlyeq}\beta(z) \quad \wedge \quad \bigwedge_{z\in S_i \& z\notin w^{<}} \neg\Diamond_{\preccurlyeq}\beta(z)$$

$$\delta(w) := \Box_{\preccurlyeq}\Big(\bigvee_{z \in w^{<}} \beta(z) \quad \vee \bigvee_{z \in \mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)} \Big(\alpha(z) \wedge \rho_{\mathsf{a}}(z) \wedge \rho_{\mathsf{e}}(z) \wedge \gamma(i)\Big)\Big)$$

We can now define $\beta(w)$:

$$\beta(w) := \alpha(w) \wedge \rho_{\mathsf{a}}(w) \wedge \rho_{\mathsf{e}}(w) \wedge \rho'_{\preccurlyeq}(w) \wedge \gamma(i) \wedge \delta(w)$$

The formula $\rho'(w)$ says that w $\mathsf{R}_{\preccurlyeq}$-sees a specified set of worlds from S_i, while $\gamma(i)$ avoids the case $w \in S_i$. Finally, $\delta(w)$ says that if a world z is $\mathsf{R}_{\preccurlyeq}$-seen by w, then either it belongs to the set of all the $\mathsf{R}_{\preccurlyeq}$-successors of w, or it is in the $\mathsf{R}_{\preccurlyeq}$-cluster generated by w. Now, we will show that, for any w, z from $Ch_{\mathsf{LTK}}(n)$, if $z \Vdash_V \beta(w)$, then $(w = z)$. We can have two cases.

Case 1. Assume w has depth 1 and suppose there is a point z s.t. $z \Vdash_V \beta(w)$.
(a) If $depth(z) = 1$, then the structure of $\beta(w)$ implies that the $\mathsf{R}_{\preccurlyeq}$-open submodels generated by z and w are isomorphic, so they should coincide. Hence, by the structure of $S_1(Ch_{\mathsf{LTK}}(n))$, we have $w = z$.
(b) The case $depth(z) = 2$ is impossible because $\rho_{\succcurlyeq}(w)$ is a conjunct of $\beta(w)$.
(c) If $depth(z) > 2$, then either $z\mathsf{R}_{\preccurlyeq}w$ or $\neg(z\mathsf{R}_{\preccurlyeq}w)$. The case $z\mathsf{R}_{\preccurlyeq}w$ is impossible for the structure of $S_2(Ch_{\mathsf{LTK}}(n))$ (i.e. there should be an R_{e}-cluster $\mathcal{C}$ s.t. $depth(\mathcal{C}) = 2$, $\mathcal{C} \in \mathcal{C}_{\mathsf{R}_{\mathsf{e}}}(z)^{\preccurlyeq}$ and $\mathcal{C} \notin \mathcal{C}_{\mathsf{R}_{\mathsf{e}}}(w)^{\preccurlyeq}$). Since $\rho_{\succcurlyeq}(w)$ is also a conjunct of $\beta(w)$, the case $\neg(z\mathsf{R}_{\preccurlyeq}w)$ is impossible as well.

Case 2. Assume w has depth $i+1$ and suppose there is a point z s.t. $z \Vdash_V \beta(w)$. By the structure of the conjunct $\gamma(i)$ of $\beta(w)$, we have $depth(z) > (i+1)$. By the conjunct $\rho'_{\preccurlyeq}$ we have $\forall v \in S_i(w\mathsf{R}_{\preccurlyeq}v \Rightarrow z\mathsf{R}_{\preccurlyeq}v)$. We can have two cases:
(a) If $depth(z) = i+1$, then, by the construction of $Ch_{\mathsf{LTK}}(n)$, $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w) = \mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)$ and so $w = z$.
(b) Suppose $depth(z) > i+1$; then either $z\mathsf{R}_{\preccurlyeq}w$ or $\neg(z\mathsf{R}_{\preccurlyeq}w)$. Assume $z\mathsf{R}_{\preccurlyeq}w$; then there are $\mathsf{R}_{\preccurlyeq}$-clusters $\mathcal{C}_1, \ldots, \mathcal{C}_m$ between $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)$ and $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ such that $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z), \mathcal{C}_1, \ldots, \mathcal{C}_m, \mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ is an $\mathsf{R}_{\preccurlyeq}$-chain of $\mathsf{R}_{\preccurlyeq}$-clusters (i.e. $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)\mathsf{R}_{\preccurlyeq}\mathcal{C}_1$, $\mathcal{C}_m\mathsf{R}_{\preccurlyeq}\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ and for each i, j $1 \leq i \leq j \leq m$, $\mathcal{C}_i\mathsf{R}_{\preccurlyeq}\mathcal{C}_j$). By the structure of the conjunct $\delta(w)$, each $\mathsf{R}_{\preccurlyeq}$-cluster from $\mathcal{C}_1, \ldots, \mathcal{C}_m$ is isomorphic to $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w)$ and this is impossible by the construction of $Ch_{\mathsf{LTK}}(n)$. Assume $\neg(z\mathsf{R}_{\preccurlyeq}w)$; then there are $\mathsf{R}_{\preccurlyeq}$-clusters $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that $depth(\mathcal{C}_m) = i+1$ and $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z), \mathcal{C}_1, \ldots, \mathcal{C}_m$ is an $\mathsf{R}_{\preccurlyeq}$-chain. Again, by the structure of $\delta(w)$, each $\mathsf{R}_{\preccurlyeq}$-cluster from $\mathcal{C}_1, \ldots, \mathcal{C}_m$ is isomorphic to $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(z)$ and this is impossible by the construction of $Ch_{\mathsf{LTK}}(n)$.

∎

6 Decidability for LTK with respect to inference rules

Definition 6.1

An inference rule $\mathbf{r}$ *is an expression of the form*

$$\mathbf{r} := \frac{\mathtt{A}_1(p_1,\ldots,p_n),\ldots,\mathtt{A}_m(p_1,\ldots,p_n)}{\mathtt{B}(p_1,\ldots,p_n)}$$

where any $\mathtt{A}_i(p_1,\ldots,p_n)$ *and* $\mathtt{B}(p_1,\ldots,p_n)$ *are wff built up from the letters* $p_1,\ldots,p_n$ *(in what follows, we will sometimes use the expression* $\mathtt{A}_1(p_1,\ldots,p_n),\ldots,\mathtt{A}_m(p_1,\ldots,p_n)/\mathtt{B}(p_1,\ldots,p_n)$*).*

A substitution σ is a map which assigns a formula to each propositional variable. Given a formula $\mathtt{A}$, $\sigma(\mathtt{A})$ is the result of the application of σ to $\mathtt{A}$.

Definition 6.2

Given a logic L *and an inference rule* $\mathbf{r} := \mathtt{A}_1,\ldots,\mathtt{A}_m/\mathtt{B}$*,* $\mathbf{r}$ *is said to be admissible for* L *if and only if for each substitution* σ*, if* $\sigma(\mathtt{A}_i) \in \mathsf{L}$ *for each* i*, then* $\sigma(\mathtt{B}) \in \mathsf{L}$.

Therefore, the greatest class of rules which can be implemented for a given logic, i.e. which are compatible with the set of its valid formulae, is the class of its *admissible rules*: this is the class of all those rules under which the theory itself is closed.

Definition 6.3

Given a model $\langle \mathcal{F}, V\rangle$*, a valuation* V' *is definable if and only if* $\forall p \in \mathsf{Dom}(V')$ *there is a formula* α_p *s.t.* $V'(p) = V(\alpha_p)$*.*

From the general result stated by Theorem 3.3.3 in Rybakov [22], it follows immediately:

Lemma 6.4

An inference rule $\mathbf{r} := \mathtt{A}_1,\ldots,\mathtt{A}_n/\mathtt{B}$ *is not admissible for* LTK *iff there is an* n*-characterising model* $Ch_{\mathsf{LTK}}(n) := \langle Ch(n), V\rangle$ *and a definable valuation* V_2 *s.t.* $Ch(n) \Vdash_{V_2} \bigwedge_{1\leq i\leq n} \mathtt{A}_i$ *and* $Ch(n) \nVdash_{V_2} \mathtt{B}$*.*

Now, we introduce a special kind of 3-modal Kripke-frames, which will play a central role in the proof of our main result. The structure of such frame is depicted in Figure 2.

Definition 6.5

Let $\mathcal{F}_{\mathsf{L}}$*,* $\mathcal{F}_{\mathsf{S}}$ *and* $\mathcal{F}_{\mathsf{P}}$ *be Kripke-frames with the following structure:*

(a) The frame $\mathcal{F}_{\mathsf{L}} = \langle W_{\mathcal{F}_{\mathsf{L}}}, \mathsf{R}^{\mathsf{L}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{L}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{L}}_{\mathsf{a}}\rangle$ *(LOOP-component) is as follows:* $W_{\mathcal{F}_{\mathsf{L}}}$ *is a nonempty set of worlds;* $\mathsf{R}^{\mathsf{L}}_{\preccurlyeq} = W_{\mathcal{F}_{\mathsf{L}}} \times W_{\mathcal{F}_{\mathsf{L}}}$*;* $\mathsf{R}^{\mathsf{L}}_{\mathsf{e}}$ *is an equivalence relation on* $W_{\mathcal{F}_{\mathsf{L}}}$*;* $\mathsf{R}^{\mathsf{L}}_{\mathsf{a}}$ *is some equivalence relation on* $\mathsf{R}^{\mathsf{L}}_{\mathsf{e}}$*-clusters;*

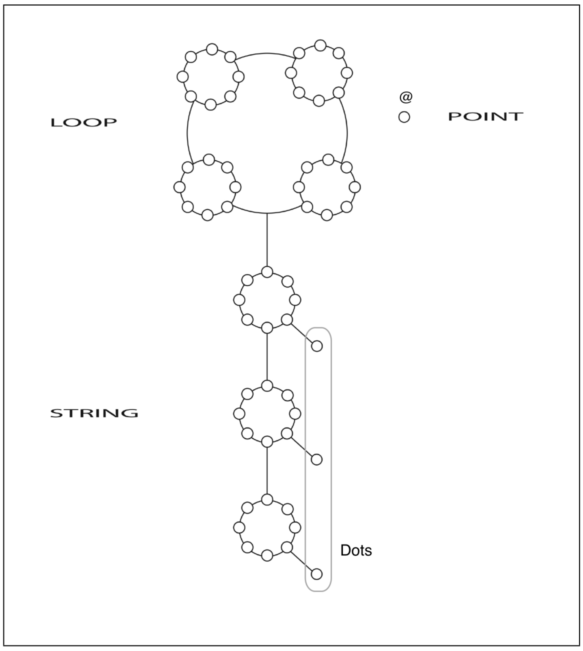

FIG. 2. Scheme of the structure of an LSP-frame.

(b) Let $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R}_{\mathsf{e}}, \mathsf{R}_{\mathsf{a}}\rangle$ *be a* finite $\mathcal{LTK}$*-frame (i.e. it is an* $\mathcal{LTK}$*-frame with a finite base set of worlds. See Definition 2.3); let* $\mathcal{C}_1, \ldots, \mathcal{C}_i$ *be an enumeration of all the* $\mathsf{R}^{\mathsf{S}}_{\preccurlyeq}$*-clusters of worlds from* $W_{\mathcal{F}}$*; let* $\mathsf{Dots} := \{w_1, \ldots, w_i\}$ *be a set of worlds such that* $\forall w_j, 1 \leq j \leq i(w_j \notin W_{\mathcal{F}})$*. The frame* $\mathcal{F}_{\mathsf{S}} = \langle W_{\mathcal{F}_{\mathsf{S}}}, \mathsf{R}^{\mathsf{S}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{S}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{S}}_{\mathsf{a}}\rangle$ *(STRING-component) has the following structure:* $W_{\mathcal{F}_{\mathsf{S}}} = W_{\mathcal{F}} \cup \mathsf{Dots}$*;* $\mathsf{R}^{\mathsf{S}}_{\preccurlyeq} = \mathsf{R}_{\preccurlyeq} \cup \{\langle w_j, z\rangle \mid w_j \in \mathsf{Dots} \ \&\ z \in \mathcal{C}_j\} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$*;* $\mathsf{R}^{\mathsf{S}}_{\mathsf{e}} = \mathsf{R}_{\mathsf{e}} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$*;* $\mathsf{R}^{\mathsf{S}}_{\mathsf{a}} = \mathsf{R}_{\mathsf{a}} \cup \{\langle w_j, w_j\rangle \mid w_j \in \mathsf{Dots}\}$.

(c) The frame $\mathcal{F}_{\mathsf{P}} := \langle W_{\mathcal{F}_{\mathsf{P}}}, \mathsf{R}^{\mathsf{P}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{P}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{P}}_{\mathsf{a}}\rangle$ *(POINT-component) is such that its base set contains only one world denoted by @,*$W_{\mathcal{F}_{\mathsf{P}}} := \{@\}$*, and all the binary relations on* $W_{\mathcal{F}_{\mathsf{P}}}$ *are universal.*

An LSP*-frame (loop-string-point frame) is a tuple* $\mathcal{F}_{\mathsf{lsp}} = \langle W_{\mathsf{lsp}}, \mathsf{R}^{\mathsf{lsp}}_{\preccurlyeq}, \mathsf{R}^{\mathsf{lsp}}_{\mathsf{e}}, \mathsf{R}^{\mathsf{lsp}}_{\mathsf{a}}\rangle$ *where* $W_{\mathcal{F}_{\mathsf{lsp}}} = W_{\mathcal{F}_{\mathsf{L}}} \cup W_{\mathcal{F}_{\mathsf{S}}} \cup W_{\mathcal{F}_{\mathsf{P}}}$*;* $\mathsf{R}^{\mathsf{lsp}}_{\preccurlyeq} = \mathsf{R}^{\mathsf{L}}_{\preccurlyeq} \cup \mathsf{R}^{\mathsf{S}}_{\preccurlyeq} \cup \mathsf{R}^{\mathsf{P}}_{\preccurlyeq} \cup \{\langle w, z\rangle \mid w \in W_{\mathcal{F}_{\mathsf{S}}} \ \&\ z \in W_{\mathcal{F}_{\mathsf{L}}}\}$*;* $\mathsf{R}^{\mathsf{lsp}}_{\mathsf{e}} = \mathsf{R}^{\mathsf{L}}_{\mathsf{e}} \cup \mathsf{R}^{\mathsf{S}}_{\mathsf{e}} \cup \mathsf{R}^{\mathsf{P}}_{\mathsf{e}}$*;* $\mathsf{R}^{\mathsf{lsp}}_{\mathsf{a}} = \mathsf{R}^{\mathsf{L}}_{\mathsf{a}} \cup \mathsf{R}^{\mathsf{S}}_{\mathsf{a}} \cup \mathsf{R}^{\mathsf{P}}_{\mathsf{a}}$ *(See Figure 2).*

THEOREM 6.6

An inference rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$ *is not admissible for* LTK *if and only if there is a finite* LSP*-frame* $\mathcal{F}_{\mathsf{lsp}}$*, whose size is computable from* $\|Var(\mathbf{r})\|$ *(where* $Var(\mathbf{r})$ *is the set of all the variables occurring in* $\mathbf{r}$*), and a model* $\mathcal{M}_{\mathsf{lsp}} = \langle \mathcal{F}_{\mathsf{lsp}}, V\rangle$ *s.t.* $\mathcal{F}_{\mathsf{lsp}} \Vdash_V \bigwedge_{1 \leq i \leq n} \mathtt{A}_i$ *and* $\mathcal{F}_{\mathsf{lsp}} \nVdash_V \mathtt{B}$.

PROOF. ($\Rightarrow$) Suppose that an inference rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$ is not admissible for LTK and let $p_1, \ldots, p_k$ be all the letters occurring in $\mathbf{r}$. Hence there are formulae $\gamma_1, \ldots, \gamma_j$, $1 \leq j \leq k$, s.t. $\bigwedge_{1 \leq i \leq m} \mathtt{A}_i(\gamma_1, \ldots, \gamma_j) \in \mathsf{LTK}$ and $\mathtt{B}(\gamma_1, \ldots, \gamma_j) \notin \mathsf{LTK}$. Let $Prop(\gamma)$ be the set of all the propositional letters occurring in $\gamma_1, \ldots, \gamma_j$.

By Lemma 6.4 there is an $n+1$-characterising model $Ch_{\mathsf{LTK}}(n+1) := \langle Ch(n+1), V\rangle$ and a new definable valuation V_2 with $\mathsf{Dom}(V_2) := Prop(\gamma) \cup \{p_{n+1}\}$, where $p_{n+1} \notin Prop(\gamma)$, s. t. $Ch(n+1) \Vdash_{V_2} \bigwedge_{1 \leq i \leq m} \mathtt{A}_i(\gamma_1, \ldots, \gamma_j)$ and $Ch(n+1) \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)$.

Take a world $w \in W_{Ch(n+1)}$ such that:

(a) $(Ch(n+1), w) \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)$

(b) $\forall v \in w^{\preccurlyeq}(v \notin V_2(p_{n+1}))$

(c) $\forall v \in W_{Ch(n+1)}(((Ch(n+1), v) \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)\ \&\ v^{\preccurlyeq} \cap V_2(p_{n+1}) = \emptyset) \Rightarrow \|w^{\preccurlyeq}\| \leq \|v^{\preccurlyeq}\|)$ (i.e. $w^{\preccurlyeq}$ is the smallest set of the kind $v^{\preccurlyeq}$ containing a world refuting $\mathtt{B}$ and such that none of its elements belongs to $V_2(p_{n+1})$).

It can be easily noticed that, since the propositional letter p_{n+1} does not occur in any γ_i, such a world w exists in $Ch_{\mathsf{LTK}}(n+1)$.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_i$ be an enumeration of all the $\mathsf{R}_{\preccurlyeq}$-clusters of worlds from $w^{\preccurlyeq}$ such that $depth(\mathcal{C}_j) \geq 2$. Now we take and fix, for each $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}_j$ a world w_j such that:

(a) $w_j \notin \mathcal{C}_j$

(b) $w_j^{\preccurlyeq} = \{w_j\} \cup \bigcup \mathcal{C}_j^{\preccurlyeq}$

(c) $w_j \in V_2(p_{n+1})$

The existence of such a world for each $\mathsf{R}_{\preccurlyeq}$-cluster is guaranteed by the construction of $Ch_{\mathsf{LTK}}(n+1)$. In fact, since $w_j \in V_2(p_{n+1})$ while none of the worlds from any $\mathcal{C}_j$ belongs to $V_2(p_{n+1})$, we have that for any j, $\mathcal{C}_{\mathsf{R}_{\preccurlyeq}}(w_j)$ is not a submodel of $\mathcal{C}_j$. Let $\mathsf{Dots} = \{w_1, \ldots, w_i\}$ be the set of those worlds w_j. Take and fix a world $@ \in W_{Ch(n+1)}$ such that:

(a) $@ \notin w^{\preccurlyeq} \cup \mathsf{Dots}$

(b) $@^{\preccurlyeq} = \{@\}$

Let $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}}} := \langle \mathcal{F}_{\mathsf{lsp}}, V_2\rangle$ be an open submodel of $Ch_{\mathsf{LTK}}(n+1)$ where $W_{\mathcal{F}_{\mathsf{lsp}}} := w^{\preccurlyeq} \cup \mathsf{Dots} \cup \{@\}$. Since $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}}}$ is a generated submodel of $Ch_{\mathsf{LTK}}(n+1)$, we have $\mathcal{F}_{\mathsf{lsp}} \Vdash_{V_2} \bigwedge_{1 \leq i \leq m} \mathtt{A}_i(\gamma_1, \ldots, \gamma_j)$ and $\mathcal{F}_{\mathsf{lsp}} \nVdash_{V_2} \mathtt{B}(\gamma_1, \ldots, \gamma_j)$. Moreover, by Definition 6.5, $\mathcal{F}_{\mathsf{lsp}}$ is an LSP-frame. Though $\mathcal{F}_{\mathsf{lsp}}$ is finite, the number of worlds from its base set is not known. To reduce such number, we apply the technique used in the proof of Theorem 3.2 in a slightly different way. Consider the STRING-component $\mathcal{F}_{\mathsf{S}}$ of $\mathcal{F}_{\mathsf{lsp}}$ (cf. Definition 6.5, item (b)). For each $\mathtt{D} \in Sub(\mathtt{B})$, we consider the $\mathsf{R}_{\preccurlyeq}$-maximal world $v \in W_{\mathcal{F}_{\mathsf{S}}}$ such that $(\mathcal{F}_{\mathsf{lsp}}, v) \nVdash_{V_2} \mathtt{D}$. We can have two cases: either $v \in \mathsf{Dots}$ and hence $v = w_j$ for some j, or $v \in \mathcal{C}_j$ for some j. In both cases, by $\mathcal{C}_{\mathtt{D}}$ we denote the set $\bigcup \mathcal{C}_j \cup \{w_j\}$. Likewise, by $\mathcal{C}_{\neg\mathtt{D}}$ we denote the set $\bigcup \mathcal{C}_j \cup \{w_j\}$ (such that there

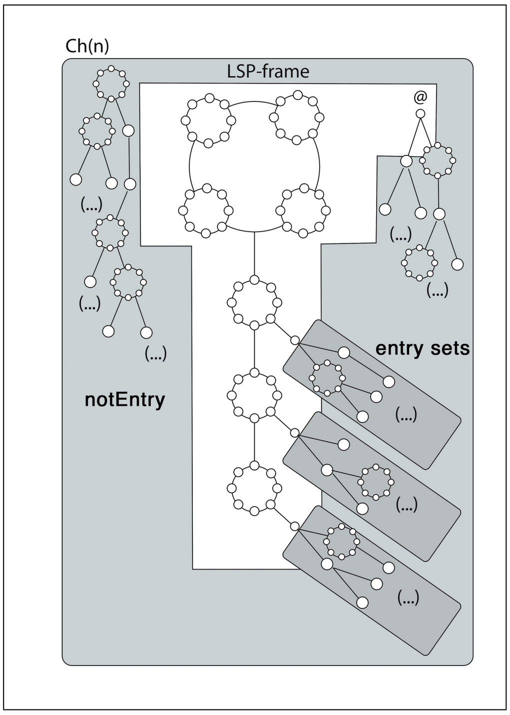

FIG. 3. Scheme of the structure of $Ch(n)$ and the sets Entry and notEntry.

is an $\mathsf{R}_{\preccurlyeq}$-maximal world v refuting $\mathtt{D}$ and either $v \in \mathcal{C}_j$ or $v = w_j$). Then we define a subframe $\mathcal{F}_{\mathsf{lsp}'} := \langle W_{\mathcal{F}_{\mathsf{lsp}'}}, \mathsf{R}^{\mathsf{lsp}'}_{\preccurlyeq}, \mathsf{R}^{\mathsf{lsp}'}_{\mathsf{e}}, \mathsf{R}^{\mathsf{lsp}'}_{\mathsf{a}} \rangle$ where:

$$W_{\mathcal{F}_{\mathsf{lsp}'}} := \bigcup_{\mathtt{B} \in Sub(\mathtt{A})} \mathcal{C}_{\mathtt{D}} \cup \bigcup_{\mathtt{B} \in Sub(\mathtt{A})} \mathcal{C}_{\neg\mathtt{D}} \cup W_{\mathcal{F}_{\mathsf{L}}} \cup \{@\}$$

Let $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}'}} := \langle \mathcal{F}_{\mathsf{lsp}'}, V_3 \rangle$ be a model s.t. $V_3 = V_2 \upharpoonright W_{\mathcal{F}_{\mathsf{lsp}'}}$. It is easy to verify that $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}'}}$ refutes $\mathbf{r}$, $\mathcal{F}_{\mathsf{lsp}'}$ is an LSP-frame. Moreover, the number of worlds from $W_{\mathcal{F}_{\mathsf{lsp}'}}$ is finite and computably bounded by the size of $Var(\mathbf{r})$ (cf. item (c) page 17). Therefore part ($\Rightarrow$) of the theorem has been proved.

($\Leftarrow$) Suppose that we have an inference rule $\mathbf{r} := \mathtt{A}_1, \ldots, \mathtt{A}_m/\mathtt{B}$, an LSP-frame $\mathcal{F}_{\mathsf{lsp}}$ and a model $\mathcal{M}_{\mathcal{F}_{\mathsf{lsp}}} := \langle \mathcal{F}_{\mathsf{lsp}}, S \rangle$ such that $\mathcal{F}_{\mathsf{lsp}} \Vdash_S \bigwedge_{1 \le i \le n} \mathtt{A}_i$ and $\mathcal{F}_{\mathsf{lsp}} \nVdash_S \mathtt{B}$. Let $Prop(W_{\mathcal{F}_{\mathsf{lsp}}}) := \{p_w \mid w \in W_{\mathcal{F}_{\mathsf{lsp}}}\}$ and $\mathsf{VAR} := Prop(W_{\mathcal{F}_{\mathsf{lsp}}}) \cup Var(\mathbf{r})$. We define a new valuation S_2 for $\mathcal{F}_{\mathsf{lsp}}$ in the following way:

(a) $\mathsf{Dom}(S_2) = \mathsf{VAR}$
(b) $\forall p_w \in Prop(W_{\mathcal{F}_{\mathsf{lsp}}})(S_2(p_w) = \{w\})$
(c) $\forall x \in Var(\mathbf{r})(S_2(x) = S(x))$

Clearly the new model $\langle \mathcal{F}_{\mathsf{lsp}}, S_2 \rangle$ still refutes $\mathtt{B}$, but not any $\mathtt{A}_i$. We construct, following the procedure explained in Section 4, the model $Ch_{\mathsf{LTK}}(n) := \langle Ch(n), V \rangle$, where $n = \|\mathsf{VAR}\|$. It is easy to see that the model $\langle \mathcal{F}_{\mathsf{lsp}}, S_2 \rangle$ formerly defined is (isomorphic to) an open submodel of $Ch_{\mathsf{LTK}}(n)$. We will construct, now, a new definable valuation V_2 for $Ch(n)$ refuting $\mathbf{r}$. The basic idea is finding a way to extend the valuation S_2 from $\mathcal{F}_{\mathsf{lsp}}$ to the whole frame $Ch(n)$. Recall that by Lemma 5.2 we know that each world from the base set of $Ch_{\mathsf{LTK}}(n)$ is definable (recall that for any world w, by $\beta(w)$ we denote that particular formula defining w).

Let @ be the name of that world from $W_{\mathcal{F}_{\mathsf{lsp}}}$ such that:

(a) $@^{\preccurlyeq} = \{@\}$

(b) $\forall w \in W_{\mathcal{F}_{\mathsf{lsp}}}$ $((w\mathsf{R_e}@$ *or* $w\mathsf{R}_{\preccurlyeq}@$ *or* $w\mathsf{R_a}@) \;\Rightarrow\; w = @)$ (See Figure 3.)

We define the set of all those worlds from $W_{Ch(n)}$ that are not $\mathsf{R}_{\preccurlyeq}$-related to any point from $[W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}]$ (see Figure 3), i.e. we set
$\mathsf{notEntry} := \{v \mid v \nVdash_V \Diamond_{\preccurlyeq}\beta(w),\ \forall w \in [W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}]\}$
Let Dots be a subset of $W_{\mathcal{F}_{\mathsf{lsp}}}$ as defined in Definition 6.5, i.e.:
$\mathsf{Dots} = \{w_j \mid w_j \in [W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}]\ \&\ \forall z \in W_{\mathcal{F}_{\mathsf{lsp}}}\ (z\mathsf{R}_{\preccurlyeq}w_j \;\Rightarrow\; w_j = z)\}$
Take and fix, for each $\mathsf{R}_{\preccurlyeq}$-cluster $\mathcal{C}_j$ of worlds from $[W_{\mathcal{F}_{\mathsf{lsp}}} - \{@\}] - \mathsf{Dots}$ such that $depth(\mathcal{C}_j) \geq 2$, a representative world z_j belonging to $\mathcal{C}_j$. Let Rep be the set containing all those representative elements.
For each representative world z_j from Rep, we shall define, now, an *entry-set* (see Figure 3). It contains all those worlds v from $W_{Ch(n)} - W_{\mathcal{F}_{\mathsf{lsp}}}$ which are $\mathsf{R}_{\preccurlyeq}$-predecessors of z_j and such that z_j is the $\mathsf{R}_{\preccurlyeq}$-deepest world belonging to $[W_{\mathcal{F}_{\mathsf{lsp}}} - \mathsf{Dots}]$ which is $\mathsf{R}_{\preccurlyeq}$-accesible from v: $\forall z_j \in \mathsf{Rep}$

$\mathsf{Entry}(z_j) := \{w \mid w \notin W_{\mathcal{F}_{\mathsf{lsp}}}\ \&\ w \Vdash_V \Diamond_{\preccurlyeq}\beta(z_j)\ \&\ \forall v \in [W_{\mathcal{F}_{\mathsf{lsp}}} - \mathsf{Dots}]((v\mathsf{R}_{\preccurlyeq}z_j\ \&\ \neg(z_j\mathsf{R}_{\preccurlyeq}v)) \Rightarrow w \nVdash_V \Diamond_{\preccurlyeq}\beta(v))\}$ (See Figure 3.)
For each representative world z_j from Rep, we define a formula $\phi(z_j)$ that is true only at those worlds belonging to $\mathsf{Entry}(z_j)$. $\forall z_j \in \mathsf{Rep}$:

$$\phi(z_j) := \bigwedge_{v \in W_{\mathcal{F}_{\mathsf{lsp}}}} \neg\beta(v) \wedge \Diamond_{\preccurlyeq}\beta(z_j) \wedge \bigwedge_{v \in W_{\mathcal{F}_{\mathsf{lsp}}} \& v\mathsf{R}_{\preccurlyeq} z_j \& \neg(z_j \mathsf{R}_{\preccurlyeq} v)} \neg\Diamond_{\preccurlyeq}\beta(v)$$

It can be easily verified that, given a world v, it belongs to $\mathsf{Entry}(z_j)$, for some $z_j \in \mathsf{Rep}$, if and only if $\phi(z_j)$ is true at v under V. Recall that for any $z_j \in \mathsf{Rep}$, by w_j we denote the world from Dots such that z_j is one of its immediate $\mathsf{R}_{\preccurlyeq}$-successors.

To define the valuation V_2, let $\mathsf{Dom}(V_2) := \mathsf{VAR}$;
$\forall p \in \mathsf{VAR}$, $V_2(p) :=$

$$\bigcup_{v \in W_{\mathcal{F}_{\mathsf{lsp}}} \& p \in V(v)} V(\beta(v)) \cup \bigcup_{v \in \mathsf{notEntry} \& @ \in V(p)} V(\beta(v)) \cup \bigcup_{z_j \in \mathsf{Rep} \& z_j \in V(p)} V(\phi(z_j))$$

Obviously the valuation V_2 is definable, in fact, for each $p \in \mathsf{VAR}$, there is a formula α_p such that $V_2(p) = V(\alpha_p)$, namely, $\forall p \in \mathsf{VAR}$:

$$\alpha_p := \bigvee_{v \in W_{\mathcal{F}_{\mathsf{lsp}}} \& p \in V(v)} \beta(v) \vee \bigvee_{v \in \mathsf{notEntry} \& @ \in V(p)} \beta(v) \vee \bigvee_{z_j \in \mathsf{Rep} \& z_j \in V(p)} \phi(z_j)$$

Next step is to show that the inference rule $\mathbf{r}$ is not valid in the new model $\langle Ch(n), V_2 \rangle$. It is sufficient to show that the following claim holds: for any formula $\mathtt{A}$ on $\mathcal{L}^{\mathsf{LTK}}$ containing only letters from VAR

$$\mathcal{F}_{\mathsf{lsp}} \Vdash_{S_2} \mathtt{A} \Leftrightarrow Ch(n) \Vdash_{V_2} \mathtt{A}.$$

Notice that the three statements below follow immediately by the definition of V_2:

(a) $\forall w \in W_{\mathcal{F}_{\mathsf{lsp}}}$ $(Ch(n), w) \Vdash_{V_2} \mathtt{A} \Leftrightarrow (\mathcal{F}_{\mathsf{lsp}}, w) \Vdash_{S_2} \mathtt{A}$ (the model $\langle \mathcal{F}_{\mathsf{lsp}}, S_2 \rangle$ being isomorphic to $\langle \mathcal{F}_{\mathsf{lsp}}, V_2 \rangle$ which is an open submodel of $\langle Ch(n), V_2 \rangle$);

(b) $\forall z_i \in \mathsf{Rep}$, $\forall v \in \mathsf{Entry}(z_i)$ $(Ch(n), v) \Vdash_{V_2} \mathtt{A} \Leftrightarrow (\mathcal{F}_{\mathsf{lsp}}, w_i) \Vdash_{S_2} \mathtt{A}$;

(c) $\forall v \in \mathsf{notEntry}$ $(Ch(n), v) \Vdash_{V_2} \mathtt{A} \Leftrightarrow (\mathcal{F}_{\mathsf{lsp}}, @) \Vdash_{S_2} \mathtt{A}$.

Since $W_{Ch(n)} = W_{\mathsf{lsp}} \cup \mathsf{notEntry} \cup \bigcup_{z_j \in \mathsf{Rep}} \mathsf{Entry}(z_j)$, the model $\langle Ch(n), V_2 \rangle$ refutes $\mathbf{r}$. ∎

COROLLARY 6.7
The logic LTK *is decidable with respect to inference rules.*

The author would like to thank Vladimir V. Rybakov for his scientific advising and the numerous improvements he suggested.

References

[1] Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, semantic approach. *Bulletin of the Section of Logic*, 34(1):13–21, 2005.

[2] Clare Dixon, Michael Fisher, and Mike Wooldridge. Resolution for temporal logics of knowledge. *Journal of Logic and Computation*, 8(3):345–372, 1998.

[3] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge.* MIT Press, 1995.

[4] Harvey Friedman. One hundred and two problems in mathematical logic. *Journal of Symbolic Logic*, 40(2):113–129, June 1975.

[5] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, North-Holland, New York - Amsterdam, 2003.

[6] Dov M. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on tense frames. In U. Mönnich, editor, *Aspects of Philosophical Logic: Some Logical Forays into Central Notions of Linguistics and Philosophy*, pages 67–89. Reidel, Dordrecht, 1981.

[7] Silvio Ghilardi. Unification in intuitionistic logic. *Journal of Symbolic Logic*, 64(2):859–880, 1999.

[8] Silvio Ghilardi. Best solving modal equations. *Annals of Pure and Applied Logic*, 10:183–198, 2000.

[9] Silvio Ghilardi. A resolution/tableaux algorithm for projective approximations in ipc. *Logic Journal of the IGPL*, 10(3):227–241, 2002.

[10] M.I. Golovanov, A.V. Kosheleva, and V. Rybakov. Logic of visibility, perception, and knowledge and admissible inference rules. *Logic Journal of the IGPL*, 13(2):201–209, 2005.

[11] M.I. Golovanov, Vladimir V. Rybakov, and E.M. Yurasova. A necessary condition for rules to be admissible in temporal tomorrow logic. *Bulletin of the Section of Logic*, 32(4):213–220, 2003.

[12] Joseph Y. Halpern, Ron Van Der Meyden, and Moshe Y. Vardi. Complete axiomatization for reasoning about knowledge and time. *SIAM Journal on Computing*, 33(3):674–703, 2004.

[13] Ronald Harrop. Concerning formulas of the types $\mathsf{A} \rightarrow \mathsf{B} \vee \mathsf{C}$, $\mathsf{A} \rightarrow \exists x(\mathsf{B}(x))$. *Journal of Symbolic Logic*, 25(1):27–32, 1960.

[14] Rosalie Iemhoff. A(nother) characterization of intuitionistic propositional logic. *Annals of Pure and Applied Logic*, 113(1-3):161–173, 2001.

[15] Rosalie Iemhoff. On the admissible rules of intuitionistic propositional logic. *Journal of Symbolic Logic*, 66(1):281–294, 2001.

[16] Rosalie Iemhoff. Intermediate logics and visser's rules. *Notre Dame Journal of Formal Logic*, 46(1):65–81, 2005.

[17] Emil Jeȓábek. Admissible rules of modal logics. *J Logic Computation*, 15(4):411–431, 2005.

[18] Vladimir V. Rybakov. A criterion for admissibility of rules in the modal system S4 and the intuitionistic logic. *Algebra and Logic*, 23(5):369–384, 1984 (English Translation).

[19] Vladimir V. Rybakov. Bases of admissible rules of the logics S4 and Int. *Algebra and Logic*, 24(1):55–68, 1985 (English Translation).

[20] Vladimir V. Rybakov. Intermediate logics preserving admissible inference rules of heyting calculus. *Mathematical Logic Quarterly*, 39:403–415, 1993.

[21] Vladimir V. Rybakov. Hereditarily structurally complete modal logics. *Journal of Symbolic Logic*, 60(1):266–288, 1995.

[22] Vladimir V. Rybakov. *Admissible Logical Inference Rules*, volume 136 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, North-Holland, New York - Amsterdam, 1997.

[23] Richard H. Thomason. Combinations of tense and modality. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic: Volume II: Extensions of Classical Logic*, pages 135–165. Reidel, Dordrecht, 1984.

[24] Mike Wooldridge and Alessio Lomuscio. A computationally grounded logic of visibility, perception, and knowledge. *Logic Journal of the IGPL*, 9(2):257–272, 2001.

Received submission date

An Axiomatisation for the Multi-modal Logic of Knowledge and Linear Time LTK

ERICA CALARDO, *Department of Computing and Mathematics, Manchester Metropolitan University, John Dalton Building, Chester Street, Manchester M1 5GD, UK.*
E-mail: E.Calardo@mmu.ac.uk

VLADIMIR V. RYBAKOV, *Department of Computing and Mathematics, Manchester Metropolitan University, John Dalton Building, Chester Street, Manchester M1 5GD, UK.*
E-mail: V.Rybakov@mmu.ac.uk

Abstract

The paper aims at providing the multi-modal propositional logic LTK with a sound and complete axiomatisation. This logic combines temporal and epistemic operators and focuses on modeling the behaviour of a set of agents operating in a system on the background of a temporal framework. Time is represented as linear and discrete, whereas knowledge is modeled as an S5-like modality. A further modal operator intended to represent environment knowledge is added to the system in order to achieve the expressive power sufficient to describe the piece of information available to the agents at each moment in the flow of time.

Keywords: Modal logic, Multi-modal logic, Temporal logic, Epistemic logic, Combined logics.

1 Introduction

Nowadays research that focuses on modelling human reasoning and agents' behaviour in a system is a very active area. Multi-modal languages provide a combination of highly expressive power and intuitive semantic tools and they are, therefore, widely used in the field. Recently they have also been applied in Artificial Intelligence and Computer Science in the attempt to formalise, for instance, reasoning about the behaviour of programs (cf. Goldblatt [9, 8]).

The main feature of modal logics is that they enable the switch from *extensional* languages (expressing only *facts, statements* which can be either true or false) to *intensional* ones. Modal logics deal with sentences that are qualified by modalities such as *can, could, might, may, must* etc. and they are often constructed by adding one or more modal operators (usually $\Box$ and $\Diamond$) to a classical propositional system. Likewise, multi-modal logics are obtained by adding more than one modal operator to an existing logical system.

Although traditionally read as expressing necessity and possibility, modal operators have numerous possible interpretations. The choice would then be suggested by the context one is to describe. In the case of tense logics, one can interpret the modal proposition $\Box p$ as *it will always be the case that* p, and its dual $\Diamond p$ as *at some point in the future it will be the case that* p. Such language is, therefore, effective whenever a description of the flow of time,

Vol. 15 No. 3,
doi:10.1093/jigpal/jzm010

towards both future and past is needed. Epistemic logics, on the other hand, are suitable to formalize reasoning about agents not possessing complete information (see Fagin *et al.* [5], Rybakov [12]). However, such systems may have an expressive limitation, for it may be difficult to deal with modifications in the pieces of information each agent possesses as well as to give an account of a changing environment. Adding a dynamic dimension to such systems is therefore almost a necessity. The most natural way partially to improve on such a deficiency is to insert epistemic logics in a temporal framework. Hence we would generate a multi-modal system combining tense and knowledge operators (see Fagin *et al.* [5], Halpern *et al.* [10]).

Moreover, systems generated by joining operators representing both time and knowledge have already proved themselves to be particularly effective in describing the interaction between agents through the flow of time (see Fagin *et al.* [5], Gabbay *et al.* [6], Halpern *et al.* [10], [3, 2]). These systems are generated by adding to an existing propositional system two sets of modalities: one to model the flow of time, the other to describe the agents' knowledge. The interaction of such modalities gives a precise account of the dynamic development of agents' knowledge. Though interesting and promising, this approach has not yet been fully investigated, due to the complexity and the extent of the subject. Nevertheless, in the last decade, the theories developed have have been successfully applied both in the study of human reasoning and in computing. These theories are concentrated on the development of systems modelling reasoning about knowledge and space, reasoning under uncertainty or with bounded resources, multi-agent reasoning and other aspects of artificial intelligence.

However, several difficulties arise whenever one is to prove an axiomatic system to be sound and complete with respect to a class of multi-modal frames. According to Bennett *et al.* [1] and Kurucz [11], if there is no interaction between modalities a transfer of properties (such as *finite model property*, *decidability*, etc.) from the component simple modal logics to the newly generated multi-modal system does apply. However, as soon as such interaction takes place it is not straightforward anymore to prove that the combined system is conservative with respect to the properties of its components. In some cases the opposite may apply. Nevertheless, despite such difficulties, interaction between modalities is necessary fully to exploit the power of multi-modal languages.

This paper aims at providing the multi-modal logic LTK (formerly introduced in [3] and [2]) with a finite, sound and complete axiomatisation with interacting modalities. In our previous works this logic has been defined semantically and proved to be decidable with respect to its theorems [3], whereas a weaker version of it is also decidable with respect to its admissible inference rules [2]. We start by recalling the semantic framework of our work introducing a special kind of multi-modal Kripke-frames aiming at modelling a set of agents operating in a system where the time is considered as linear and discrete. Besides a temporal operator and a knowledge operator for each agent operating in the system, our language provides a further epistemic modality, which is intended to represent environmental knowledge, i.e. a modality which makes our logic sufficiently powerful to express the piece of information available to the agents at each moment in the flow of time. We provide a sound axiomatic system and we proceed by following the standard approach of constructing canonical models, generated submodels and making filtration (see Gabbay [7]). Since standard tools are not sufficient to show completeness, in Section 3 we prove Lemma 3.15, which is the core of the whole work (cf. Goldblatt [8]) and makes a substancial use of our multi-modal version of the axiom presented by Dummett and Lemmon [4].

2 Syntax and Semantics

The alphabet of the language $\mathcal{L}^{\mathsf{LTK}}$ includes a countable set of propositional letters $P := \{p_1, \ldots, p_n, \ldots\}$, round brackets (,) and the boolean operations $\{\rightarrow, \bot\}$ as well as a set of modal operators $\{\Box_{\preccurlyeq}, \mathsf{K_e}, \mathsf{K}_1, \ldots, \mathsf{K}_k\}$. Well formed formulae (wff's) are defined as follows: each propositional letter $p \in P$ is a wff and if $\mathtt{A}$ is a wff, then so are $\Box_{\preccurlyeq}\mathtt{A}, \mathsf{K_e}\mathtt{A}, \mathsf{K}_i\mathtt{A}$. We assume $\Diamond_{\preccurlyeq}$, $\Diamond_{\mathsf{e}}$ and $\Diamond_i$ to be abbreviations for $\neg\Box_{\preccurlyeq}\neg$, $\neg\mathsf{K_e}\neg$ and $\neg\mathsf{K}_i\neg$ respectively. The boolean operations $\neg, \wedge, \vee$ are defined in the usual way by means of $\rightarrow$ and $\bot$.

The intended meaning of the modal operators formerly introduced is: (a) $\Box_{\preccurlyeq}\mathtt{A}$: *the fact* $\mathtt{A}$ *is true from now on*; (b) $\mathsf{K_e}\mathtt{A}$: $\mathtt{A}$ is true everywhere in the *environment*; (c) $\mathsf{K}_i\mathtt{A}$: *the agent i operating in the system knows* $\mathtt{A}$ *in the current moment* in the sense that all the information points accessible to agent i provide the information $\mathtt{A}$.

Formulae in the language $\mathcal{L}^{\mathsf{LTK}}$ allow occurrences of temporal operators in the scope of the epistemic modalities $\mathsf{K}_1, \ldots, \mathsf{K}_k$, leading to the possibility of expressing formulae such as $\mathsf{K}_i\Diamond_{\preccurlyeq}\mathtt{A}$, interpreted as *agent i knows that eventually it will be the case that* $\mathtt{A}$. To prevent agents from having pre-knowledge concerning future events we introduce a weaker language $\mathcal{L}^{-}_{\mathsf{LTK}}$. A formula $\mathtt{A}$ is *local* if and only if it does not contain any occurrence of the modal operator $\Box_{\preccurlyeq}$, i.e. each propositional letter is local and if $\mathtt{A}$ is local, then so are $\mathsf{K_e}\mathtt{A}$ and $\mathsf{K}_i\mathtt{A}$ for each i. Well formed formulae are defined as they are in the former case, with the only exception of formulae containing a modal operator K_i for some i: if $\mathtt{A}$ is a wff, then $\mathsf{K}_i\mathtt{A}$ is a wff, provided that $\mathtt{A}$ is *local*.

By the expression $Fma(\mathcal{L}^{\mathsf{LTK}})$ we denote the set of all the wff's on $\mathcal{L}^{\mathsf{LTK}}$ and by the term *formula* we always refer to a member of $Fma(\mathcal{L}^{\mathsf{LTK}})$. Clearly $Fma(\mathcal{L}^{-}_{\mathsf{LTK}}) \subset Fma(\mathcal{L}^{\mathsf{LTK}})$.

Although we assume the reader to be familiar with *possible world semantics*, we provide few basic definitions necessary to understand the particular case we will work with.

Definition 2.1
A k-modal Kripke-frame is a tuple $\mathcal{F} = \langle W, \mathsf{R}_1, \ldots, \mathsf{R}_k\rangle$ where W is a non-empty set of worlds and each R_j is some binary relation on $W \times W$. Given a frame $\mathcal{F}$, by $W_{\mathcal{F}}$ we denote its base set.

Given a Kripke-frame $\mathcal{F}$, a model $\mathcal{M}$ on $\mathcal{F}$ is a tuple $\mathcal{M} = \langle \mathcal{F}, V\rangle$ where V is a valuation of a set P of propositional letters in $\mathcal{F}$.

Definition 2.2
Given a Kripke-frame $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_1, \ldots, \mathsf{R}_k\rangle$, for any R_i, an R_i-cluster of worlds is a subset $\mathcal{C}_{\mathsf{R}_i}$ of $W_{\mathcal{F}}$ s.t.: $\forall w \forall z \in \mathcal{C}_{\mathsf{R}_i}$ $(w\mathsf{R}_i z \;\&\; z\mathsf{R}_i w)$ and $\forall z \in W_{\mathcal{F}} \forall w \in \mathcal{C}_{\mathsf{R}_i}((w\mathsf{R}_i z \;\&\; z\mathsf{R}_i w) \Rightarrow z \in \mathcal{C}_{\mathsf{R}_i})$.

An R_i-cluster is said to be: **degenerate** if it consists of one single R_i-irreflexive world; **simple** if it consists of a single R_i-reflexive world; **proper** if it contains at least two R_i-reflexive worlds.

For any R_i, $\mathcal{C}_{\mathsf{R}_i}(w)$ is the R_i-cluster s.t. $w \in \mathcal{C}_{\mathsf{R}_i}(w)$. Given two R_i-clusters $\mathcal{C}_m$ and $\mathcal{C}_j$ the expression $\mathcal{C}_m\mathsf{R}_i\mathcal{C}_j$ is an abbreviation for $\forall w \in \mathcal{C}_m \forall z \in \mathcal{C}_j(w\mathsf{R}_i z)$.

We use a special kind of multi-modal Kripke frames called $\mathcal{LTK}$-frames, where the prefix $\mathcal{LTK}$ is an acronym for *Linear Time and Knowledge*. These structures aim at modelling a set of agents operating in a temporal framework.

Definition 2.3
An $\mathcal{LTK}$-frame (*Linear Time and Knowledge frame*) is a k+2-modal Kripke-frame $\mathcal{F} := \langle W_{\mathcal{F}}, \mathsf{R}_{\preccurlyeq}, \mathsf{R_e}, \mathsf{R}_1, \ldots, \mathsf{R}_k\rangle$, where $W_{\mathcal{F}}$ is the disjoint union of certain

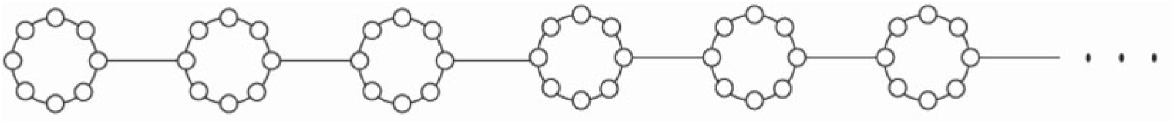

FIG. 1. Scheme of the structure of an $\mathcal{LTK}$-frame: here each big circle represents both a moment in the time line and an environmental cluster, whereas each small circle is intended to represent a single information point.

non empty sets $\mathcal{C}_n$, for $n \in \mathbb{N}$: $W_{\mathcal{F}} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$. The binary relations $\mathsf{R}_{\preccurlyeq}$, R_{e}, and R_j are as follows:

(a) $\mathsf{R}_{\preccurlyeq}$ is the linear, reflexive and transitive relation on $W_{\mathcal{F}}$ such that: $\forall v \forall z \in W_{\mathcal{F}}(v\mathsf{R}_{\preccurlyeq}z$ iff $\exists i, j \in \mathbb{N}((v \in \mathcal{C}_i)$ & $(z \in \mathcal{C}_j)$ & $(i \leq j)))$
(b) R_{e} is a universal relation on any $\mathcal{C}_i \in W_{\mathcal{F}}$: $\forall v \forall z \in W_{\mathcal{F}}(v\mathsf{R}_{\mathsf{e}}z \Leftrightarrow \exists i \in \mathbb{N}(v \in \mathcal{C}_i$ & $z \in \mathcal{C}_i))$;
(c) each R_j is some equivalence relation on each $\mathcal{C}_i$.

Each world can be interpreted as a single information point. The linear temporal relation $\mathsf{R}_{\preccurlyeq}$ links such information points so that, given two worlds v and z, the expression $v\mathsf{R}_{\preccurlyeq}z$ means either that v and z are both available at a moment n, or that z will be available in the future with respect to v. Hence two information points are concurrent if they belong to the same $\mathsf{R}_{\preccurlyeq}$-cluster (time-cluster) and an $\mathsf{R}_{\preccurlyeq}$-cluster can be seen as a moment in the time line (cf. Figure 1). Although time is usually perceived as continuous, it may as well be thought as discrete. In this context the property of discretenss means that given any two distinct points in the time line, there might be only a finite amounts of moments between them (though each moment may contain an infinite amount of information points). Therefore the relation $\mathsf{R}_{\preccurlyeq}$ is discrete with respect to time-clusters. This is actually the way in which computers work. Moreover, the temporal line has a first point starting from which it proceeds towards the future. The most important assumption is to consider the flow of time as linear and hence not branching. This implies that we may not quantify over possible though not actual temporal paths. In other words, what is relevant is only the actual path the world goes through. Such strong theoretical deterministic assumption may be practically justified by the observation that, in analogy to the human situation, all the agents operating in the system are not aware of the prefixed unicity of their temporal path and they act as heading to a not-determined future.

The relation R_{e} is defined at each moment in the time line and it links all the information points belonging to the same *environment*. In our specific interpretation, only one environment is possible at each moment and hence time-clusters and environment-clusters do coincide (see the Appendix for the general case of multiple concurrent environments and further discussion).

The relation R_i links all the information points accessible by agent i in a given environment. Any information point provides the agents with some information. $\mathcal{LTK}$-frames are a suitable tool to interpret the language $\mathcal{L}^{-}_{\mathsf{LTK}}$ as well as $\mathcal{L}^{\mathsf{LTK}}$. The only difference would be that in the case of $\mathcal{L}^{-}_{\mathsf{LTK}}$, all the facts available to the agents are *local* and therefore do not concern any future event. Nevertheless at each world a certain number of statements about the future could be true, but this piece of information would not be available to any agent.

Besides the standard ones, the main features of $\mathcal{LTK}$-frames are the following properties:

PM.1: $v\mathsf{R_e}z \Rightarrow (v\mathsf{R}_{\preccurlyeq}z \;\&\; z\mathsf{R}_{\preccurlyeq}v)$ i.e. the information points available in the same environment are concurrent

PM.2: $v\mathsf{R}_iz \Rightarrow v\mathsf{R_e}z$ i.e. the information points available to agent i must be in the same environment (hence at the same moment)

PM.3: $(v\mathsf{R}_{\preccurlyeq}z \;\&\; z\mathsf{R}_{\preccurlyeq}v) \Rightarrow v\mathsf{R_e}z$ i.e. concurrent information points are in the same environment

A model $\mathcal{M}$ on $\mathcal{F}$ is a pair $\langle\mathcal{F}, V\rangle$, where $\mathcal{F}$ is an $\mathcal{LTK}$-frame and V is a map (valuation) which associates to each propositional letter $p \in P$ a set of worlds from the base set of $\mathcal{F}$. The valuation V can be extended in the standard way from the set P onto all the well formed formulae built up on P. In particular, $\forall v \in W_{\mathcal{F}}$,

(a) $(\mathcal{F}, v) \Vdash_V p \Leftrightarrow v \in V(p)$;
(b) $(\mathcal{F}, v) \Vdash_V \Box_{\preccurlyeq}\mathtt{A} \Leftrightarrow \forall z \in W_{\mathcal{F}}(v\mathsf{R}_{\preccurlyeq}z \Rightarrow (\mathcal{F}, z) \Vdash_V \mathtt{A})$;
(c) $(\mathcal{F}, v) \Vdash_V \mathsf{K_e}\mathtt{A} \Leftrightarrow \forall z \in W_{\mathcal{F}}(v\mathsf{R_e}z \Rightarrow (\mathcal{F}, z) \Vdash_V \mathtt{A})$;
(d) For each j, $(\mathcal{F}, v) \Vdash_V \mathsf{K}_j\mathtt{A} \Leftrightarrow \forall z \in W_{\mathcal{F}}(v\mathsf{R}_jz \Rightarrow (\mathcal{F}, z) \Vdash_V \mathtt{A})$.

If $\mathcal{M} = \langle\mathcal{F}, V\rangle$ is a model on a frame $\mathcal{F}$, a formula $\mathtt{A}$ is said to be *true in the model* $\mathcal{M}$ *at the world* v if $(\mathcal{F}, v) \Vdash_V \mathtt{A}$; $\mathtt{A}$ is *true in the model* $\mathcal{M}$, notation $\mathcal{F} \Vdash_V \mathtt{A}$, if $\forall v \in W_{\mathcal{F}}, (\mathcal{F}, v) \Vdash_V \mathtt{A}$; $\mathtt{A}$ is *valid in the frame* $\mathcal{F}$, notation $\mathcal{F} \Vdash \mathtt{A}$, if, for any valuation V for $\mathcal{F}$ (that is for any model $\mathcal{M}_{\mathcal{F}}$ on $\mathcal{F}$), $\mathcal{F} \Vdash_V \mathtt{A}$. Given a class of frames $\mathbb{F}$, $\mathtt{A}$ *is valid on* $\mathbb{F}$ (and we say $\mathtt{A}$ to be $\mathbb{F}$-valid) if $\forall \mathcal{F} \in \mathbb{F}, \mathcal{F} \Vdash \mathtt{A}$.

Definition 2.4
Let $\mathcal{LTK}$ be the class of all $\mathcal{LTK}$-frames. The logic LTK is the set of all $\mathcal{LTK}$-valid formulae: $\mathsf{LTK} := \{\mathtt{A} \in Fma(\mathcal{L}^{\mathsf{LTK}}) \mid \mathcal{F} \Vdash \mathtt{A} \;\&\; \mathcal{F} \in \mathcal{LTK}\}$. If $\mathtt{A}$ belongs to LTK, then $\mathtt{A}$ is a theorem of LTK. Likewise $\mathsf{LTK}^- := \{\mathtt{A} \in Fma(\mathcal{L}^-_{\mathsf{LTK}}) \mid \mathcal{F} \Vdash \mathtt{A} \;\&\; \mathcal{F} \in \mathcal{LTK}\}$

Axioms of $\mathcal{AS}_{\mathsf{LTK}}$ (Schemata)

Axioms of CPC (classical propositional calculus)

$K_{\Box_{\preccurlyeq}}: \Box_{\preccurlyeq}(\mathtt{A} \to \mathtt{B}) \to (\Box_{\preccurlyeq}\mathtt{A} \to \Box_{\preccurlyeq}\mathtt{B})$
$T_{\Box_{\preccurlyeq}}: \Box_{\preccurlyeq}\mathtt{A} \to \mathtt{A}$
$4_{\Box_{\preccurlyeq}}: \Box_{\preccurlyeq}\mathtt{A} \to \Box_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{A}$
$3_{\Box_{\preccurlyeq}}: \Box_{\preccurlyeq}(\mathtt{A}\wedge\Box_{\preccurlyeq}\mathtt{A} \to \mathtt{B}) \vee \Box_{\preccurlyeq}(\mathtt{B}\wedge\Box_{\preccurlyeq}\mathtt{B} \to \mathtt{A})$

$K_{\mathsf{K}_\xi}: \mathsf{K}_\xi(\mathtt{A} \to \mathtt{B}) \to (\mathsf{K}_\xi\mathtt{A} \to \mathsf{K}_\xi\mathtt{B}), \quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$
$T_{\mathsf{K}_\xi}: \mathsf{K}_\xi\mathtt{A} \to \mathtt{A}, \quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$
$4_{\mathsf{K}_\xi}: \mathsf{K}_\xi\mathtt{A} \to \mathsf{K}_\xi\mathsf{K}_\xi\mathtt{A}, \quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$
$5_{\mathsf{K}_\xi}: \neg\mathsf{K}_\xi\mathtt{A} \to \mathsf{K}_\xi\neg\mathsf{K}_\xi\mathtt{A}, \quad \xi \in \{\mathsf{e}, 1, \ldots, k\}$

$M.1: \Box_{\preccurlyeq}\mathtt{A} \to \mathsf{K_e}\mathtt{A}$
$M.2: \mathsf{K_e}\mathtt{A} \to \mathsf{K}_i\mathtt{A}, \quad 1 \leq i \leq k$
$Dum_{\Box_{\preccurlyeq}}: \Box_{\preccurlyeq}(\Box_{\preccurlyeq}(\mathsf{K_e}\mathtt{A} \to \Box_{\preccurlyeq}\mathtt{A}) \to \mathsf{K_e}\mathtt{A}) \to (\Diamond_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{A} \to \Box_{\preccurlyeq}\mathtt{A})$

Inference Rules of $\mathcal{AS}_{\mathsf{LTK}}$:

$$MP: \frac{\mathtt{A}, \quad \mathtt{A} \to \mathtt{B}}{\mathtt{B}} \qquad Nec: \frac{\mathtt{A}}{\Box_{\preccurlyeq}\mathtt{A}}$$

It is easy to notice that we can derive a necessitation rule for the modalities $\mathsf{K_e}, \mathsf{K}_1, \ldots, \mathsf{K}_k$ by means of the axioms M.1 - M.2 and the rule *Nec* .

Definition 2.5
A **deduction** $\mathcal{D}$ in an axiomatic system $\mathcal{AS}$ is a finite sequence of formulae $\mathtt{A}_1, \ldots, \mathtt{A}_n$ s.t. each $\mathtt{A}_i$ is either an instance of an axiom schema from $\mathcal{AS}$ or it has been obtained from a sequence of formulae $\mathtt{B}_1, \ldots, \mathtt{B}_k$ occurring before $\mathtt{A}$ in $\mathcal{D}$ via application of an inference rule from $\mathcal{AS}$. A formula $\mathtt{A}$ is a **theorem** in $\mathcal{AS}$, denoted by $\vdash_{\mathcal{AS}} \mathtt{A}$, if there is a deduction $\mathcal{D} = \mathcal{D}_1, \ldots, \mathcal{D}_k$ with $\mathcal{D}_k = \mathtt{A}$.

Definition 2.6
$\mathsf{LTK}_{\mathsf{ax}} := \{\mathtt{A} \in Fma(\mathcal{L}^{\mathsf{LTK}}) \mid \ \vdash_{\mathcal{AS}_{\mathsf{LTK}}} \mathtt{A}\}$.
$\mathsf{LTK}^{-}_{\mathsf{ax}} := \mathsf{LTK}_{\mathsf{ax}} \cap Fma(\mathcal{L}^{-}_{\mathsf{LTK}})$.

Our axioms for the time modality $\Box_{\preccurlyeq}$ give rise to an $\mathsf{S}4.3$ modal system, known to be sound and complete with respect to the class of linear orders. Formerly we stated that each agent operating in the system is provided with a certain knowledge background. In order to give a simple account of it, we associate each agent to an $S5$-modal system. The assumptions we make are the usual ones:

- If agent i knows $\mathtt{A}$, then the fact $\mathtt{A}$ is provided by all the resources she/he has access to;
- Positive Introspection: if someone knows something, she/he is also aware of it;
- Negative Introspection: if someone ignores something, she/he is aware of it.

Moreover, we assume each agent to be logicallly omniscient (knowing both all the tautologies and all the consequences implicit in her/his knowledge).

The same assumptions appear to be more natural when it comes to model the behaviour of the environment. In our system, the axioms involving the environment modality play a central role. In the interaction between different modalities, the operator K_{e} works like a bridge connecting the others, which otherwise would not interact at all. The axioms M.1 and M.2 state that if something is always true toward the future, then it is also true at the current moment/environment (M.1) and hence each agent knows it (M.2).

More specifically, Axiom M.1 aims at achieving property PM.1, whereas Axiom M.2 is responsable for PM.2. Axiom $Dum_{\Box_{\preccurlyeq}}$ entails the property PM.3 in a less evident and straightforward way. However, this is probably the most interesting one, for it is the one regulating the peculiar relation linking $\Box_{\preccurlyeq}$ to K_{e}. Indeed, as it is made clear by Lemma 2.7, axiom $Dum_{\Box_{\preccurlyeq}}$ achieves two things:

(a) making temporal and environmental clusters coincide;
(b) ensuring a discrete order of temporal clusters.

Theorem 2.7 (Soundness)
$\forall \mathtt{A} \in Fma(\mathcal{L}^{\mathsf{LTK}}) \ \ (\mathtt{A} \in \mathsf{LTK}_{\mathsf{ax}} \Rightarrow \mathtt{A} \in \mathsf{LTK})$

Proof. (by induction on the length of the deduction $\mathcal{D} = \mathcal{D}_1, \ldots, \mathcal{D}_j$ of a theorem $\mathtt{A} \in \mathsf{LTK}_{\mathsf{ax}}$). Suppose $j=1$, then $\mathtt{A}$ is an axiom from $\mathcal{AS}_{\mathsf{LTK}}$. We provide a proof only for the axioms M.1, M.2 and $Dum_{\Box_{\preccurlyeq}}$. (a) Suppose there are an $\mathcal{LTK}$-frame $\mathcal{F}$, a valuation V for $\mathcal{F}$, and a world $v \in W_{\mathcal{F}}$ such that $(\mathcal{F}, v) \nVdash_V \Box_{\preccurlyeq}\mathtt{A} \to \mathsf{K}_{\mathsf{e}}\mathtt{A}$. Then $(\mathcal{F}, v) \Vdash_V \Box_{\preccurlyeq}\mathtt{A}$ and $(\mathcal{F}, v) \nVdash_V \mathsf{K}_{\mathsf{e}}\mathtt{A}$. Hence for each world $z \in \{t \mid v\mathsf{R}_{\preccurlyeq}t\}$, $z \Vdash_V \mathtt{A}$ but there is a world $u \in \{t \mid v\mathsf{R}_{\mathsf{e}}t\}$ such that $u \nVdash \mathtt{A}$. Since by definition $\{t \mid v\mathsf{R}_{\preccurlyeq}t\} \supseteq \{t \mid v\mathsf{R}_{\mathsf{e}}t\}$, this leads to a contradiction. Using a similar argument, it can be easily seen that Axiom M.2 is valid too.

(b) Suppose that Axiom $Dum_{\Box_\preccurlyeq}$ is not valid. Then there are an $\mathcal{LTK}$-frame $\mathcal{F}$, a valuation V for $\mathcal{F}$, and a world $v \in W_\mathcal{F}$ such that $(\mathcal{F}, v) \nVdash_V \Box_\preccurlyeq(\Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \to \mathsf{K_e}\mathtt{A}) \to (\Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A} \to \Box_\preccurlyeq\mathtt{A})$, and hence:

$$(\mathcal{F}, v) \Vdash_V \Box_\preccurlyeq(\Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \to \mathsf{K_e}\mathtt{A}) \tag{2.1}$$

and

$$(\mathcal{F}, v) \nVdash_V (\Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \tag{2.2}$$

Condition (2.1) implies that $\forall z \in W_\mathcal{F}$ $(v\mathsf{R}_\preccurlyeq z \Rightarrow (\mathcal{F}, z) \Vdash_V \Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A}) \to \mathsf{K_e}\mathtt{A})$. This means that for each $\mathsf{R}_\preccurlyeq$-successor z of v, at least one of the following conditions should hold:

(2.1.1).: $(\mathcal{F}, z) \nVdash_V \Box_\preccurlyeq(\mathsf{K_e}\mathtt{A} \to \Box_\preccurlyeq\mathtt{A})$, then there is a world t such that $z\mathsf{R}_\preccurlyeq t$ & $(\mathcal{F}, t) \Vdash_V \mathsf{K_e}\mathtt{A}$ & $(\mathcal{F}, t) \nVdash_V \Box_\preccurlyeq\mathtt{A}$;

(2.1.2).: $(\mathcal{F}, z) \Vdash_V \mathsf{K_e}\mathtt{A}$.

Let us analise condition (2.2): if $(\mathcal{F}, v) \nVdash_V (\Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A} \to \Box_\preccurlyeq\mathtt{A})$, we have that both of the following conditions (2.2.1) and (2.2.2) must hold:

(2.2.1): $(\mathcal{F}, v) \Vdash_V \Diamond_\preccurlyeq\Box_\preccurlyeq\mathtt{A}$;

(2.2.2): $(\mathcal{F}, v) \nVdash_V \Box_\preccurlyeq\mathtt{A}$;

From condition (2.2.1) and (2.2.2) it follows that there is a world $\mathsf{R}_\preccurlyeq$-accessible from v in which $\mathtt{A}$ is not true, while there is another point $\mathsf{R}_\preccurlyeq$-accessible from v starting from which $\mathtt{A}$ holds true everywhere toward the future. Since each $\mathcal{LTK}$-frame is a linear and discrete order with respect to $\mathsf{R}_\preccurlyeq$-clusters, there is a world v_2 such that $v\mathsf{R}_\preccurlyeq v_2$ and $(\mathcal{F}, v_2) \nVdash_V \mathtt{A}$ and for each world z_2 such that $v_2\mathsf{R}_\preccurlyeq z_2$ & $\neg(z_2\mathsf{R}_\preccurlyeq v_2)$, $(\mathcal{F}, z_2) \Vdash \mathtt{A}$, and hence $(\mathcal{F}, z_2) \Vdash \Box_\preccurlyeq\mathtt{A}$. Trivially, condition (2.1.2) does not hold at v_2, for $\mathsf{R_e}$ is reflexive.

Then condition (2.1.1) should hold. This implies that there is a world t_2 such that $v_2\,\mathsf{R}_\preccurlyeq\, t_2$, $(\mathcal{F}, t_2) \Vdash_V \mathsf{K_e}\mathtt{A}$ and $(\mathcal{F}, t_2) \nVdash_V \Box_\preccurlyeq\mathtt{A}$. Hence $t_2\,\mathsf{R}_\preccurlyeq\, v_2$ (for, by the way we chose v_2, if $v_2\,\mathsf{R}_\preccurlyeq\, t_2$ and $\neg(t_2\mathsf{R}_\preccurlyeq v_2)$, we should have $(\mathcal{F}, t_2) \Vdash_V \Box_\preccurlyeq\mathtt{A}$). Moreover $\neg(t_2\mathsf{R_e}v_2)$. However, this is in contradiction with Definition 2.3, for in $\mathcal{LTK}$-frames $\mathsf{R}_\preccurlyeq$-clusters and $\mathsf{R_e}$-clusters should coincide.

Finally, if $lg(\mathcal{D}) = n + 1$, it could be easily shown that each inference rule preserves validity. ■

3 Canonical models and Generated subframes

We briefly recall few standard definitions and results concerning canonical models and generated submodels.

Definition 3.1
Given an axiomatic system $\mathcal{AS}$ on a language $\mathcal{L}$, a set $\Delta \subset Fma(\mathcal{L})$ is:

(a) $\mathcal{AS}$-consistent iff $\Delta \nvdash_{\mathcal{AS}} \bot$;
(b) $\mathcal{L}$-complete iff $\forall \mathtt{A} \in Fma(\mathcal{L})$ $\mathtt{A} \in \Delta$ or $\neg\mathtt{A} \in \Delta$;
(c) $\mathcal{AS}$-maximal iff Δ is $\mathcal{AS}$-consistent and $\mathcal{L}$-complete.

Definition 3.2
Let L be a consistent normal k-modal logic on a language $\mathcal{L}$ containing the modal operators $\Box_1, \ldots, \Box_k$. An n-**canonical model** $\mathcal{M}_n^\mathsf{c} = \langle W_n^\mathsf{c}, \mathsf{R}_1^\mathsf{c}, \ldots, \mathsf{R}_k^\mathsf{c}, V_n^\mathsf{c}\rangle$ for L is such that:

(a) W_n^c is the set of all the possible L-maximal sets w.r.t. those formulae built up from the propositional letters $p_1, \ldots, p_n$;

(b) $\forall v, z \in W_n^{\mathsf{c}}, \quad v\mathsf{R}_i^{\mathsf{c}} z \iff \{\mathtt{A} \mid \Box_i \mathtt{A} \in v\} \subseteq z,\ 1 \leq i \leq k;$
(c) $V_n^{\mathsf{c}}(p_i) = \{v \in W_n^{\mathsf{c}} \mid p_i \in v\}, \quad 1 \leq i \leq n.$

Lemma 3.3
Let L be a consistent normal k-modal logic and let $\mathcal{M}_n^{\mathsf{c}} = \langle W_n^{\mathsf{c}}, \mathsf{R}_1^{\mathsf{c}}, \ldots, \mathsf{R}_k^{\mathsf{c}}, V_n^{\mathsf{c}} \rangle$ be an n-canonical model for L. Then $\forall v \in W_n^{\mathsf{c}}, \ \forall \mathtt{A}(p_1, \ldots, p_n) \in Fma(\mathcal{L})(\Box_i \mathtt{A} \in v \iff \forall z \in W_n^{\mathsf{c}}(v\mathsf{R}_i^{\mathsf{c}} z \Rightarrow \mathtt{A} \in z))$.

Lemma 3.4 (Truth Lemma)
Let L be a consistent normal k-modal logic and let $\mathcal{M}_n^{\mathsf{c}} = \langle W_n^{\mathsf{c}}, \mathsf{R}_1^{\mathsf{c}}, \ldots, \mathsf{R}_k^{\mathsf{c}}, V_n^{\mathsf{c}} \rangle$ be an n-canonical model for L. Then $\forall v \in W_n^{\mathsf{c}}, \ \forall \mathtt{A}(p_1, \ldots, p_n) \in Fma(\mathcal{L})$

$$(\mathcal{F}_n^{\mathsf{c}}, v) \Vdash_{V_n^{\mathsf{c}}} \mathtt{A} \iff \mathtt{A} \in v$$

where $\mathcal{F}_n^{\mathsf{c}}$ denotes the n-canonical frame on which $\mathcal{M}_n^{\mathsf{c}}$ is built.

Take and fix for the rest of the paper a formula $\mathtt{B}(p_1, \ldots, p_n) \notin \mathsf{LTK}_{\mathsf{ax}}$. Hence the set $\{\neg \mathtt{B}\}$ is $\mathcal{AS}_{\mathsf{LTK}}$-consistent and it follows that there exists an $\mathcal{AS}_{\mathsf{LTK}}$-maximal set w w.r.t. all the formulae built up from $p_1, \ldots, p_n$ such that $\mathtt{B} \notin w$. Therefore there is an n-canonical model for $\mathcal{M}^{\mathsf{c}} = \langle \mathcal{F}_n^{\mathsf{c}}, V_n^{\mathsf{c}} \rangle$ for $\mathsf{LTK}_{\mathsf{ax}}$ (where $\mathcal{F}_n^{\mathsf{c}} = \langle W_n^{\mathsf{c}}, \mathsf{R}_{\preccurlyeq}^{\mathsf{c}}, \mathsf{R}_{\mathsf{e}}^{\mathsf{c}}, \mathsf{R}_1^{\mathsf{c}}, \ldots, \mathsf{R}_k^{\mathsf{c}} \rangle$) such that $w \in W_n^{\mathsf{c}}$ and, by Lemma 3.4, $(\mathcal{F}_n^{\mathsf{c}}, w) \nVdash_{V_n^{\mathsf{c}}} \mathtt{B}$. Although this model shows some interesting properties, it is not built on an $\mathcal{LTK}$-frame.

However, the binary relations in the n-canonical frame have the following properties:

(a) $\mathsf{R}_{\mathsf{e}}^{\mathsf{c}}, \ \mathsf{R}_i^{\mathsf{c}}$ are reflexive, symmetric and transitive.
(b) $\mathsf{R}_{\preccurlyeq}^{\mathsf{c}}$ is reflexive, transitive and weakly connected.
(c) $\forall v, z \in W_n^{\mathsf{c}} \ \ (v\mathsf{R}_{\mathsf{e}}^{\mathsf{c}} z \Rightarrow (v\mathsf{R}_{\preccurlyeq}^{\mathsf{c}} z \ \& \ z\mathsf{R}_{\preccurlyeq}^{\mathsf{c}} v))$. Notice that the opposite direction does not hold.

Definition 3.5
An n-modal K-frame $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_1, \ldots, \mathsf{R}_n \rangle$ is a subframe of an m-modal K-frame $\mathcal{S} = \langle W_{\mathcal{S}}, \mathsf{S}_1, \ldots, \mathsf{S}_m \rangle$ if $n = m$, $W_{\mathcal{F}} \subseteq W_{\mathcal{S}}$ and each R_i is the restriction of S_i on $W_{\mathcal{F}}$, i.e. $\mathsf{R}_i = \mathsf{S}_i \upharpoonright W_{\mathcal{F}}$.

Definition 3.6
An n-modal K-frame $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_1, \ldots, \mathsf{R}_n \rangle$ is a *generated* subframe of an m-modal K-frame $\mathcal{S} = \langle W_{\mathcal{S}}, \mathsf{S}_1, \ldots, \mathsf{S}_m \rangle$ (notation $\mathcal{F} \sqsubseteq \mathcal{S}$) if $\mathcal{F}$ is a subframe of $\mathcal{S}$ and $\forall v \in W_{\mathcal{F}} \ \ \forall z \in W_{\mathcal{S}}$ if there is a relation S_j such that $v\mathsf{S}_j z$ in $\mathcal{S}$, then $z \in W_{\mathcal{F}}$. A model $\langle \mathcal{F}, V \rangle$ is a generated submodel of $\langle \mathcal{S}, S \rangle$ if $\mathcal{F} \sqsubseteq \mathcal{S}$ and V is the restriction of S on $W_{\mathcal{F}}$ (i.e. $V = S \upharpoonright W_{\mathcal{F}}$).

Lemma 3.7 (Generated subframes)
If $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is a generated submodel of $\mathcal{M}_2 = \langle \mathcal{F}_2, V_2 \rangle$, then $\forall v \in W_{\mathcal{F}}$, $(\mathcal{F}, v) \Vdash_V \mathtt{A} \iff (\mathcal{F}_2, v) \Vdash_{V_2} \mathtt{A}$.

Definition 3.8
Given a Kripke-frame $\mathcal{F} = \langle W_{\mathcal{F}}, \mathsf{R}_1, \ldots, \mathsf{R}_k \rangle$ and a world w in $W_{\mathcal{F}}$, $w^{\mathsf{R}_i \leq} := \{z \mid w\mathsf{R}_i z\}$ and $w^{\mathsf{R}_i <} := \{z \mid w\mathsf{R}_i z \ \& \ \neg(z\mathsf{R}_i w)\}$. Given a R_i-cluster $\mathcal{C}$, $\mathcal{C}^{\mathsf{R}_i \leq} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{R}_i \mathcal{C}_j\}$ and $\mathcal{C}^{\mathsf{R}_i <} := \{\mathcal{C}_j \mid \mathcal{C}\mathsf{R}_i \mathcal{C}_j \ \& \ \neg(\mathcal{C}_j \mathsf{R}_i \mathcal{C})\}$ ($w^{\mathsf{R}_{\preccurlyeq} \leq}$ and $\mathcal{C}^{\mathsf{R}_{\preccurlyeq} \leq}$ shall henceforth be referred to as $w^{\preccurlyeq}$ and $\mathcal{C}^{\preccurlyeq}$ respectively).

Consider the generated submodel of $\langle \mathcal{F}_n^{\mathsf{c}}, V_n^{\mathsf{c}} \rangle$ generated by $w^{\preccurlyeq}$ (recall that w is that world refuting $\mathtt{B}$ in the n-canonical model) and denote it by $\langle \mathcal{F}_{w^{\preccurlyeq}}, V_n^{\mathsf{c}} \rangle$. Hence $(\mathcal{F}_{w^{\preccurlyeq}}, w) \not\Vdash_{V_n^{\mathsf{c}}} \mathtt{B}$ which entails $\mathcal{F}_{w^{\preccurlyeq}} \not\Vdash \mathtt{B}$.

As the following lemma will clarify, the generated submodel $\langle \mathcal{F}_{w^{\preccurlyeq}}, V_n^{\mathsf{c}} \rangle$ shows some interesting properties shared with $\mathcal{LTK}$-frames:

Lemma 3.9
$\mathcal{F}_{w^{\preccurlyeq}}$ has the following properties:

(a) The relations $\mathsf{R}_{\mathsf{e}}^{\mathsf{c}}, \mathsf{R}_1^{\mathsf{c}}, \dots, \mathsf{R}_k^{\mathsf{c}}$ are equivalence relations;
(b) The relation $\mathsf{R}_{\preccurlyeq}^{\mathsf{c}}$ is reflexive, transitive and connected;
(c) $\forall v, z \in W_F \ v\mathsf{R}_{\mathsf{e}}^{\mathsf{c}} z \Rightarrow (v\mathsf{R}_{\preccurlyeq}^{\mathsf{c}} z \ \& \ z\mathsf{R}_{\preccurlyeq}^{\mathsf{c}} v)$;
(d) $\forall v, z \in W_F \ v\mathsf{R}_i^{\mathsf{c}} z \Rightarrow v\mathsf{R}_{\mathsf{e}}^{\mathsf{c}} z$;

Proof. (c) Suppose $\neg(v\mathsf{R}_{\preccurlyeq}^{\mathsf{c}} z)$. Then there is a formula $\Box_{\preccurlyeq}\mathtt{D} \in v$ s.t. $\mathtt{D} \notin z$. By Axiom M.1 it follows $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in v$ and hence $\neg(v\mathsf{R}_{\mathsf{e}}^{\mathsf{c}} z)$. The same case arises if we assume $\neg(z\mathsf{R}_{\preccurlyeq}^{\mathsf{c}} v)$. (d) Suppose $\neg(v\mathsf{R}_{\mathsf{e}}^{\mathsf{c}} z)$. Then there is a formula $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in v$ s.t. $\mathtt{D} \notin z$. By Axiom M.2 it follows $\mathsf{K}_i\mathtt{D} \in v$ for each i and hence $\neg(v\mathsf{R}_i^{\mathsf{c}} z)$ ■

A good way to achieve the required property of discreteness (i.e. given any two disctint worlds in a model, there could be only a finite amount of *moments* between them) is to make a filtration of the base set of the model in order to have it finite. Although it is a standard technique, it requires a careful and appropriate selection of the *filtration set*. The well known results concerning this method are recalled below and they would allow us to show the following:

Lemma 3.10 (Filtration Lemma)
$\forall \mathtt{D} \in \Gamma \ \forall v \in W_n^{\mathsf{c}} \ (v \Vdash_{V_n^{\mathsf{c}}} \mathtt{D} \Leftrightarrow [v] \Vdash_{V^\Gamma} \mathtt{D})$

We start by defining the filtration set Γ as the union of several sets:

- $\Gamma_0 := \mathit{Sub}(\mathtt{B})$, where $\mathit{Sub}(\mathtt{B})$ is the set of all the *subformulae* of $\mathtt{B}$
- $\Gamma_1 := \mathit{Sub}\{\mathit{Dum}_{\Box_{\preccurlyeq}}(\mathtt{D}) \mid \mathtt{D} \in \Gamma_0\}$ (where the notation $\mathit{Dum}_{\Box_{\preccurlyeq}}(\mathtt{D})$ is intended to denote the instance of the axiom $\mathit{Dum}_{\Box_{\preccurlyeq}}$ by the formula $\mathtt{D}$)
- $\Gamma_2 := \{\mathsf{K}_{\mathsf{e}}\Box_{\preccurlyeq}\mathtt{D} \mid \Box_{\preccurlyeq}\mathtt{D} \in \Gamma_0 \cup \Gamma_1\}$
- $\Gamma_3 := \{\mathsf{K}_i\mathsf{K}_{\mathsf{e}}\mathtt{D} \mid \mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2\}$ for $1 \le i \le k$

Let the filtration set Γ be the union of the formerly defined sets:

$$\Gamma := \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

Then we define a new relation between worlds as:

$$\forall v, z \in W_n^{\mathsf{c}} \ \ (v \sim z \Leftrightarrow \forall \mathtt{D} \in \Gamma(v \Vdash_{V_n^{\mathsf{c}}} \mathtt{D} \Leftrightarrow z \Vdash_{V_n^{\mathsf{c}}} \mathtt{D}))$$

and we generate equivalence classes with respect to the relation $\sim$:

$$[v] := \{z \mid v \sim z\}$$

The Γ-filtered model $\mathcal{M}^\Gamma$ is defined as $\langle \mathcal{F}^\Gamma, V^\Gamma \rangle$ where $\mathcal{F}^\Gamma = \langle W^\Gamma, \mathsf{R}_{\preccurlyeq}^\Gamma, \mathsf{R}_{\mathsf{e}}^\Gamma, \mathsf{R}_1^\Gamma, \dots, \mathsf{R}_k^\Gamma \rangle$ and:

(a) $W^\Gamma := \{[v] \mid v \in W_n^{\mathsf{c}}\}$
(b) $\mathsf{R}_{\mathsf{e}}^\Gamma$ and each R_i^Γ are standard $S5$ filtration relations, i.e.:
$[v]\mathsf{R}_\xi^\Gamma[z] \Leftrightarrow \forall \mathsf{K}_\xi\mathtt{D} \in \Gamma((\mathcal{F}_n^{\mathsf{c}}, v) \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_\xi\mathtt{D} \Leftrightarrow (\mathcal{F}_n^{\mathsf{c}}, z) \Vdash_{V_n^{\mathsf{c}}} \mathsf{K}_\xi\mathtt{D})$ for $\xi \in \{\mathsf{e}, 1, \dots, k\}$

(c) $\mathsf{R}^{\Gamma}_{\preccurlyeq}$ is a standard $\mathsf{S}4.3$ filtration relation, namely:
$[v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z] \Leftrightarrow \forall\Box_{\preccurlyeq}\mathtt{D} \in \Gamma((\mathcal{F}^{\mathsf{c}}_{n}, v) \Vdash_{V^{\mathsf{c}}_{n}} \Box_{\preccurlyeq}\mathtt{D} \Rightarrow (\mathcal{F}^{\mathsf{c}}_{n}, z) \Vdash_{V^{\mathsf{c}}_{n}} \Box_{\preccurlyeq}\mathtt{D})$
(d) $\forall p_i \in \{p_1, \ldots, p_n\}\ V^{\Gamma}(p_i) := \{[v] \mid v \in V^{\mathsf{c}}_{n}(p_i)\}$

It remains only to show that such model satisfies the two filtration conditions for each binary relation $\mathsf{R}^{\Gamma}_{\xi}$:

F1 $v\mathsf{R}^{\mathsf{c}}_{\xi}z \Rightarrow [v]\mathsf{R}^{\Gamma}_{\xi}[z]$, where $\xi \in \{\preccurlyeq, \mathsf{e}, 1\ldots k\}$;
F2.1 $[v]\mathsf{R}^{\Gamma}_{\xi}[z] \Rightarrow \forall\mathsf{K}_{\xi}\mathtt{D} \in \Gamma(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\xi}\mathtt{D} \Rightarrow z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D})$, for $\xi \in \{\mathsf{e}, 1\ldots k\}$;
F2.2 $[v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z] \Rightarrow \forall\Box_{\preccurlyeq}\mathtt{D} \in \Gamma(v \Vdash_{V^{\mathsf{c}}_{n}} \Box_{\preccurlyeq}\mathtt{D} \Rightarrow z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D})$

Lemma 3.11
The following properties hold true in the model $\mathcal{M}^{\Gamma}$:

(a) The relation $\mathsf{R}^{\Gamma}_{\mathsf{e}}$ satisfies $F1$ and $F2.1$.
(b) Each relation R^{Γ}_{i} satisfies $F1$ and $F2.1$.
(c) The relation $\mathsf{R}^{\Gamma}_{\preccurlyeq}$ satisfies $F1$ and $F2.2$.

Proof. (a) F1. Suppose there are two worlds v and z such that $v\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}z$. Then $v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \Rightarrow z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D}$. Since $v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \to \mathsf{K}_{\mathsf{e}}\mathsf{K}_{\mathsf{e}}\mathtt{D}$ we have $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \Rightarrow (v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathsf{K}_{\mathsf{e}}\mathtt{D}) \Rightarrow (z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Suppose $(z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Then $(z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathsf{K}_{\mathsf{e}}\mathtt{D})$. Since $\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}$ is symmetric by Lemma 3.9, we have $z\mathsf{R}^{\mathsf{c}}_{\mathsf{e}}v$ and hence $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Therefore, by our definition of $\mathsf{R}^{\Gamma}_{\mathsf{e}}$, it follows $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$.

F2.1. Suppose $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z]$. Then $\forall\mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma\ (v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \Leftrightarrow z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Since $z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \to \mathtt{D}$ we have $z \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D}$. ■

The proof of cases (b) and (c) is similar.

Hence by the standard results concerning filtrations, we can state the following:

Lemma 3.12
$\forall\mathtt{D} \in \Gamma\ \forall v \in W^{\mathsf{c}}_{n}\ ((\mathcal{F}^{\mathsf{c}}_{n}, v) \Vdash_{V^{\mathsf{c}}_{n}} \mathtt{D} \Leftrightarrow (\mathcal{F}^{\Gamma}, [v]) \Vdash_{V^{\Gamma}} \mathtt{D})$.

Corollary 3.13
$(\mathcal{F}^{\Gamma}, [w]) \nVdash_{V^{\Gamma}} \mathtt{B}$.

Once more again, we can show that our current Γ-filtered model $\mathcal{M}^{\Gamma}$ is conservative with respect to the properties stated by Lemma 3.

Lemma 3.14
In the model $\mathcal{M}^{\Gamma}$ the following holds:

(a) $\mathsf{R}^{\Gamma}_{\mathsf{e}}$ and each R^{Γ}_{i} are reflexive, symmetric and transitive.
(b) $\mathsf{R}^{\Gamma}_{\preccurlyeq}$ is reflexive, transitive and connected.
(c) $\forall[v],[z] \in W^{\Gamma}\ \ ([v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z] \Rightarrow ([v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z]\ \&\ [z]\mathsf{R}^{\Gamma}_{\preccurlyeq}[v]))$.
(d) $\forall[v],[z] \in W^{\Gamma}\ \ ([v]\mathsf{R}^{\Gamma}_{i}[z] \Rightarrow [v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z])$.

Proof. (a) Trivially $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \Leftrightarrow (v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Hence $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[v]$ and $\mathsf{R}^{\Gamma}_{\mathsf{e}}$ is reflexive.

Suppose $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z]$ and $[z]\mathsf{R}^{\Gamma}_{\mathsf{e}}[u]$. Hence $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \Leftrightarrow (z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$ and $(z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \Leftrightarrow (u \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$, which entails $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \Leftrightarrow (u \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$. Hence $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[u]$.

Suppose $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z]$. Then $(v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \Leftrightarrow (z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$ and hence $(z \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D}) \Leftrightarrow (v \Vdash_{V^{\mathsf{c}}_{n}} \mathsf{K}_{\mathsf{e}}\mathtt{D})$ which means $[z]\mathsf{R}^{\Gamma}_{\mathsf{e}}[v]$.

(b) For the properties of reflexivity and transitivity see the previous case. Since by F1 ($v\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}z$) implies ($[v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z]$) and $\mathsf{R}^{\mathsf{c}}_{\preccurlyeq}$ is connected (see Lemma 3.14), it follows that $\mathsf{R}^{\Gamma}_{\preccurlyeq}$ is connected.

(c) Suppose $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z]$ and either $\neg([v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z])$ or $\neg([z]\mathsf{R}^{\Gamma}_{\preccurlyeq}[v])$. If $\neg([v]\mathsf{R}^{\Gamma}_{\preccurlyeq}[z])$, then there is a formula $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma$ such that $v \Vdash_{V^{\mathsf{c}}_n} \Box_{\preccurlyeq}\mathtt{D}$ and $z \nVdash_{V^{\mathsf{c}}_n} \Box_{\preccurlyeq}\mathtt{D}$. By Axiom $4_{\Box_{\preccurlyeq}}$ it follows $v \Vdash_{V^{\mathsf{c}}_n} \Box_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{D}$, hence by Axiom M.1 we have $v \Vdash_{V^{\mathsf{c}}_n} \mathsf{K}_{\mathsf{e}}\Box_{\preccurlyeq}\mathtt{D}$. But $\mathsf{K}_{\mathsf{e}}\Box_{\preccurlyeq}\mathtt{D} \in \Gamma$ by construction[1], therefore since $[v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z]$ we have $z \Vdash_{V^{\mathsf{c}}_n} \mathsf{K}_{\mathsf{e}}\Box_{\preccurlyeq}\mathtt{D}$ and, by reflexivity, $z \Vdash_{V^{\mathsf{c}}_n} \Box_{\preccurlyeq}\mathtt{D}$, which is a contradiction. We reach a similar contradiction if we assume $\neg([z]\mathsf{R}^{\Gamma}_{\preccurlyeq}[v])$.

(d) Suppose $[v]\mathsf{R}^{\Gamma}_{i}[z]$ for some i and $\neg([v]\mathsf{R}^{\Gamma}_{\mathsf{e}}[z])$. Then there is a formula $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma$ s.t. $v \Vdash_{V^{\mathsf{c}}_n} \mathsf{K}_{\mathsf{e}}\mathtt{D}$ and $z \nVdash_{V^{\mathsf{c}}_n} \mathsf{K}_{\mathsf{e}}\mathtt{D}$. Again, by the axioms $4_{\mathsf{K}_{\mathsf{e}}}$ and M.2 we obtain $v \Vdash_{V^{\mathsf{c}}_n} \mathsf{K}_i\mathsf{K}_{\mathsf{e}}\mathtt{D}$. The formula $\mathsf{K}_i\mathsf{K}_{\mathsf{e}}\mathtt{D}$ belongs to Γ by construction[2], therefore, given $[v]\mathsf{R}^{\Gamma}_{i}[z]$, we have $z \Vdash_{V^{\mathsf{c}}_n} \mathsf{K}_i\mathsf{K}_{\mathsf{e}}\mathtt{D}$ and, by reflexivity, $z \Vdash_{V^{\mathsf{c}}_n} \mathsf{K}_{\mathsf{e}}\mathtt{D}$, which gives rise to a contradiction. ■

Properties of filtered relations. From Lemma 3.14 follows that the new binary relations possess certain properties, namely all the knowledge modalities are reflexive, symmetric and transitive, whereas the time relation is connected as well as reflexive and transitive.

Both properties PM.1 and PM.2 hold true:

$$\forall v, z \in W^{\Gamma} \ \ (v\mathsf{R}^{\Gamma}_{\mathsf{e}}z \ \Rightarrow \ (v\mathsf{R}^{\Gamma}_{\preccurlyeq}z \ \& \ z\mathsf{R}^{\Gamma}_{\preccurlyeq}v))$$

which means that two information points (worlds) are simultaneous whenever they are from the same environment. But this is something we were able to state even in the previous stages of our construction. The main achievement is that now we have a very important property. In fact, since the base set of the filtered frame is finite, trivially the time relation $\mathsf{R}^{\Gamma}_{\preccurlyeq}$ gives rise to a discrete linear order of temporal ($\mathsf{R}^{\Gamma}_{\preccurlyeq}$-) clusters, which in this context means that given any two distinct worlds, there is only a finite number of moments between them.

However we do not have the property PM.3 yet:

$$\forall v, z \in W^{\Gamma} \ \ ((v\mathsf{R}^{\Gamma}_{\preccurlyeq}z \ \& \ z\mathsf{R}^{\Gamma}_{\preccurlyeq}v) \ \Rightarrow \ v\mathsf{R}^{\Gamma}_{\mathsf{e}}z)$$

In other words in this frame we may have $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper clusters of $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters, which in our intended interpretation means that it can be the case that two points, though at the same moment of the flow of time could belong to different environments (see Figure 2). Unfortunately, this is not the case for $\mathcal{LTK}$-frames, therefore another transformation seems to be necessary to prove our axiomatic system to be complete w.r.t. this kind of structures. To achieve this goal, we will construct another frame. The idea is to unravel each $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster, without using the standard technique of *bulldozing*, which would give rise to an infinite, but non discrete (with respect to tempooral clusters) frame. In other words we will define a well ordering on $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters inside each $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster, in order to construct a new frame. Such frame will be obtained by substituting each $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster with the

---

[1]Indeed if $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma$, then there are only two possibilities: either $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma_0$ or $\Box_{\preccurlyeq}\mathtt{D} \in \Gamma_1$ and in both cases $\mathsf{K}_{\mathsf{e}}\Box_{\preccurlyeq}\mathtt{D} \in \Gamma_2$ and hence it belongs to Γ as well.

[2]IIn fact if $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma$, then either $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma_0$ or $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma_1$ or else $\mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma_2$; hence $\mathsf{K}_i\mathsf{K}_{\mathsf{e}}\mathtt{D} \in \Gamma_3$ and it belongs to Γ as well.

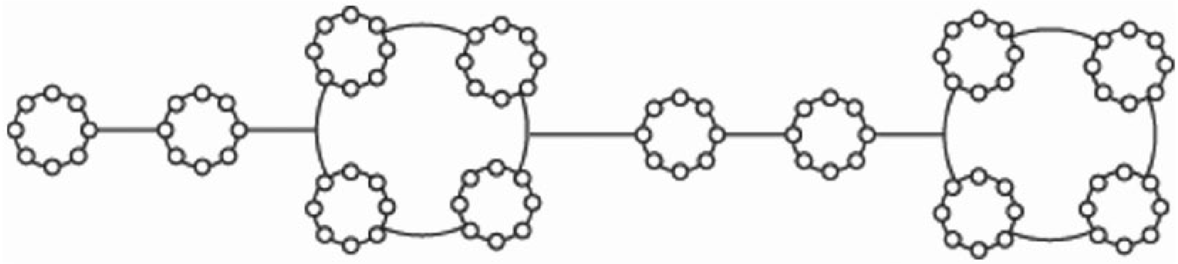

FIG. 2. Scheme of the structure of the frame $\mathcal{F}^{\Gamma}$.

finite ordered line formerly defined. The only troubling case is for formulae of the form $\Box_{\preccurlyeq}\mathtt{D}$ from $Sub(\mathtt{B})$. There could be, for example, a world v in an $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster such that it falsifies $\Box_{\preccurlyeq}\mathtt{D}$ and another world, say z which is the only point $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-accessible from v falsifying $\mathtt{D}$. If such world z belonged to the same $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster as v and it were not $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-accessible from v in the new unravelled frame, then we would have a lack of truth values for formulae from $Sub(\mathtt{B})$. However this situation is made impossible by the subsequent lemma, stating that whenever a world v in an $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-proper cluster falsifies $\Box_{\preccurlyeq}\mathtt{D}$, $\mathtt{D}$ is falsified either in the same environment-cluster to which v belongs, or in another world z which is *strictly above* with respect to v, i.e. $v\mathsf{R}^{\Gamma}_{\preccurlyeq}z$ & $\neg(z\mathsf{R}^{\Gamma}_{\preccurlyeq}v)$.

LEMMA 3.15
$\forall\Box_{\preccurlyeq}\mathtt{D} \in Sub(\mathtt{B})\forall v \in W^{\Gamma}$ if $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{D}$ and v is not final, then there is a world $z \in W^{\Gamma}$ s.t. $v\mathsf{R}^{\Gamma}_{\preccurlyeq}z$, $z \nVdash_{V^{\Gamma}} \mathtt{D}$ and either $v\mathsf{R}^{\Gamma}_{\mathsf{e}}z$ or $\neg(z\mathsf{R}^{\Gamma}_{\preccurlyeq}v)$.

PROOF. Suppose there are a formula $\Box_{\preccurlyeq}\mathtt{D} \in Sub(\mathtt{B})$ and a non final world v such that $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{D}$. There are only two possible cases.

Case 1. $v \Vdash_{V^{\Gamma}} \Diamond_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{D}$. Hence, since the instance of Axiom $Dum_{\Box_{\preccurlyeq}}$ with respect to the formula $\mathtt{D}$ is true in the model $\mathcal{M}^{\Gamma}$ (for the way we defined Γ, and in particular Γ_1), we have $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}(\Box_{\preccurlyeq}(\mathsf{K}_{\mathsf{e}}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}) \to \mathsf{K}_{\mathsf{e}}\mathtt{D})$, and then $v \Vdash_{V^{\Gamma}} \Diamond_{\preccurlyeq}(\Box_{\preccurlyeq}(\mathsf{K}_{\mathsf{e}}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}) \wedge \Diamond_{\mathsf{e}}\neg\mathtt{D})$.

Therefore there exists a world z such that:

$$(v\mathsf{R}^{\Gamma}_{\preccurlyeq}z) \ \ \& \ \ (z \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}(\mathsf{K}_{\mathsf{e}}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}) \wedge \Diamond_{\mathsf{e}}\neg\mathtt{D}) \tag{3.1}$$

Let us suppose by contradiction that

(a) The formula $\mathtt{D}$ is true in each world from the $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-cluster of v, i.e. $\forall t \in W^{\Gamma}$ $(v\mathsf{R}^{\Gamma}_{\mathsf{e}}t \Rightarrow t \Vdash_{V^{\Gamma}} \mathtt{D})$.
(b) There is no world *strictly above* w.r.t. v in which $\mathtt{D}$ is false, i.e. $\forall t \in W^{\Gamma}$ $(v\mathsf{R}^{\Gamma}_{\preccurlyeq}t \ \& \ \neg(t\mathsf{R}^{\Gamma}_{\preccurlyeq}v) \Rightarrow t \Vdash_{V^{\Gamma}} \mathtt{D})$.

From (b) and (3.1) follows that $z\mathsf{R}^{\Gamma}_{\preccurlyeq}v$. By (3.1) we also have $z \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}(\mathsf{K}_{\mathsf{e}}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D})$, hence $v \Vdash_{V^{\Gamma}} \mathsf{K}_{\mathsf{e}}\mathtt{D} \to \Box_{\preccurlyeq}\mathtt{D}$. But from (a) follows that $v \Vdash_{V^{\Gamma}} \mathsf{K}_{\mathsf{e}}\mathtt{D}$, therefore $v \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{D}$, which is a contradiction.

Case 2. $v \nVdash_{V^{\Gamma}} \Diamond_{\preccurlyeq}\Box_{\preccurlyeq}\mathtt{D}$. Hence $v \Vdash_{V^{\Gamma}} \Box_{\preccurlyeq}\Diamond_{\preccurlyeq}\neg\mathtt{D}$. This implies that $\mathtt{D}$ is false at least in some world from the final $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster and such world is *strictly above* w.r.t. v, which is non final by assumption. ■

4 Completeness

Consider the frame $\mathcal{F}^{\Gamma}$.

(a) Fix a well ordering $\mathcal{C}_1, \dots, \mathcal{C}_f$ among $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-clusters such that $i < l$ if and only if $(\mathcal{C}_i \mathsf{R}^{\Gamma}_{\preccurlyeq} \mathcal{C}_m) \& \neg(\mathcal{C}_m \mathsf{R}^{\Gamma}_{\preccurlyeq} \mathcal{C}_i)$.
(b) Fix some well ordering among $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters inside any $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster, so that each $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-cluster would be taken once and only only once.

Hence each single world from the base set of $\mathcal{F}^{\Gamma}$ would be displayed as $\langle v_j, i \rangle$, meaning that v belongs to the j-th $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-cluster inside the i-th $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster.

Given that the number of $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-clusters is f, we stipulate that the index f denotes that a world $\langle v_j, \mathsf{f} \rangle$ belongs to the *final* $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-cluster $\mathcal{C}_f$.

(c) Define a new frame $\mathcal{S} = \langle W_{\mathcal{S}}, \mathsf{S}_{\preccurlyeq}, \mathsf{S}_{\mathsf{e}}, \mathsf{S}_1, \dots, \mathsf{S}_k \rangle$ in the following way:

- $W_{\mathcal{S}} = \bigcup_{v \in W^{\Gamma}} \langle v_j, i \rangle$;
- $\langle v_j, i \rangle \mathsf{S}_{\xi} \langle z_m, l \rangle \iff v \mathsf{R}^{\Gamma}_{\xi} z$, for $1 \leq \xi \leq k$;
- $\langle v_j, i \rangle \mathsf{S}_{\mathsf{e}} \langle z_m, l \rangle \iff i = l \;\&\; j = m$;
- $\langle v_j, i \rangle \mathsf{S}_{\preccurlyeq} \langle z_m, l \rangle \iff (j \leq m \;\&\; i = l)$ or $i < l$ or $l = \mathsf{f}$, i.e. z is $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final;

(d) Let $\mathcal{M}_{\mathcal{S}} = \langle \mathcal{S}, V^{\mathcal{S}} \rangle$ be a model such that $\forall p \in \{p_1, \dots, p_n\}$ $V^{\mathcal{S}}(p) = \{\langle v_j, i \rangle \mid v \in V^{\Gamma}(p)\}$. Then clearly the following holds:

Lemma 4.1
$\forall \mathtt{D} \in Sub(\mathtt{B}) \forall v \in W^{\Gamma} \quad ((\mathcal{F}^{\Gamma}, v) \Vdash_{V^{\Gamma}} \mathtt{D} \;\Leftrightarrow\; (\mathcal{S}, \langle v_j, i \rangle) \Vdash_{V^{\mathcal{S}}} \mathtt{D})$.

Proof. (a) (By induction on the length of $\mathtt{D}$). Trivially, if $lg(\mathtt{D}) = 1$, $\mathtt{D}$ has the form p, $v \in V^{\Gamma}(p)$ and hence $\langle v_j, i \rangle \in V^{\mathcal{S}}(p)$. Therefore $\langle v_j, i \rangle \Vdash_{V^{\mathcal{S}}} p$.

Suppose $\mathtt{D}$ has the form $\mathsf{K}_{\mathsf{e}}\mathtt{A}$. Then $v \Vdash_{V^{\Gamma}} \mathsf{K}_{\mathsf{e}}\mathtt{A}$ if and only if $\forall z \in W^{\Gamma}$ $(v \mathsf{R}^{\Gamma}_{\mathsf{e}} z \Rightarrow z \Vdash_{V^{\Gamma}} \mathtt{A})$. By inductive hypothesis (IH) $\langle z_m, l \rangle \Vdash_{V^{\mathcal{S}}} \mathtt{A}$. Since $v \mathsf{R}^{\Gamma}_{\mathsf{e}} z$ implies that both v and z belong to the same $\mathsf{R}^{\Gamma}_{\preccurlyeq}$- and $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters, it follows $i = l$ and $j = m$, and hence $\langle v_j, i \rangle \mathsf{S}_{\mathsf{e}} \langle z_m, l \rangle$, which means $\langle v_j, i \rangle \Vdash_{V^{\mathcal{S}}} \mathsf{K}_{\mathsf{e}}\mathtt{A}$.

Suppose $\langle v_j, i \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$. Then there is a world $\langle z_m, l \rangle$ such that $\langle v_j, i \rangle \mathsf{S}_{\preccurlyeq} \langle z_m, l \rangle$ and $\langle z_m, l \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$. This means that either $i < l$, or $(i = l \;\&\; j \leq m)$ or else $l = \mathsf{f}$. Each of these cases implies $v \mathsf{R}^{\Gamma}_{\preccurlyeq} z$. By IH we have $z \nVdash_{V^{\Gamma}} \mathtt{A}$ and therefore $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{A}$.

(b) Assume $v \nVdash_{V^{\Gamma}} \Box_{\preccurlyeq}\mathtt{A}$. Suppose v is not $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final. Then, by lemma 3.15, there is a world z such that $v \mathsf{R}^{\Gamma}_{\preccurlyeq} z$, $z \nVdash_{V^{\Gamma}} \mathtt{A}$ and either $v \mathsf{R}^{\Gamma}_{\mathsf{e}} z$ or $\neg(z \mathsf{R}^{\Gamma}_{\preccurlyeq} v)$. If $v \mathsf{R}^{\Gamma}_{\mathsf{e}} z$, it follows that both v and z have the same indices for the $\mathsf{R}^{\Gamma}_{\preccurlyeq}$- and $\mathsf{R}^{\Gamma}_{\mathsf{e}}$-clusters they belong to, i.e. they are displayed as $\langle v_i, i \rangle$ and $\langle z_i, i \rangle$. Hence $\langle v_i, i \rangle \mathsf{S}_{\preccurlyeq} \langle z_i, i \rangle$. By IH $\langle z_i, i \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$ and therefore $\langle v_i, i \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$. Else if $\neg(z \mathsf{R}^{\Gamma}_{\preccurlyeq} v)$, given that the worlds v and z are displayed as $\langle v_j, i \rangle$ and $\langle z_m, l \rangle$, it follows that $i < l$, and hence $\langle v_j, i \rangle \mathsf{S}_{\preccurlyeq} \langle z_m, l \rangle$. Again, by IH we have $\langle z_m, l \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$ and therefore $\langle v_j, i \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$. Finally suppose v is $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final. Then there is a world z which is $\mathsf{R}^{\Gamma}_{\preccurlyeq}$-final as well and it is such that $v \mathsf{R}^{\Gamma}_{\preccurlyeq} z$ and $z \nVdash_{V^{\Gamma}} \mathtt{A}$. Since z is displayed as $\langle z_m, \mathsf{f} \rangle$, it follows that $\langle v_j, \mathsf{f} \rangle \mathsf{S}_{\preccurlyeq} \langle z_m, \mathsf{f} \rangle$. By IH $\langle z_m, \mathsf{f} \rangle \nVdash_{V^{\mathcal{S}}} \mathtt{A}$ and therefore $\langle v_j, \mathsf{f} \rangle \nVdash_{V^{\mathcal{S}}} \Box_{\preccurlyeq}\mathtt{A}$.

Corollary 4.2
$\mathcal{S} \nVdash_{V^{\mathcal{S}}} \mathtt{B}$

The frame $\mathcal{S}$ has the structure depicted in Figure 3. If we regard environment-clusters (S_{e}-clusters) as single worlds, this frame has the structure of a *reflexive balloon*.

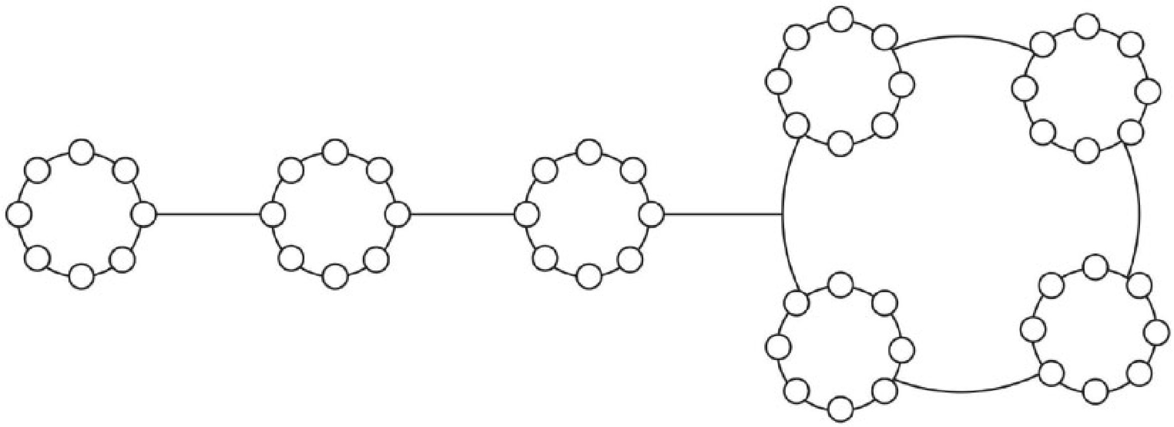

Fig. 3. Scheme of the structure of $\mathcal{S}$: a particular case of reflexive balloon.

Whenever the $\mathsf{S}_{\preccurlyeq}$-final cluster of our model is an $\mathsf{S}_{\preccurlyeq}$-proper cluster of $\mathsf{S_e}$-clusters (i.e. not *simple*), the frame we have obtained is not an $\mathcal{LTK}$-frame. We recall that these frames have no $\mathsf{S}_{\preccurlyeq}$-proper clusters of $\mathsf{S_e}$-clusters inside, and hence our final construction cannot be considered as a member of such class. However, this is not a problem, for it follows from [2] and [3] that these structures are nothing but *p-morphic* images of $\mathcal{LTK}$-frames.

Therefore we can state the following:

Theorem 4.3
$\forall \mathtt{B} \in Fma(\mathcal{L}^{\mathsf{LTK}})$ if $\mathtt{B} \notin \mathsf{LTK_{ax}}$ then there is an $\mathcal{LTK}$-frame $\mathcal{F}$ such that $\mathcal{F} \nVdash \mathtt{B}$.

Corollary 4.4. (Soundness and Completeness)

(a) $\mathsf{LTK_{ax}} = \mathsf{LTK}$
(b) $\mathsf{LTK^{-}_{ax}} = \mathsf{LTK}^{-}$

Corollary 4.5
$\mathsf{LTK^{1}_{ax}}$ (the version of $\mathsf{LTK_{ax}}$ with only one agent operating in the system) has the effective finite model property and it is decidable w.r.t. admissible inference rules.

Appendix

So far we have presented a semantic framework for reasoning about time and knowledge which can be useful whenever the flow of time is considered as linear and discrete and only one situation (environment) is possible at each moment. However, we might be interested in generalising such approach and presenting a system based on more general theoretical assumptions. A *generalised* $\mathcal{LTK}$-frame can be thought as a structure which is identical to an $\mathcal{LTK}$-frame except for the fact that it allows distinct environment-clusters to be concurrent (see Figure 4)[3]. This aspect may result of use whenever we aim at reasoning about simultaneous alternatives to a given state of affairs without assuming the time as branching.

[3]Following the terminology previously used, a *generalized* $\mathcal{LTK}$-frame can be understood as an $\mathcal{LTK}$-frame lacking the property PM.3.

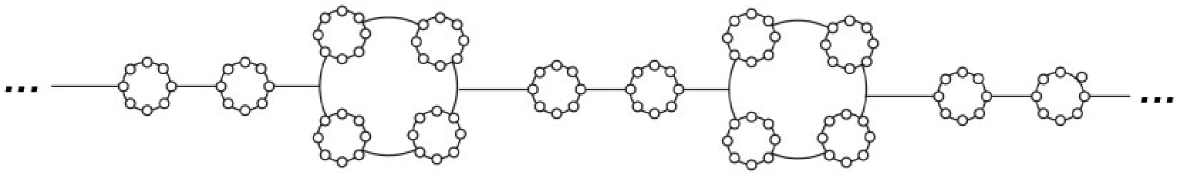

FIG. 4. Scheme of the structure of a generalized $\mathcal{LTK}$-frame.

The logic GLTK generated by this class of generalised frames can be easily proven to be decidable with respect to its theorems.

To prove this claim it is sufficient merely to modify our previous proof in the following way:

(a) Let $\mathcal{AS}_{\mathsf{GLTK}}$ be an axiomatic system obtained by deleting Axiom $Dum_{\Box_{\preccurlyeq}}$ from $\mathcal{AS}_{\mathsf{LTK}}$ and let $\mathsf{GLTK}_{\mathsf{ax}}$ be the set of theorems generated by $\mathcal{AS}_{\mathsf{GLTK}}$.
(b) Trivially, delete part (b) from Theorem 2.7;
(c) Change the filtration set Γ to $\Gamma^- := \Gamma_0 \cup \Gamma_2^- \cup \Gamma_3^-$ where:

- $\Gamma_0 := Sub(\mathtt{B})$
- $\Gamma_2^- := \{\mathsf{K_e}\Box_{\preccurlyeq}\mathtt{D} \mid \Box_{\preccurlyeq}\mathtt{D} \in \Gamma_0\}$
- $\Gamma_3^- := \{\mathsf{K}_i\mathsf{K_e}\mathtt{D} \mid \mathsf{K_e}\mathtt{D} \in \Gamma_0 \cup \Gamma_2^-\}$ for $1 \le i \le k$

At the end of the process of filtration, we obtain a model based on a finite *generalised* $\mathcal{LTK}$-frame.

References

[1] Brandon Bennett, Clare Dixon, Michael Fisher, Enrico Franconi, Ian Horroks, and Maarten De Rijke. Combination of modal logics. *Artificial Intelligence Review*, 17:1–20, 2002.
[2] Erica Calardo. Admissible inference rules in the linear logic of knowledge and time LTK. *Logic Journal of the IGPL*, 14(1):15–34, 2006.
[3] Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, semantic approach. *Bulletin of the Section of Logic*, 34(1):13–21, 2005.
[4] Michael A. E. Dummett and Edward John Lemmon. Modal logics between S4 and S5. *Z. Math. Logik Grundlagen Math.*, 5:250–264, 1959.
[5] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge*. MIT Press, 1995.
[6] Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, North-Holland, New York - Amsterdam, 2003.
[7] Dov M. Gabbay. A general filtration method for modal logics. *Journal of Philosophical Logic*, 1(1):29–34, 1972.
[8] Robert Goldblatt. *Logics of time and computation*. Center for the Study of Language and Information, 1987.
[9] Robert Goldblatt. Mathematical modal logic: a view of its evolution. *Journal of Applied Logic*, 1(5-6):309–392, October 2003.

[10] Joseph Y. Halpern, Ron Van Der Meyden, and Moshe Y. Vardi. Complete axiomatization for reasoning about knowledge and time. *SIAM Journal on Computing*, 33(3):674–703, 2004.
[11] Agi Kurucz. Combining modal logics. In *Handbook of Modal Logic*. Elsevier, (to appear).
[12] Vladimir V. Rybakov. Refined common knowledge logics or logics of common information. *Archive for mathematical Logic*, 42:179–200, 2003.

Received 9 March 2007

Appendix B

International Conferences (contributed papers)

(i) Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, a semantic approach (contributed paper). In *European Logic Colloquium*, Athens, Greece, July 28–August 3 2005.

(ii) Erica Calardo. Inference in modal and multi-modal logic (poster). In *Doctoral Symposium at Conference on Principles of Knowledge Representation and Reasoning*, Lake District, UK, June 2–5 2006.

(iii) Erica Calardo and Vladimir V. Rybakov. A sound and complete axiomatization for the linear logic of knowledge and time LTK (contributed paper). In *European Logic Colloquium*, Nijmegen, the Netherlands, July 26–August 2 2006.

(iv) Erica Calardo. Admissible rules for the multi modal logic of knowledge and time LTK (contributed paper). In *13th International Congress of*

Logic Methodology and Philosophy of Science (CLMPS2007), Beijing, China, August 9–15 2007.

(v) Erica Calardo. Admissible rules in the multi-modal logic of knowledge and time LTK (contributed paper). In *2nd World Congress and School on Universal Logic (UNILOG2007)*, Xi'an, China, August 16–22 2007.

Logic Colloquium 2005

Combining Time and Knowledge, A Semantic Approach

Erica Calardo and Vladimir V. Rybakov

Department of Computing and Mathematics, Manchester Metropolitan University, All Saints, Oxford Rd., M15 6BH Manchester, UK

E.Calardo@mmu.ac.uk, V.Rybakov@mmu.ac.uk

The language of multi-modal logic is a good tool to study the interaction of agents' knowledge in the flow of time. Many different axiomatic systems for multi-modal propositional logics involving time and knowledge modalities have been introduced. We study a logic of this kind, but generated in a semantic way by introducing a certain class $\mathbb{TK}$ of multi-modal frames, called $\mathcal{LTK}$-frames. In these frames the flow of time is linear and agents are operating synchronously. One agent - the *wise agent* - plays a special role: the *wise agent* knowledge is the universal modality on time clusters in these frames. The multi-modal logic LTK is the set of all $\mathcal{LTK}$-valid formulae. It has been already proved by us that LTK has the *fmp*, and hence LTK is decidable.

Our present research is devoted to extending this result to admissible inference rules. We consider the logic LTK_1, a weaker variant of LTK, with only one agent besides the *wise agent*. Our approach involves the construction of n-characterising models $Ch_{\mathsf{LTK}}(n)$ for LTK_1 and the description of free temporal algebras $\mathcal{F}_w(\mathsf{LTK}_1)$ by means of these models. A rule r is admissible in LTK_1 if and only if r is valid in the models $Ch_{\mathsf{LTK}}(n)$ with respect to all definable valuations. Using this approach we prove

Theorem 1 *The logic LTK_1 is decidable w.r.t. admissible inference rules.*

Corollary 2 *The quasi-equational theory of free temporal algebra $\mathcal{F}_w(\mathsf{LTK}_1)$ is decidable.*

Inference in Modal and Multi-Modal Logics

Erica Calardo

Department of Computing and Mathematics, Manchester Metropolitan University,
John Dalton Building, Chester Street, Manchester M1 5GD, UK.
E-mail: E.Calardo@mmu.ac.uk
Supervisor: Dr V.V. Rybakov

1 Background

How do human beings reason? How do they interact? There will never be, probably, a fully satisfactory answer to such questions, but if we were to enumerate the tools developed so far to try to give an answer, we would find that Modal and Multi-modal logics are doubtless among the best ones. Since they combine a highly expressive power with several handle semantic tools (as the easily understandable Possible World Semantics [1]), they suddenly become far more effective than classical propositional systems for this purpose. Propositional modal logic have been investigated since the dawn of philosophical, hence logical, research. The study of modal logic began with Aristotle and his attempt to analise statements containing words like *possible* and *necessary.* Recently modal logics have found several applications in Artificial Intelligence and Computer Science in the attempt to formalise, for instance, reasoning about the behavior of programs (cf [2]).

The main feature of modal logics is that they enable to switch from *extensional* languages (expressing only *facts, statements* which can be either true or false) to *intensional* ones. Modal logics deal with sentences that are qualified by modalities such as can, could, might, may, must et c. A modality is, therefore, any word that could be added to a statement p to modify it: the new statement says something about the *mode of truth* of the old one (cf [2]). Modal logics are often constructed by adding one or more sentential operators (usually $\Box$ and $\Diamond$) to a classical propositional system. Likewise, multi-modal logics are obtained by adding more than one modal operator to an existing logical system.

Traditionally the modal operators $\Box$ and $\Diamond$ are interpreted as expressing necessity and possibility respectively, so that an expression like $\Box p$ would be read as *it is necessary that* p. But this is only one of the many possible interpretations: modal languages are extremely flexible. Our choice would then be suggested by the context we are to describe. Temporal logics, for instance, are obtained by joining tense operators to an existing logic, usually the classical propositional calculus (cf. [3]). In the case of temporal logics, we can interpret the modal proposition $\Box p$ as *it will always be the case that* p, and its dual $\Diamond p$ as *at some point in the future it will be the case that* p. Therefore, the language of temporal logic is particularly effective when we want to describe the flow of time, towards both future and past. Epistemic logics, on the other hand, are suitable to formalize reasoning about agents not possessing a complete information. They interpret modal operators in terms of knowledge (the expression $\mathsf{K}_{\mathsf{a}}p$ standing for *the agent a knows that* p, see [4]). However, such systems have an expressive limitation. They cannot handle modifications in the pieces of information each agent possess, nor can they give an account of a changing environment. Adding a dynamic dimension to such systems is therefore almost a necessity. The most natural way to partially fix such deficiency is to insert epistemic logics in a temporal framework. Hence we would generate a multi-modal system.

Moreover systems generated by joining operators representing both time and knowledge have already proved themselves to be particularly effective in describing the interaction of agents through the flow of time ([4, 5, 6, 7, 8]). They are generated by adding to an existing propositional system two sets of modalities: one to model the flow of time, the other to describe agents' knowledge. The interaction of such modalities gives a precise account of the dynamic development of agents' knowledge. Though highly interesting and fascinating, this approach has not yet been fully investigated, due to the complexity and the extent of the subject. Nevertheless, in the last decade, the theories developed have found many fruitful applications in the study of human reasoning and computing. These are concerned with the development of systems modelling reasoning about knowledge and space, reasoning under uncertainty or with bounded resources, multi-agent reasoning and other aspects of artificial intelligence.

However, despite the power of multi-modal propositional logics, multi modal languages can only express formulae which are static in a way: the statements only fix a fact, and cannot handle a changing environment. But this is exactly what is required in the case of human reasoning, computation and multi-agent environment. In fact, we are usually more interested in deducing what follows logically given some premises, rather than knowing logical truths. For this reason, inference rules, or logical consecutions, are a core instrument.

For instance, rules can describe properties of modal frames in some cases in which using formulae may be difficult. A good example is Gabbay's *irreflexive rule* (cf. [9]):

$$\mathbf{ir} := \frac{\neg(p \to \Diamond p) \to \mathtt{A}}{\mathtt{A}}$$

(where p does not occur in the formula $\mathtt{A}$). This rule states that each world of a model, where $\mathtt{A}$ is not valid, should be irreflexive.

The greatest class of rules that can be applied to a certein logic is that of *admissible consecutions.* Such a class contains all those rules under which the logic itself is closed, i.e. all those rules that can be applied to a given logic while preserving its set of theorems. So far, the research in this field has investigated many modal and superintuitionistic logics (see, for instance, Ghilardi [10, 11, 12], Golovanov et al. [13], Iemhoff [14, 15], Jeřábek [16], Rybakov [17, 18, 19]). The investigation began with Harrop's observation (cf. [20]) that we can enlarge an axiomatic system by adding admissible, though not derivable, inference rules.This approach led Friedman (see [21]) to ask whether there is an algorithm to recognise the rules admissible in IPC, the intuitionistic propositional calculus. This question and its analogues for modal logic has been solved by Rybakov [19, 22, 23], and a robust mathematical theory has been developed[1].

However, for the case of multi-modal logics, not much is known concerning admissible inference rules, though there have been some attempts to approach the problem (cf. for instance Golovanov et al. [25]). Nowadays, logics of this kind are an active research area and the axiomatic systems that have been constructed and examined are numerous (cf. Halpern et al. [8]).

2 Our research

In our research, we would like to extend the investigation to a multi-modal propositional logic, LTK (Linear Time and Knowledge), which combines tense and knowledge modalities. This logic is semantically defined as the set of all $\mathcal{LTK}$-valid formulae, where $\mathcal{LTK}$-frames are multi-modal Kripke-frames combining a linear and discrete representation of the flow of time with special $S5$-like modalities, defined at each time cluster and representing knowledge. Initial results of our research have been published in [5] and a more comprehensive paper has been accepted for publication ([6]). They have also been presented by the author at *European Logic Colloquium 2005* (Athens).

So far the aim of our research has been to show that the multi-modal propositional system LTK is decidable with respect to admissible inference rules, i.e. we have found an algorithm which, given a rule $\mathbf{r}$, checks if $\mathbf{r}$ is admissible for LTK. This have been proved in the following way:

a. we start by showing that LTK has the *effective finite model property* and hence it is decidable with respect to theorems;

b. we construct special countable n-

[1] For a more detailed historical account see Rybakov [19], Iemhoff [24].

characterising models for LTK;

c. we show that an inference rule **r** is admissible in LTK if and only if it is valid in all the frames of a special kind, whose size is computable and bounded by the size of **r**. Hence, we prove that LTK is decidable w.r.t. inference rules.

3 Further Work

In the light of the results obtained, we plan to extend our research to finding a sound and complete axiomatisation for our logic system LTK. Although many axiomatic system have been presented in [8], we remind the reader that such system is original and as far as we are concerned it has never been investigated.

This would enable us to prove several more important properties, especially Kripke Completeness, and it would confer more generality to the work done up to now. We would approach the matter with both standard tools (as canonical models, filtration and unravelling) and new methods. The idea is to model each knowledge modality as an $S5$-modal system, while the time would be represented as an $S4.3$-system. Several new axioms mixing and modelling the interactions among modalities would be introduced.
construct an algorithm recognising admissibility for inference rules with meta-variables.

This would enable us to prove several more important properties, especially Kripke Completeness, and it wuld confer more generality to the work done up to now. We would approach the matter with both standard tools (as *canonical models, filtration* and *unravelling*) and new methods. The idea is to model each knowledge modality as an $S5$-modal system, while the time would be represented as an $S4.3$-system. Several new axioms mixing and modelling the interactions among modalities would be introduced.

Methodology. More specifically, the structure of our proof and the techniques involved would be as follows.

a. Syntax. Introduction of our Hilbert-style axiomatic system: language, axiom schemata, rules and syntactical definitions.

b. n-Canonical Model.

c. Open submodels.

d. Filtration. This technique is used to reduce the amount of worlds in an infinite frame to a finite number. It is particularly effective when the canonical frame of a given logic system does not have the required properties, like our case. Applying this technique we would prove the *fmp* (finite model property).

e. Unravelling. This is a transformation which would very likely help us reaching our goal. It unravels all the loops inside our model, so that the final construct is finite, linear and discrete.

Moreover we are planning to extend our research to finding a *basis* fpr admissible inference rules in LTK. A basis can be understood as the smallest set of rules from which we can derive all vthe admissible consecutions. More formally, we say:

Definition 3.1 *A collection $\mathcal{B}$ of admissible inference rules for some logic L is said to be a* basis *for all admissible rules of L iff every rule r is admissible for L if and only if r is a consequence of $\mathcal{B}$ in L.*

There are several difficulties hidden in the scheme presented. If in a multi-modal system there are no interactions between modalities, many important properties do transfer, like decidability and finite model properties. As soon as we have interaction between modalities, this is no longer true (see [26]). It might happen that such combined logics turned out to be undecidable, hence useless from the computational and knowledge representation point of view. That is why it is so important a deep and precise analysis of combination and interactions between modalities. Multi-modal logics are powerful and useful and in order to be applied to Computer Science and Artificial Intelligence need to be investigated deeply and carefully. Our result is therefore original and it allows a little step towards a complete investigation of the area.

The interdisciplinary nature of our project is clear. It concerns philosophy, knowledge representation, mathematics, artificial intelligence and computation as well. Modern applications of logic in computer science and artificial intelligence often require languages able to represent knowledge about dynamic systems (such as program executions, information flows, expert, distributed and multi-agent systems, temporal databases, consulting by Web, et c.). Distinct designed logics, e.g. modal and temporal logics for multi-agent reasoning, serve these applications in a very efficient way, and we would absorb and develop some of these techniques to represent logical consequences in artificial intelligence and computation.

References

[1] Saul Kripke. Semantical considerations on modal logic. *Acta Philosophica Fennica*, 16:83–94, 1963.

[2] Robert Goldblatt. Mathematical modal logic: a view of its evolution. *Journal of Applied Logic*, 1(5-6):309–392, October 2003.

[3] Robert Goldblatt. *Logics of time and computation.* Center for the Study of Language and Information, 1987.

[4] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge.* MIT Press, 1995.

[5] Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, semantic approach. *Bulletin of the Section of Logic*, 34(1):13–21, 2005.

[6] Erica Calardo. Admissible inference rules in the linear logic of knowledge and time LTK. *Logic Journal of the IGPL*, in press.

[7] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, North-Holland, New York - Amsterdam, 2003.

[8] Joseph Y. Halpern, Ron Van Der Meyden, and Moshe Y. Vardi. Complete axiomatization for reasoning about knowledge and time. *SIAM Journal on Computing*, 33(3):674–703, 2004.

[9] Dov M. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on tense frames. In U. Mönnich, editor, *Aspects of Philosophical Logic: Some Logical Forays into Central Notions of Linguistics and Philosophy*, pages 67–89. Reidel, Dordrecht, 1981.

[10] Silvio Ghilardi. Unification in intuitionistic logic. *Journal of Symbolic Logic*, 64(2):859–880, 1999.

[11] Silvio Ghilardi. Best solving modal equations. *Annals of Pure and Applied Logic*, 10:183–198, 2000.

[12] Silvio Ghilardi. A resolution/tableaux algorithm for projective approximations in ipc. *Logic Journal of the IGPL*, 10(3):227–241, 2002.

[13] M.I. Golovanov, Vladimir V. Rybakov, and E.M. Yurasova. A necessary condition for rules to be admissible in temporal tomorrow logic. *Bulletin of the Section of Logic*, 32(4):213–220, 2003.

[14] Rosalie Iemhoff. A(nother) characterization of intuitionistic propositional logic. *Annals of Pure and Applied Logic*, 113(1-3):161–173, 2001.

[15] Rosalie Iemhoff. Intermediate logics and visser's rules. *Notre Dame Journal of Formal Logic*, 46(1):65–81, 2005.

[16] Emil Jeřábek. Admissible rules of modal logics. *J Logic Computation*, 15(4):411–431, 2005.

[17] Vladimir V. Rybakov. Intermediate logics preserving admissible inference rules of heyting calculus. *Mathematical Logic Quarterly*, 39:403–415, 1993.

[18] Vladimir V. Rybakov. Hereditarily structurally complete modal logics. *Journal of Symbolic Logic*, 60(1):266–288, 1995.

[19] Vladimir V. Rybakov. *Admissible Logical Inference Rules*, volume 136 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, North-Holland, New York - Amsterdam, 1997.

[20] Ronald Harrop. Concerning formulas of the types $a \to b \vee c, a \to \exists x(b(x))$. *Journal of Symbolic Logic*, 25(1):27–32, 1960.

[21] Harvey Friedman. One hundred and two problems in mathematical logic. *Journal of Symbolic Logic*, 40(2):113–129, June 1975.

[22] Vladimir V. Rybakov. A criterion for admissibility of rules in the modal system S4 and the intuitionistic logic. *Algebra and Logic*, 23(5):369–384, 1984 (English Translation).

[23] Vladimir V. Rybakov. Bases of admissible rules of the logics S4 and Int. *Algebra and Logic*, 24(1):55–68, 1985 (English Translation).

[24] Rosalie Iemhoff. A(nother) characterization of intuitionistic propositional logic. *Annals of Pure and Applied Logic*, 113(1-3):161–173, 2001.

[25] M.I. Golovanov, A.V. Kosheleva, and V. Rybakov. Logic of visibility, perception, and knowledge and admissible inference rules. *Logic Journal of the IGPL*, 13(2):201–209, 2005.

[26] Brandon Bennett, Clare Dixon, Michael Fisher, Enrico Franconi, Ian Horroks, and Maarten De Rijke. Combination of modal logics. *Artificial Intelligence Review*, 17:1–20, 2002.

A SOUND AND COMPLETE AXIOMATIZATION FOR THE LINEAR LOGIC OF KNOWLEDGE AND TIME LTK

ERICA CALARDO

The multi-modal propositional logic LTK combines linear time and knowledge and is semantically defined as the set of all the formulae valid on $\mathcal{LTK}$-frames, the prefix $\mathcal{LTK}$ standing for *Linear Time and Knowledge*. $\mathcal{LTK}$-frames are multi-modal Kripke-frames that combine a linear and discrete representation of the flow of time with special $S5$-like modalities, defined at each time cluster and representing agents' knowledge.

The logic LTK has already been proved to be decidable with respect both to its theorems and to its admissible inference rules (see [2, 1]).

Our research aims at finding a sound and complete axiomatization of LTK. We have developed an axiomatic system $\mathcal{AS}_{\mathsf{LTK}}$ in which the axioms describing the flow of time give rise to an $S4.3$ modal system, whereas the ones intended to describe agents' knowledge produce a series of $S5$ modal systems. Moreover, several axioms have been added to the system in order to regiment the interactions between distinct modalities. This is the most important part of our approach because there are several problems behind the interactions of different modalities. However, we proved the following result:

THEOREM 1. $\mathsf{LTK} = \mathsf{LTK}_{\mathsf{ax}}$.

REFERENCES

[1] ERICA CALARDO, *Admissible inference rules in the linear logic of knowledge and time* LTK, ***Logic Journal of the IGPL***, vol. 14 (2006), no. 1, pp. 15–34.

[2] ERICA CALARDO and VLADIMIR V. RYBAKOV, *Combining time and knowledge, semantic approach*, ***Bulletin of the Section of Logic***, vol. 34 (2005), no. 1, pp. 13–21.

[3] ROBERT GOLDBLATT, ***Logics of time and computation***, Center for the Study of Language and Information, 1987.

[4] JOSEPH Y. HALPERN, RON VAN DER MEYDEN, and MOSHE Y. VARDI, *Complete axiomatization for reasoning about knowledge and time*, ***SIAM Journal on Computing***, vol. 33 (2004), no. 3, pp. 674–703.

DEPARTMENT OF COMPUTING AND MATHEMATICS, MANCHESTER METROPOLITAN UNIVERSITY, JOHN DALTON BUILDING, CHESTER STREET, MANCHESTER M1 5GD, UK.

E-mail: E.Calardo@mmu.ac.uk

Basis of Admissible rules for the multi modal logic of knowledge and time LTK_1

Erica Calardo

e.calardo@gmail.com
Manchester Metropolitan University, Dep. of Computing and Mathematics
Chester Street, Manchester M1 5GD, UK

Multi-modal propositional systems generated by joining operators representing both time and knowledge have already proved themselves to be particularly effective in describing the interaction of agents through the flow of time ([2, 1, 3, 4, 6]). These logics are generated by adding to an existing propositional system two sets of modalities: one to model the flow of time, the other to describe agents' knowledge. The interaction of such modalities gives a precise account of the dynamic development of agents' knowledge.
However, despite the power of multi-modal propositional logics, multi modal languages can only express formulae which are static in a way: the statements only fix a fact, and cannot handle a changing environment. But this is exactly what is required in the case of human reasoning, computation and multi-agent environment. In fact, we are usually more interested in deducing what follows logically given some premises, rather than knowing logical truths. For this reason, inference rules, or logical consecutions, are a core instrument.

Our research aims at investigating a multi-modal propositional logic, LTK_1 (Linear Time and Knowledge), which combines tense and knowledge modalities. This logic is semantically defined as the set of all $\mathcal{LTK}$-valid formulae, where $\mathcal{LTK}$-frames are multi-modal Kripke-frames combining a linear and discrete representation of the flow of time with special $S5$-like modalities, defined at each time cluster and representing agents' knowledge.

So far we have proved that the logic LTK_1 (i) has the finite model property [2]; (ii) is decidable with respect to its admissible inference rules [1]; (iii) has a finite axiomatisation[1].

Our latest results is to show that LTK_1 has a finite basis for admissible inference rules, i.e. all those rules under which the logic itself is closed.

References

[1] Erica Calardo. Admissible inference rules in the linear logic of knowledge and time LTK. *Logic Journal of the IGPL*, 14(1):15–34, 2006.

[2] Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, semantic approach. *Bulletin of the Section of Logic*, 34(1):13–21, 2005.

[3] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge.* MIT Press, 1995.

[4] Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, North-Holland, New York - Amsterdam, 2003.

[5] Robert Goldblatt. *Logics of time and computation.* Center for the Study of Language and Information, 1987.

[6] Joseph Y. Halpern, Ron Van Der Meyden, and Moshe Y. Vardi. Complete axiomatization for reasoning about knowledge and time. *SIAM Journal on Computing*, 33(3):674–703, 2004.

[1] result presented by E.Calardo and V.V.Rybakov at *European Logic Colloquium* 2007, Nijmegen, The Netherlands

Admissible rules in the multi modal logic of knowledge and time LTK

Erica Calardo

e.calardo@gmail.com
Manchester Metropolitan University, Dep. of Computing and Mathematics
Chester Street, Manchester M1 5GD, UK

Multi-modal logics are one of the best tools developed so far to describe the behaviour of agents throughout the flow of time. Since they combine their highly expressive power and flexibility with several handle semantic tools (as the Possible World Semantics), they are far more effective than classical propositional systems for this purpose.

Multi-modal logics are obtained by adding more than one modal operator to an existing logical system. This feature guarantees flexibility in the sense that the interpretation of the modal operators can be chosen according to the situation to be described. In our research we join tense and knowledge operators to the classical propositional calculus. The interaction of these two sets of modalities gives a precise account of the dynamic development of agents' knowledge (see [1, 2, 3], Fagin *et al.* [4], Gabbay *et al.* [5], Halpern *et al.* [6]).

Although highly interesting and expressive, combined modal logics, however, present several difficulties if seen as a class. It is known, in fact, that if there is no interaction between modalities a transfer of properties (such as *finite model property*, *decidability*, etc.) from the component simple modal logics to the newly generated multi-modal system does apply. But as soon as such interaction takes place it is no longer straightforward to prove that the combined system is conservative with respect to the properties of its components (see Bennett *et al.* [7] and Kurucz [8]). Indeed, in some cases the opposite may apply. Nevertheless, despite such difficulties, interaction between modalities is necessary fully to exploit the power of multi-modal languages.

Moreover, although highly expressive, multi-modal languages have a limitation. The formulae which are expressible by means of multi modal languages are static: they can only state facts and hence they cannot handle a changing environment. But this is exactly what is required in the case of human reasoning, computation and multi-agent environment. In fact, we are usually more interested in discovering what follows given some premises, rather than deducing logical truths. For this reason, inference rules, or logical consecutions, are a core instrument.

An admissible consecution for a given logic is a rule which can be applied to the logic itself while preserving its set of theorems. So far, the research in this field has investigated many modal and superintuitionistic logics. The investigation began with Harrop's observation that we can enlarge an axiomatic system by adding admissible, though not derivable, inference rules. This approach led Friedman to ask whether there is an algorithm to recognise the rules admissible in IPC, the intuitionistic propositional calculus. This question and its analogues for modal logic has been solved by Rybakov [9], and a robust mathematical theory has been developed.

However, for the case of multi-modal logics, not much is known concerning admissible inference rules, although there have been several attempts to approach the problem.

In our research we extend the investigation concerning admissible inference rules to a multi-modal propositional logic, LTK (Linear Time and Knowledge), which combines tense and knowledge modalities and which allows interactions between modal operators. This logic is semantically defined as the set of all $\mathcal{LTK}$-valid formulae, where $\mathcal{LTK}$-frames are multi-modal Kripke-frames combining a linear and discrete representation of the flow of time with special S5-like modalities, defined at each time cluster and representing agents' knowledge.

So far we have proved that LTK is decidable with respect both to its theorems [1], and to its admissible inference rules [2]. Moreover, LTK has been proved to have a finite, sound and complete axiomatisation [3]. Our latest result is that LTK has a finite basis for admissible inference rules.

References

[1] Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, semantic approach. *Bulletin of the Section of Logic*, 34(1):13–21, 2005.

[2] Erica Calardo. Admissible inference rules in the linear logic of knowledge and time LTK. *Logic Journal of the IGPL*, 14(1):15–34, 2006.

[3] Erica Calardo and Vladimir V. Rybakov. An axiomatisation for the multi-modal logic of knowledge and linear time LTK. *Logic Journal of the IGPL*, (to appear).

[4] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge*. MIT Press, 1995.

[5] Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, North-Holland, New York - Amsterdam, 2003.

[6] Joseph Y. Halpern, Ron Van Der Meyden, and Moshe Y. Vardi. Complete axiomatization for reasoning about knowledge and time. *SIAM Journal on Computing*, 33(3):674–703, 2004.

[7] Brandon Bennett, Clare Dixon, Michael Fisher, Enrico Franconi, Ian Horroks, and Maarten De Rijke. Combination of modal logics. *Artificial Intelligence Review*, 17:1–20, 2002.

[8] Agi Kurucz. Combining modal logics. In *Handbook of Modal Logic*, volume 3. Elsevier, 2006.

[9] Vladimir V. Rybakov. *Admissible Logical Inference Rules*, volume 136 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, North-Holland, New York - Amsterdam, 1997.

Bibliography

[1] Fabio Bellissima and Paolo Pagli. *La Veritá Trasmessa.* Sansoni, 1993.

[2] Brandon Bennett, Clare Dixon, Michael Fisher, Enrico Franconi, Ian Horrocks, and Maarten De Rijke. Combination of modal logics. *Artificial Intelligence Review*, 17:1–20, 2002.

[3] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic.* Cambridge University Press, 2001.

[4] Erica Calardo. Admissible inference rules in the linear logic of knowledge and time LTK. *Logic Journal of the IGPL*, 14(1):15–34, 2006.

[5] Erica Calardo. Inference in modal and multi-modal logic (poster). In *Doctoral Symposium at Conference on Principles of Knowledge Representation and Reasoning*, Lake District, UK, June 2–5 2006.

[6] Erica Calardo. Admissible rules for the multi modal logic of knowledge and time LTK (contributed paper). In *13th International Congress of Logic Methodology and Philosophy of Science (CLMPS2007)*, Beijing, China, August 9–15 2007.

[7] Erica Calardo. Admissible rules in the multi-modal logic of knowledge and time LTK (contributed paper). In *2nd World Congress and School on Universal Logic (UNILOG2007)*, Xi'an, China, August 16–22 2007.

[8] Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, a semantic approach (contributed paper). In *European Logic Colloquium*, Athens, Greece, July 28–August 3 2005.

[9] Erica Calardo and Vladimir V. Rybakov. Combining time and knowledge, semantic approach. *Bulletin of the Section of Logic*, 34(1):13–21, 2005.

[10] Erica Calardo and Vladimir V. Rybakov. A sound and complete axiomatization for the linear logic of knowledge and time LTK (contributed paper). In *European Logic Colloquium*, Nijmegen, the Netherlands, July 26–August 2 2006.

[11] Erica Calardo and Vladimir V. Rybakov. An Axiomatisation for the Multi-modal Logic of Knowledge and Linear Time LTK. *Logic Journal of the IGPL*, 15(3):239–254, 2007.

[12] Rudolf Carnap. *Meaning and Necessity: a Study in Semantics and Modal Logic*. University of Chicago Press, 1996.

[13] Giovanna Corsi. Counterparts and possible worlds. a study on quantified modal logics. *Preprint*, 21:1–61, 2001.

[14] Giovanna Corsi. A unified completeness theorem for quantified modal logics. *Journal of Symbolic Logic*, 67(4):1483–1510, 2002.

[15] Giovanna Corsi and Gabriele Tassi. Intuitionistic logic freed of all metarules. *Journal of Symbolic Logic*, 72(4):1204–1218, 2007.

[16] Michael A. E. Dummett and Edward John Lemmon. Modal logics between S4 and S5. *Z. Math. Logik Grundlagen Math.*, 5:250–264, 1959.

[17] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge*. MIT Press, 1995.

[18] Melvin Fitting and R. L. Mendelsohn. *First Order Modal Logic*. Kluwer Academic Publishers, London, 1998.

[19] Gottlob Frege. On sense and reference. In P. Geach and M. Black, editors, *Translations from the Philosophical Writings of Gottlob Frege*, pages 25–50. Blackwell Publishers, Oxford, 1952.

[20] Harvey Friedman. One hundred and two problems in mathematical logic. *Journal of Symbolic Logic*, 40(2):113–129, June 1975.

[21] Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, North-Holland, New York - Amsterdam, 2003.

[22] Dov M. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on tense frames. In U. Mönnich, editor, *Aspects of Philosophical Logic: Some Logical Forays into Central Notions of Linguistics and Philosophy*, pages 67–89. Reidel, Dordrecht, 1981.

[23] Silvio Ghilardi. Unification in intuitionistic logic. *Journal of Symbolic Logic*, 64(2):859–880, 1999.

[24] Silvio Ghilardi. Best solving modal equations. *Annals of Pure and Applied Logic*, 10:183–198, 2000.

[25] Silvio Ghilardi. A resolution/tableaux algorithm for projective approximations in IPC. *Logic Journal of the IGPL*, 10(3):227–241, 2002.

[26] Robert Goldblatt. *Logics of time and computation.* Center for the Study of Language and Information, 1987.

[27] Robert Goldblatt. Mathematical modal logic: a view of its evolution. *Journal of Applied Logic*, 1(5-6):309–392, October 2003.

[28] M.I. Golovanov, A.V. Kosheleva, and V. Rybakov. Logic of Visibility, Perception, and Knowledge and Admissible Inference Rules. *Logic Journal of the IGPL*, 13(2):201–209, 2005.

[29] M.I. Golovanov, Vladimir V. Rybakov, and E.M. Yurasova. A necessary condition for rules to be admissible in temporal tomorrow logic. *Bulletin of the Section of Logic*, 32(4):213–220, 2003.

[30] Joseph Y. Halpern, Ron Van Der Meyden, and Moshe Y. Vardi. Complete axiomatization for reasoning about knowledge and time. *SIAM Journal on Computing*, 33(3):674–703, 2004.

[31] Joseph Y. Halpern and Moshe Y. Vardi. The complexity of reasoning about knowledge and time, i: lower bounds. *Journal of Computer and System Sciences*, 38(1):195–237, 1989.

[32] Ronald Harrop. Concerning formulas of the types $a \to b \vee c, a \to \exists x(b(x))$. *Journal of Symbolic Logic*, 25(1):27–32, 1960.

[33] Jakko Hintikka. Modality and quantification. *Theoria*, 27:117–128, 1961.

[34] Rosalie Iemhoff. A(nother) characterization of intuitionistic propositional logic. *Annals of Pure and Applied Logic*, 113(1-3):161–173, 2001.

[35] Rosalie Iemhoff. *Provability Logic and Adsmissible Rules.* PhD the-

sis, Institute for Logic, Language and Computation, Amsterdam, The Netherlands, 2001.

[36] Rosalie Iemhoff. Intermediate logics and Visser's rules. *Notre Dame Journal of Formal Logic*, 46(1):65–81, 2005.

[37] Rosalie Iemhoff. On the rules of intermediate logics. *Archive for mathematical Logic*, 45(5):581–599, 2006.

[38] Emil Jeȓábek. Admissible rules of modal logics. *Journal of Logic Computation*, 15(4):411–431, 2005.

[39] Emil Jeřábek. Complexity of admissible rules. *Archive for Mathematical Logic*, 46(2):73–92, 2007.

[40] S. Kanger. A note on quantification and modalities. *Theoria*, 23:133–134, 1957.

[41] Saul Kripke. A completeness theorem in modal logic. *Journal of Symbolic Logic*, 24:1–14, 1959.

[42] Saul Kripke. Semantical considerations on modal logic. *Acta Philosophica Fennica*, 16:83–94, 1963.

[43] Saul Kripke. *Naming and Necessity.* Blackwell Publishers, 1980.

[44] Agi Kurucz. Combining modal logics. In J. Van Benthem, P. Blackburn, and F. Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, pages 497–564. Elsevier, 2007.

[45] Richard Montague. Logical necessity, physical necessity, ethics and quantifiers. *Inquiry*, 43:259–269, 1960.

[46] Alvin Plantinga. *The Nature of Necessity.* Oxford University Press, Oxford, 1974.

[47] Alvin Plantinga. De re et de dicto. In D. Silvestrini, editor, *Individui e Mondi Possibili. Problemi di Semantica Modale*, pages 167–190. Feltrinelli, Milano, 1979.

[48] Ian Pratt-Hartmann. Total knowledge. In *17th National Conference on Artificial Intelligence (AAAI-00)*, pages 423–428, 2000.

[49] Ian Pratt-Hartmann, Marco Aiello, and Johan van Benthem. *Handbook of Spatial Logics*, chapter What is Spatial Logic?, pages 1–11. Springer, 2007.

[50] Arthur N. Prior. *Papers on Time and Tense.* Oxford University Press, 2003.

[51] Vladimir V. Rybakov. A criterion for admissibility of rules in the modal system S4 and the intuitionistic logic. *Algebra and Logic*, 23(5):369–384, 1984 (English Translation).

[52] Vladimir V. Rybakov. Bases of admissible rules of the logics S4 and Int. *Algebra and Logic*, 24(1):55–68, 1985 (English Translation).

[53] Vladimir V. Rybakov. Intermediate logics preserving admissible inference rules of heyting calculus. *Mathematical Logic Quarterly*, 39:403–415, 1993.

[54] Vladimir V. Rybakov. Hereditarily structurally complete modal logics. *Journal of Symbolic Logic*, 60(1):266–288, 1995.

[55] Vladimir V. Rybakov. *Admissible Logical Inference Rules*, volume 136 of

Studies in Logic and the Foundations of Mathematics. Elsevier, North-Holland, New York - Amsterdam, 1997.

[56] Vladimir V. Rybakov. Refined common knowledge logics or logics of common information. *Archive for mathematical Logic*, 42:179–200, 2003.

[57] Vladimir V. Rybakov. Admissible inference rules in temporal linear logics based at integer numbers. In *9th Asian Logic Confernce*, Novosibirsk, 16–19 August 2005.

[58] Vladimir V. Rybakov. Inference in linear temporal logics. In *First World Congress and School on Universal Logic*, Montreux, Switzerland, March 26th–April 3rd 2005.

[59] Vladimir V. Rybakov. Inference in temporal next-time. In H. Artemov, H. Barringer, A. d'Avila Garcez, L. Lamb, and J. Woods, editors, *We Will Show Them: Essays in Honour of Dov Gabbay*, pages 615–637. Kings College Publications, London, 2005.

[60] Vladimir V. Rybakov. Logical consecutions in discrete linear temporal logic. *Journal of Symbolic Logic*, 70(4):1137–1149, 2005.

[61] Vladimir V. Rybakov. Logical consecutions in intransitive temporal linear logic of finite intervals. *Journal of Logic and Computation*, 15(5):663–678, 2005.

[62] Vladimir V. Rybakov. Intransitive linear temporal logic based on integer numbers. In Goncharov S. et al., editor, *Proceedings of the Asian Logic 9th Conference*, pages 233–253. World Scientific Publishing Co., 2006.

[63] Vladimir V. Rybakov. Linear temporal logic with until and before on integer numbers, deciding algorithms. In Dima Grigorviec and et al., editors, *Computer Science - Theory and Applications, Lecture Notes in Computer Scince*, volume 3967, pages 322–334. Springer, 2006.

[64] M. Sato. A study of kripke-style methods of some modal logics by gentzen's sequential method. *Publications of the Research Institute for Mathematical Sciences*, 13(2), 1977.

[65] Krister Segerberg. *An Essay In Classical Modal Logic*, volume 13 of *Filosofiska Studier*. Mimeograph, Uppsala, 1971.

[66] Edith Spaan. Nexttime is not necessary (extended abstract). In *Theoretical aspects of reasoning about knowledge*, Morgan Kaufmann Ser. Represent. Reason., pages 241–256, San Mateo, CA, 1990. Morgan Kaufmann.

[67] Richard H. Thomason. Modality and reference. *Noûs*, pages 359–372, 1968.

[68] A.S. Troelstra and H. Schwichtenberg. *Basic proof theory (2nd ed.)*. Cambridge University Press, New York, NY, USA, 2000.

[69] Yde Venema. Temporal logic. In L. Goble, editor, *The Blackwell Guide to Philosophical Logic*, chapter 10, pages 203–223. Blackwell Publishers, Malden, USA, 2001.

[70] Michael Wooldridge. *An Introduction To MultiAgent Systems*. John Wiley & Sons, LTD, Chichester, UK, 2002.

[71] Mike Wooldridge and Alessio Lomuscio. A Computationally Grounded

Logic of Visibility, Perception, and Knowledge. *Logic Journal of the IGPL*, 9(2):257–272, 2001.

[72] A. Zakharyaschev, M. adn Chagrov. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Clarendon Press, Oxford, 1997.